Random walks are completely determined by their trace on the positive half-line

Mateusz Kwaśnicki

Wrocław University of Science and Technology

mateusz.kwasnicki@pwr.edu.pl

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Main Theorem

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A few remarks

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V. Vigon
Simplifiez vos Lévy en titillant la factorisation de Wiener–Hopf
PhD thesis, INSA de Rouen, 2001

Random walks

- A **random walk** X_n is a sequence of partial sums of i.i.d. random variables:

$$X_n = \Delta X_1 + \Delta X_2 + \ldots + \Delta X_n,$$

where $\Delta X_1, \Delta X_2, \ldots$ are independent and identically distributed on $\mathbb{R}$.

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- We write $A \stackrel{\mathrm{d}}{=} B$ if $\mathbb{P}(A > t) = \mathbb{P}(B > t)$ for all $t \in \mathbb{R}$.
- Of course if $X_1 \stackrel{\mathrm{d}}{=} Y_1$, then $X_n \stackrel{\mathrm{d}}{=} Y_n$ for all $n = 1, 2, \ldots$; in this case we say that X_n and Y_n are **identical**.

Main theorem

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If X_n and Y_n are non-trivial random walks, and

$$\mathbb{P}(X_n > t) = \mathbb{P}(Y_n > t)$$

for all $n = 1, 2, \ldots$ and all $t \in (0, \infty)$, then the same is true for all $t \in \mathbb{R}$ (that is, X_n and Y_n are identical).

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- This was proved by Chaumont and Doney under additional conditions on X_n and Y_n:
  - if X_1 has exponential moments; or
  - if $\mathbb{P}(X_1 > t)$ is completely monotone on $(0, \infty)$; or
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  - if X_1 has analytic density function on $(0, \infty)$.
- This covers a majority of interesting examples.
- It is often enough to take $n = 1, 2$ in the assumption.

Simple reformulation

Theorem (equivalent version)

If X_n and Y_n are non-trivial random walks, and

$$\max\{0, X_n\} \stackrel{\mathrm{d}}{=} \max\{0, Y_n\}$$

for all $n = 1, 2, \ldots$, then

$$X_n \stackrel{\mathrm{d}}{=} Y_n$$

for all $n = 1, 2, \ldots$

Some fluctuation theory

- Define $\overline{X}_n = \max\{0, X_1, X_2, \ldots, X_n\}$.

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- **Spitzer's formula**: if $|w| < 1$ and $\operatorname{Im} z \geqslant 0$, then

$$\sum_{n=0}^{\infty} \left(\mathbb{E} \exp(iz\overline{X}_n)\right) w^n = \exp\left(\sum_{n=0}^{\infty} \frac{\mathbb{E} \exp(iz \max\{0, X_n\})}{n} w^n\right).$$

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- Knowing the distributions of $\overline{X}_n$ is thus equivalent to knowing the distributions of $\max\{0, X_n\}$.

Another reformulation

Theorem (equivalent version)

If X_n and Y_n are non-trivial random walks, and

$$\overline{X}_n \stackrel{\mathrm{d}}{=} \overline{Y}_n$$

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- Let N be the smallest number n such that $\overline{X}_n > 0$
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If X_n and Y_n are non-trivial random walks with equal upward space-time Wiener–Hopf factors, then X_n and Y_n are identical.

Lévy processes

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Corollary

If X_t and Y_t are non-trivial Lévy processes, and

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- Conjectured by Vigon, proved under extra assumptions by Chaumont and Doney

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Theorem (extended version)

If μ and ν are non-trivial measures and

$$\mu^n(A) = \nu^n(A)$$

for all Borel $A \subseteq (0, \infty)$ and $n = 1, 2, \ldots$, then $\mu = \nu$.

Change of notation

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- Denote:

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- Now $\mu = \alpha + \beta$ and $\nu = \alpha + \gamma$.

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$$\Big\Downarrow \text{ (simple algebra)}$$

$$\mathbb{1}_{(0,\infty)}(\alpha^n * \beta) = \mathbb{1}_{(0,\infty)}(\alpha^n * \gamma) \text{ for all } n = 0, 1, \dots$$

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$$\Downarrow \text{(complex analysis)}$$

$$\beta = \gamma.$$

- We prove that

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- Induction with respect to n.
- For $n = 0$:

$$\mathbb{1}_{(0,\infty)}(\beta^k) = 0 = \mathbb{1}_{(0,\infty)}(\gamma^k).$$

- Suppose that

$$\mathbb{1}_{(0,\infty)}(\alpha^n * \beta^k) = \mathbb{1}_{(0,\infty)}(\alpha^n * \gamma^k) \text{ for } n = 0, 1, \ldots, N-1$$
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- By the binomial formula,

$$\begin{aligned}
(\alpha+\beta)^{N+1} - (\alpha+\gamma)^{N+1} \\
&= \alpha^{N+1} - \alpha^{N+1} && (j=0) \\
&+ (N+1)\big(\alpha^N * \beta - \alpha^N * \gamma\big) && (j=1) \\
&+ \sum_{j=2}^{N} \binom{N+1}{j} \big(\alpha^{N+1-j} * \beta^j - \alpha^{N+1-j} * \gamma^j \big) \\
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\overbrace{(\alpha+\beta)^{N+1} - (\alpha+\gamma)^{N+1}}^{\text{zero on } (0,\infty) \text{ by the assumption}} &= \alpha^{N+1} - \alpha^{N+1} && (j=0) \\
&+ (N+1)\big(\alpha^N * \beta - \alpha^N * \gamma\big) && (j=1) \\
&+ \sum_{j=2}^{N} \binom{N+1}{j} \big(\, \alpha^{N+1-j} * \beta^j - \alpha^{N+1-j} * \gamma^j \,\big) \\
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&\quad + (N+1)\big(\alpha^N * \beta - \alpha^N * \gamma\big) && (j=1) \\
&\quad + \sum_{j=2}^{N} \binom{N+1}{j} \big(\underbrace{\alpha^{N+1-j} * \beta^j - \alpha^{N+1-j} * \gamma^j}_{\text{zero on } (0,\infty) \text{ by the induction hypothesis}}\big) \\
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&\qquad + \underbrace{\beta^{N+1} - \gamma^{N+1}}_{\text{zero on } (0,\infty)}. && (j = N+2)
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- Thus, $0 = (N+1)\big(\alpha^N * \beta - \alpha^N * \gamma\big)$ on $(0, \infty)$.

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- Larger values of k: induction within induction.

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$$\begin{array}{c} \mathbb{1}_{(0,\infty)}(\alpha^N * \beta^K) \\ \| \\ \mathbb{1}_{(0,\infty)}\big((\,\alpha^N * \beta^{K-1}\,) * \beta\big) \end{array}$$

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- Then, $\sigma * \beta = (\mathbb{1}_{(-\infty,0]}\sigma) * \beta + (\mathbb{1}_{(0,\infty)}\sigma) * \beta$

$$\mathbb{1}_{(0,\infty)}(\alpha^N * \beta^K)$$

$$||$$

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$$\mathbb{1}_{(0,\infty)}(\alpha^N * \beta^k) = \mathbb{1}_{(0,\infty)}(\alpha^N * \gamma^k) \text{ for } k = 1, 2, \ldots, K-1.$$

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- Equivalently: it is not possible to have

$$\mathbb{1}_{(0,\infty)}(\alpha^n * (\beta - \gamma)) = 0 \text{ for all } n = 0, 1, \ldots$$

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- We proceed by contradiction.

- We know that $\alpha^n * (\beta - \gamma)$ is concentrated on $(-\infty, 0]$ for all $n = 1, 2, \ldots$

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- Define analytic extensions of characteristic functions:

$$f(z) = \int_{(0,\infty)} e^{izt}\alpha(dt) \qquad (\operatorname{Im} z \geqslant 0)$$

$$g(z) = \int_{(-\infty,0]} e^{izt}(\beta - \gamma)(dt) \qquad (\operatorname{Im} z \leqslant 0)$$

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- Let $\mathbb{C}_\pm = \{z \in \mathbb{C} : \pm \operatorname{Im} z > 0\}$ and

$$A = \{z \in \mathbb{R} : g(z) = 0\}, \qquad \text{(closed, null)}$$
$$B = \{z \in \mathbb{C}_- : g(z) = 0\}. \qquad \text{(discrete)}$$

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$$\varphi_n(z) = \begin{cases} (f(z))^n & \text{for } z \in \mathbb{C}_+ \cup \mathbb{R}, \\ \dfrac{h_n(z)}{g(z)} & \text{for } z \in \mathbb{C}_- \setminus B. \end{cases}$$

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- Thus, z it is a zero of g of degree at least nk.

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- This is not possible when $n \to \infty$.
- Therefore, φ_n has no poles in $\mathbb{C}_-$: it is analytic in $\mathbb{C} \setminus A$.

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- The functions h_n and g are bounded in $\mathbb{C}_-$. Each of them can be uniquely written as a product of:
  - an outer function $O(z)$,
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- The function $\varphi_n = (\varphi_1)^n$ is of bounded type (a.k.a. Nevanlinna class) in $\mathbb{C}_-$, and thus it has a similar unique factorisation.

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- The singular inner function S_g corresponding to g satisfies

$$|S_g(z)| = \exp\left(a_g \operatorname{Im} z - \frac{1}{\pi} \int_{\mathbb{R}} \frac{-\operatorname{Im} z}{|z-x|^2} \, \lambda_g(dx) \right)$$

for some singular measure $\lambda_g \geqslant 0$ and $a_g \geqslant 0$.

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- Thus, $na_{\varphi_1} = a_{\varphi_n} = a_{h_n} - a_g$ and $n\lambda_{\varphi_1} = \lambda_{\varphi_n} = \lambda_{h_n} - \lambda_g$.
- Taking $n \to \infty$, we see that $a_{\varphi_1} \geqslant 0$ and $\lambda_{\varphi_1} \geqslant 0$.
- That is, S_{φ_1} is bounded on $\mathbb{C}_-$.

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- An outer function O_{φ_1} in the factorisation of φ_1 satisfies

$$|O_{\varphi_1}(z)| = \exp\left(\frac{1}{\pi}\int_{-\infty}^{\infty} \frac{-\operatorname{Im} z}{|z-x|^2} \log|\varphi_1(x)|\, dx\right).$$

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- It follows that φ_1 is a bounded analytic function in $\mathbb{C}_-$.

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- We know that φ_1 is a bounded analytic function in $\mathbb{C}_-$ and in $\mathbb{C}_+$, and hence in $\mathbb{C} \setminus A$.
- Painlevé's theorem asserts that φ_1 extends to a bounded entire function.

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- Thus, f is constant.

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- Painlevé's theorem asserts that φ_1 extends to a bounded entire function.
- As a consequence, φ_1 is constant.
- Thus, f is constant.
- But f is the characteristic function of a measure α concentrated on $(0, \infty)$, it cannot be constant.