Fractional Laplace operator in the unit ball

Mateusz Kwaśnicki

Wrocław University of Science and Technology

mateusz.kwasnicki@pwr.edu.pl

Stable processes conference

Oaxaca, November 10, 2016

Outline

(1) Eigenvalues λ_n of $(-\Delta)^{\alpha/2}$ in a ball

(2) Eigenvalues μ_n of $(1-|x|^2)_+^{\alpha/2}(-\Delta)^{\alpha/2}$

(3) Detour: Jacobi diffusions

(4) Bounds for λ_n.

Based on joint work with:

- **Bartłomiej Dyda** (Wrocław)
- **Alexey Kuznetsov** (Toronto)

Definition

Let X_t denote the **isotropic α-stable Lévy process**.
Let $-\mathbf{L} = -(-\Delta)^{\alpha/2}$ be the generator of X_t:

$$-\mathbf{L}f(x) = \lim_{t \to 0^+} \frac{\mathbf{E}_x f(X_t) - f(x)}{t} .$$

Equivalently:

$$-\mathbf{L}f(x) = c_{d,\alpha} \lim_{\varepsilon \to 0^+} \int_{\mathbf{R}^d \setminus B_\varepsilon} \frac{f(y) - f(x)}{|y - x|^{d+\alpha}} \, dy.$$

Remarks:

- We always assume that $d = 1, 2, \ldots$ and $\alpha \in (0, 2)$.
- $B_r = B(0, r)$, $B = B(0, 1)$.
- Above definitions are pointwise; throughout the talk we ignore (important and delicate) questions about domains of unbounded operators.

Eigenvalue problem

$$\begin{cases} \mathbf{L}\varphi_n(x) = \lambda_n \varphi_n(x) & \text{for } x \in B, \\ \varphi_n(x) = 0 & \text{otherwise.} \end{cases}$$

Classical theorem

Solutions φ_n form an orthonormal basis in $L^2(B)$,

$$0 < \lambda_0 < \lambda_1 \leqslant \lambda_2 \leqslant \ldots$$

and $\varphi_0(x) > 0$ for $x \in B$.

Let τ be the time of first exit from B:

$$\tau = \inf\{t \geqslant 0 : X_t \notin B\}.$$

Then:

$$\mathbf{E}_x\big(\varphi_n(X_t)\mathbf{1}_{\{t<\tau\}}\big) = e^{-\lambda_n t}\varphi_n(x).$$

Theorem (consequence of Bochner's relation)

Let $V(x)$ be a **solid harmonic polynomial** of degree ℓ. Then:

$$\mathbf{L}\big[V(x)\, f(|x|)\big] = V(x)\, g(|x|) \qquad \text{in } \mathbf{R}^d$$

if and only if

$$\mathbf{L}\big[f(|y|)\big] = g(|y|) \qquad \text{in } \mathbf{R}^{d+2\ell}.$$

Remarks:

- True for arbitrary convolution operators $\mathbf{L}$ with isotropic kernels.
- Here 'solid' = 'homogeneous'.
- Examples of $V(x)$: 1, x_1, $x_1 x_2$, $x_1 x_2 \ldots x_d$, $x_1^2 - x_2^2$.
- Solid harmonic polynomials span $L^2(\partial B)$.

We choose $V_{\ell,j}(x)$ so that

- $V_{\ell,j}(x)$ is a solid harmonic polynomial of degree ℓ,
- $\ell \geqslant 0$, $j = 1, 2, \ldots, J_{d,\ell}$, where $J_{d,\ell} = \frac{d+2\ell-2}{d+\ell-2}\binom{d+\ell-2}{\ell}$,
- $V_{\ell,j}(x)$ form the basis of $L^2(\partial B)$.

Corollary

Let $\lambda_{d,n}^{\text{rad}}$ and $\phi_{d,n}^{\text{rad}}(|x|)$ be the n-th **radial** eigenvalue and eigenfunction.

The eigenvalues λ_n are given by $\lambda_{d+2\ell,n}^{\text{rad}}$, where $n, \ell \geqslant 0$.

The corresponding eigenfunctions are

$$V_{\ell,j}(x)\, \phi_{d+2\ell,n}^{\text{rad}}(|x|),$$

where $j = 1, 2, \ldots, J_{d,\ell}$.

The eigenvalues λ_n can thus be arranged in the table:

$$\begin{array}{cccc}
\lambda^{\mathrm{rad}}_{d,0} & \lambda^{\mathrm{rad}}_{d,1} & \lambda^{\mathrm{rad}}_{d,2} & \cdots \\
\lambda^{\mathrm{rad}}_{d+2,0} & \lambda^{\mathrm{rad}}_{d+2,1} & \lambda^{\mathrm{rad}}_{d+2,2} & \cdots \\
\lambda^{\mathrm{rad}}_{d+4,0} & \lambda^{\mathrm{rad}}_{d+4,1} & \lambda^{\mathrm{rad}}_{d+4,2} & \cdots \\
\vdots & & &
\end{array}$$

(with ℓ-th row repeated $J_{d,\ell}$ times).

We have $\lambda_0 = \lambda^{\mathrm{rad}}_{d,0}$. Which one is λ_1?

The eigenvalues λ_n can thus be arranged in the table:

$$\begin{array}{ccccccc}
\lambda^{\mathrm{rad}}_{d,0} & < & \lambda^{\mathrm{rad}}_{d,1} & \leqslant & \lambda^{\mathrm{rad}}_{d,2} & \leqslant & \cdots \\
\wedge & & & & & & \\
\lambda^{\mathrm{rad}}_{d+2,0} & < & \lambda^{\mathrm{rad}}_{d+2,1} & \leqslant & \lambda^{\mathrm{rad}}_{d+2,2} & \leqslant & \cdots \\
\wedge & & & & & & \\
\lambda^{\mathrm{rad}}_{d+4,0} & < & \lambda^{\mathrm{rad}}_{d+4,1} & \leqslant & \lambda^{\mathrm{rad}}_{d+4,2} & \leqslant & \cdots \\
\wedge & & & & & & \\
\vdots & & & & & &
\end{array}$$

(with ℓ-th row repeated $J_{d,\ell}$ times).

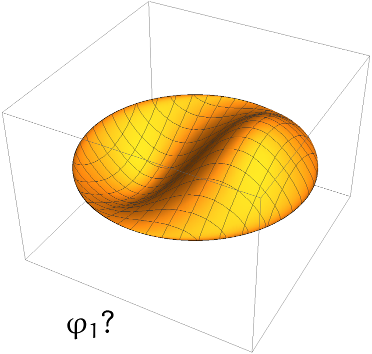
We have $\lambda_0 = \lambda^{\mathrm{rad}}_{d,0}$. Which one is λ_1?

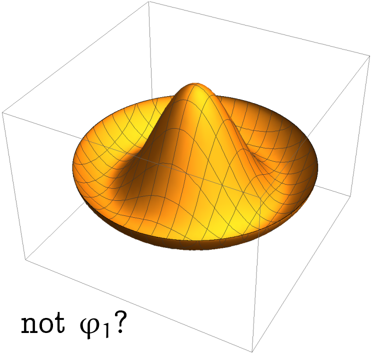
The only possible values are $\lambda_1 = \lambda^{\mathrm{rad}}_{d,1}$ and $\lambda_1 = \lambda^{\mathrm{rad}}_{d+2,0}$.

Conjecture

$$\lambda^{\mathrm{rad}}_{d+2,0} < \lambda^{\mathrm{rad}}_{d,1}$$

Equivalently: $\lambda_1 = \lambda^{\mathrm{rad}}_{d+2,0}$, or: φ_1 is antisymmetric.





Theorem

If $d \leqslant 2$, or if $\alpha = 1$ and $d \leqslant 9$, then indeed

$$\lambda^{\mathrm{rad}}_{d+2,0} < \lambda^{\mathrm{rad}}_{d,1}.$$

Remarks:

- Otherwise this is still an open problem...
- ...strongly supported by numerical bounds.
- Our method: find two-sided bounds for $\lambda^{\mathrm{rad}}_{d,n}$.

Definition

Let $P_n^{(\alpha,\beta)}(r)$ be the **Jacobi polynomial** and

$$\psi_{d,n}^{\mathrm{rad}}(|x|) = P_n^{(\frac{\alpha}{2},\frac{d}{2}-1)}(2|x|^2-1),$$

$$\mu_{d,n}^{\mathrm{rad}} = 2^\alpha \frac{\Gamma(\frac{\alpha}{2}+n+1)\Gamma(\frac{d+\alpha}{2}+n)}{n!\,\Gamma(\frac{d}{2}+n)}.$$

Theorem

$$\mathbf{L}\left[(1-|x|^2)_+^{\alpha/2}\,\psi_{d,n}^{\mathrm{rad}}(|x|)\right] = \mu_{d,n}^{\mathrm{rad}}\,\psi_{d,n}^{\mathrm{rad}}(|x|) \qquad \text{for } x \in B.$$

Remark: some special cases have been known before.

Theorem

The eigenvalues of the operator

$$\mathbf{L}\left[(1-|x|^2)_+^{\alpha/2}\, f(x)\right]$$

are given by $\mu^{\mathrm{rad}}_{d+2\ell,n}$, where $n, \ell \geqslant 0$.

The corresponding eigenfunctions are

$$\psi_{\ell,j,n}(x) = V_{\ell,j}(x)\, P_n^{(\frac{\alpha}{2},\frac{d+2\ell}{2}-1)}(2|x|^2-1),$$

where $j = 1, 2, \ldots, J_{d,\ell}$.

These eigenfunctions form an orthogonal basis in weighted $L^2(B)$ space with weight $(1-|x|^2)^{\alpha/2}\,dx$.

Once it is proved that in B:

$$\mathbf{L}\left[(1-|x|^2)_+^{\alpha/2} f(x)\right] \text{ maps polynomials of degree } n \text{ to polynomials of degree } n, \qquad (\bigstar)$$

it follows easily that:

- the eigenfunctions are polynomials;
- they are orthogonal with respect to $(1-|x|^2)_+^{\alpha/2} dx$;
- they have the form given in the theorem.

(The actual proof follows a completely different path).

Open problem

Is there a soft proof of $(\bigstar)$?

	operator	eigenfunction	eigenvalue
(1)	$\mathbf{L} f(x)$	$\varphi_{d+2\ell,j,n}$	$\lambda^{\mathrm{rad}}_{d+2\ell,n}$
(2)	$\mathbf{L}\left[(1-|x|^2)_+^{\alpha/2} f(x)\right]$	$\psi_{\ell,j,n}$	$\mu^{\mathrm{rad}}_{d+2\ell,n}$
(3)	$(1-|x|^2)_+^{\alpha/2}\,\mathbf{L} f(x)$	$(1-|x|^2)_+^{\alpha/2}\psi_{\ell,j,n}$	$\mu^{\mathrm{rad}}_{d+2\ell,n}$

These operators are generators of:

(1) X_t^B, the process X_t killed upon exiting B;

(3) time-changed X_t^B;

(2) time-changed Doob h-transform of X_t^B (corresponding to $h(x) = \mathbf{E}_x \tau = c_{d,\alpha}(1-|x|^2)^{\alpha/2}$).

The two operators on $L^2(B)$:

$$-(1-|x|^2)_+^{\alpha/2}\,\mathbf{L} \qquad \text{and} \qquad (1-|x|^2)\Delta - (2-\alpha)\nabla$$

have identical eigenfunctions!

These operators are generators of:

- time-changed X_t^B;
- d-dimensional Jacobi diffusion.

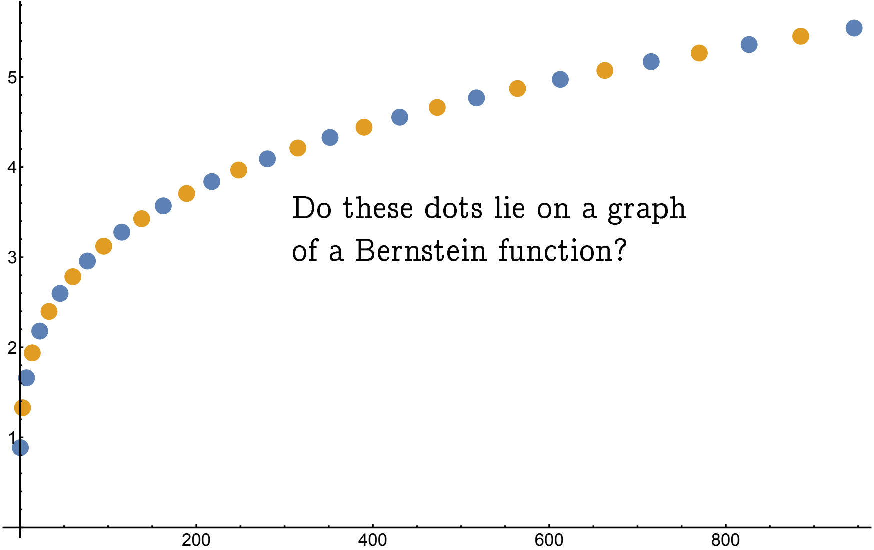
Question

Is time-changed X_t^B a subordinate Jacobi diffusion?

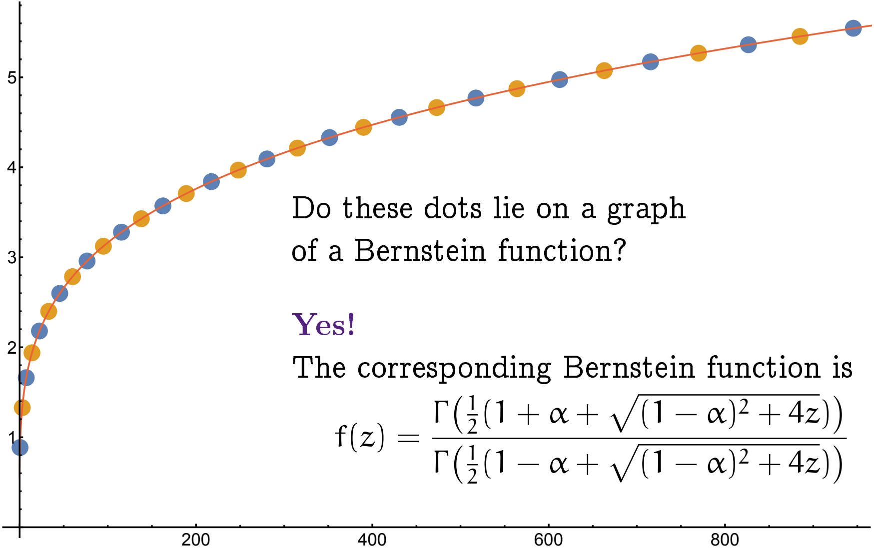
To answer this, one needs to see whether

$$\mu_{d+2\ell,n} = f((2n+\alpha)(2n+d) + (4n+2+\alpha)\ell)$$

for some Bernstein function f.



$d = 1$, $\alpha = 0.5$, $\ell = 0$ (blue) and $\ell = 1$ (yellow)



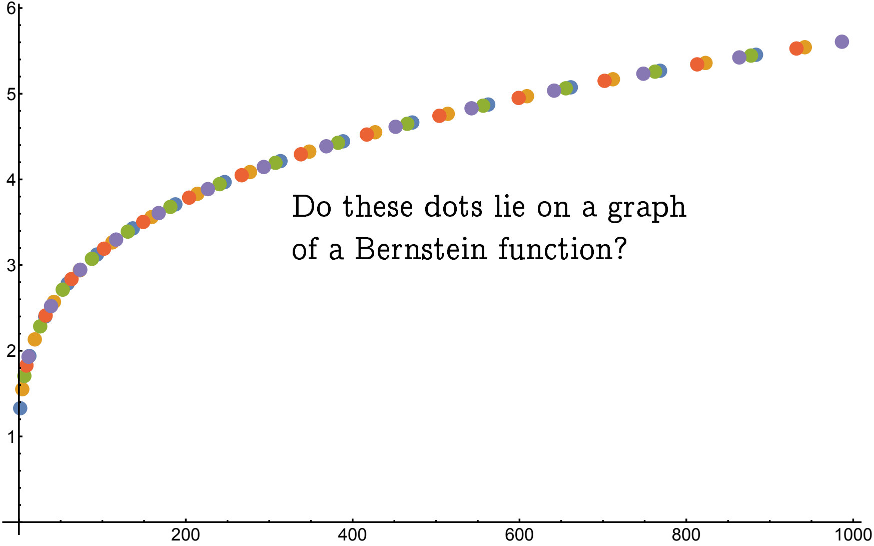
Do these dots lie on a graph of a Bernstein function?

Yes!

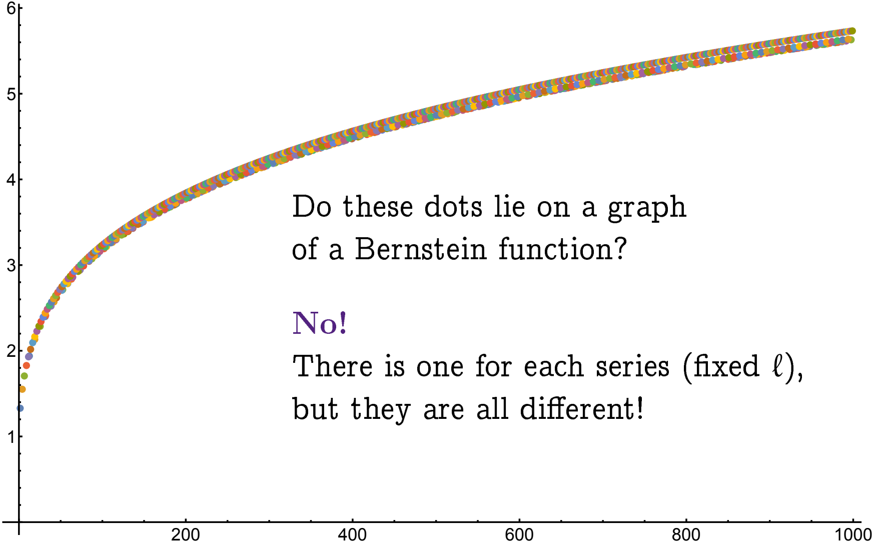
The corresponding Bernstein function is

$$f(z) = \frac{\Gamma\left(\frac{1}{2}\left(1 + \alpha + \sqrt{(1-\alpha)^2 + 4z}\right)\right)}{\Gamma\left(\frac{1}{2}\left(1 - \alpha + \sqrt{(1-\alpha)^2 + 4z}\right)\right)}$$

$d = 1$, $\alpha = 0.5$, $\ell = 0$ (blue) and $\ell = 1$ (yellow)



$d = 3$, $\alpha = 0.5$, colours correspond to $\ell = 0, 1, 2, 3, 4$



Do these dots lie on a graph of a Bernstein function?

No!

There is one for each series (fixed ℓ), but they are all different!

$d = 3$, $\alpha = 0.5$, colours correspond to $\ell = 0, 1, 2, 3, 4, \dots$

Question

Is time-changed X_t^B a subordinate Jacobi diffusion?

Disappointing theorem

Yes if $d = 1$. No if $d \geqslant 2$.

Time-changed $|X_t^B|$, however, is a subordinate Jacobi diffusion in any dimension!

Open problem

Consider time-changed asymmetric 1-dimensional stable process, with clock running at rate $(1+x)^{\rho\alpha}(1-x)^{\hat{\rho}\alpha}$. Is this process a subordinate Jacobi diffusion?

For $x \in B$ we have:

$$\mathbf{L}\left[\varphi_{d,n}^{\text{rad}}(|x|)\right] = \lambda_{d,n}^{\text{rad}}\varphi_{d,n}^{\text{rad}}(|x|)$$
$$\mathbf{L}\left[(1-|x|^2)^{\alpha/2}\,\psi_{d,n}^{\text{rad}}(|x|)\right] = \mu_{d,n}^{\text{rad}}\psi_{d,n}^{\text{rad}}(|x|).$$

Definition

$$f_{d,n}^{\text{rad}}(x) = (1-|x|^2)_+^{\alpha/2}\psi_{d,n}^{\text{rad}}(|x|).$$

We fix d and restrict attention to radial functions.

Drop ${}^{\text{rad}}_{d,}$ from the notation: $\mu_n = \mu_{d,n}^{\text{rad}}$, $f_n = f_{d,n}^{\text{rad}}$ etc.

Thus, for $x \in B$ we have:

$$(1-|x|^2)^{\alpha/2}\,\mathbf{L} f_n(x) = \mu_n f_n(x).$$

Rayleigh–Ritz variational method gives upper bounds.

The values of

$$A(n,m) = \int_B f_n(x)\,\mathbf{L} f_m(x)\,dx,$$

$$B(n,m) = \int_B f_n(x)\,f_m(x)\,dx$$

are given by closed-form expressions.

Fix N and let $\mathbb{A}$, $\mathbb{B}$ be $N \times N$ matrices with entries $A(n,m)$, $B(n,m)$, respectively.

Theorem

Let $\bar{\lambda}_n$ be the solutions of the eigenvalue problem

$$\mathbb{A}\vec{v} = \lambda\,\mathbb{B}\vec{v}.$$

Then $\lambda_n \leqslant \bar{\lambda}_n$ for $n = 0, 1, \ldots, N-1$.

Remarks:

- Since f_n are orthogonal in weighted $L^2(B)$ with weight $(1-|x|^2)^{-\alpha/2}\,dx$, in the problem
$$\mathbb{A}\vec{v} = \lambda\,\mathbb{B}\vec{v}.$$
the matrix $\mathbb{A}$ is diagonal:
$$\begin{aligned} A(n,m) &= \int_B f_n(x)\,\mathbf{L} f_m(x)\,dx \\ &= \mu_m \int_B f_n(x)\,f_m(x)\,(1-|x|^2)^{-\alpha/2}\,dx; \end{aligned}$$
the matrix $\mathbb{B}$ with entries $B(n,m) = \int_B f_n(x)\,f_m(x)\,dx$ is not diagonal.
- Quality of the bounds improve rapidly as N grows.
- Numerical methods work for relatively large N.

Aronszajn method of intermediate problems gives lower bounds.

Two eigenvalue problems in B:

$$\mathbf{L} f(x) = \lambda\, f(x),$$

$$\mathbf{L} f(x) = \mu\, (1 - |x|^2)^{-\alpha/2}\, f(x)$$

correspond to Rayleigh quotients:

$$Q(f) = \frac{\int_B f(x)\, \mathbf{L} f(x)\, dx}{\int_B (f(x))^2\, dx}\,,$$

$$Q_0(f) = \frac{\int_B f(x)\, \mathbf{L} f(x)\, dx}{\int_B (f(x))^2\, (1 - |x|^2)^{-\alpha/2}\, dx}\,.$$

Clearly, $Q_0(f) \leqslant Q(f)$, and hence $\mu_n \leqslant \lambda_n$.

The basic bound $\mu_n \leqslant \lambda_n$ is poor.

Improved bounds come from intermediate problems, corresponding to Reyleigh quotient

$$Q_N(f) = \frac{\int_B f(x)\,\mathbf{L}f(x)\,dx}{\int_B (f(x))^2\,(1-|x|^2)^{-\alpha/2}\,dx - \int_B (\mathbf{P}_N f(x))^2\,w(x)\,dx},$$

where

$$w(x) = ((1-|x|^2)^{-\alpha/2} - 1)$$

and $\mathbf{P}_N$ is the orthogonal projection in weighted $L^2(B)$ space with weight $w(x)\,dx$ onto the linear span of

$$\frac{f_{n+1}(x) - f_n(x)}{1-(1-|x|^2)^{\alpha/2}}, \qquad n = 0, 1, \ldots, N-2.$$

Recall that

$$Q_N(f) = \frac{\int_B f(x)\,\mathbf{L}f(x)\,dx}{\int_B (f(x))^2\,(1-|x|^2)^{-\alpha/2}\,dx - \int_B (\mathbf{P}_N f(x))^2\,w(x)\,dx}\,.$$

It is rather clear that $Q_0(f) \leqslant Q_1(f) \leqslant \ldots \to Q(f)$.

Theorem

The eigenvalues $\underline{\lambda}_n$ corresponding to Q_N satisfy

$$\underline{\lambda}_n \leqslant \lambda_n.$$

Surprise: one can actually calculate $\underline{\lambda}_n$!

Remarks:

- The only non-closed-form expressions here are
$$\int_B \frac{(1-|x|^2)^{\alpha/2}(1-|x|^{2n})}{1-(1-|x|^2)^{\alpha/2}}\,dx.$$
- The eigenvalues $\underline{\lambda}_n$ of the intermediate problem are equal to either μ_m or zeros of a polynomial W_n, which is the determinant of an $N \times N$ matrix (Weinstein–Aronszajn determinant).
- Quality of the bounds improve rapidly as N grows.
- Numerical methods work well for relatively small N; larger N leads to ill-conditioned matrices.

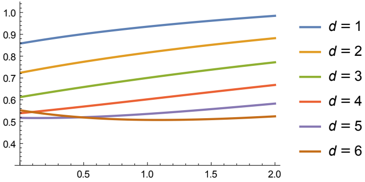
We prove the middle inequality in

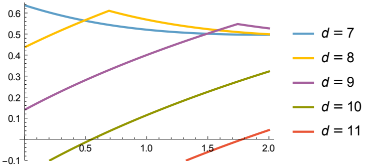
$$\lambda_{d+2,0}^{\text{rad}} \leqslant \overline{\lambda}_{d+2,0}^{\text{rad}} < \underline{\lambda}_{d,1}^{\text{rad}} \leqslant \lambda_{d,1}^{\text{rad}}$$

analytically using $N = 2$ (that is, 2×2 matrices).

Our method could work for $d \leqslant 9$ and any $\alpha \in (0, 2)$.

We managed to work out the technical details only when $d \leqslant 2$ or $\alpha = 1$.





$$\left(\underline{\lambda}_{d,1}^{\text{rad}}\right)^{1/\alpha} - \left(\overline{\lambda}_{d+2,0}^{\text{rad}}\right)^{1/\alpha}$$

Open problem 1

Is there a soft proof of the statement:

$$\mathbf{L}\left[(1-|x|^2)_+^{\alpha/2} f(x)\right] \text{ maps polynomials of degree } n \text{ to polynomials of degree } n? \qquad (\star)$$

Open problem 2

Consider an asymmetric 1-dimensional stable process, time-changed with clock running at rate $(1-x)^{\alpha_+}(1+x)^{\alpha_-}$. Is it a subordinate Jacobi diffusion?

Open problem 3

Explain why the spectrum of $(1-|x|^2)_+^{\alpha/2}\,\mathbf{L}$ is so simple.

Open problem 4

Prove that φ_1 is antisymmetric when $d \leqslant 9$.