CALCULUS II

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Lecture 1

Improper Integrals of the First Kind

In this lecture we will study Riemann integrals over infinite intervals, such as $[a,\infty)$, $(-\infty,b]$ and $(-\infty,\infty)$, where a,b are real numbers.

Definition 1. Suppose that a function f is continuous on $[a,\infty)$, where a is a real number. By an *improper integral of the first kind* of the function f over the interval $[a,\infty)$ we understand the limit

$$\int_a^\infty f(x)dx := \lim_{T\to\infty}\int_0^T f(x)dx$$

if it exists. If this limit is finite, we say that the integral *converges.* If this limit is equal to ∞ or $-\infty$, we say that the integral *diverges to* ∞, or $-\infty$, respectively. If this limit does not exist, we say that the integral *diverges.*

Remarks. In an analogous fashion we define an improper integral of the first kind of a function $f\in C(-\infty,b]$ (i.e. continuous on the interval $(-\infty,b]$), namely

$$\int_{-\infty}^b f(x)dx := \lim_{S\to-\infty}\int_S^b f(x)dx$$

if it exists. Note that the continuity assumption ensures existence of the integrals $\int_a^T f(x)dx$ and $\int_{-S}^b f(x)dx$ for all finite T,S and that is why the above definitions makes sense. In principle, we can consider improper integrals of the first kind of functions which are not continuous, but it is not necessary for our purposes.

If an improper integral I converges, diverges to ∞ or diverges to $-\infty$, we often write $I<\infty$, $I=\infty$, or $I=-\infty$, respectively.

Example 1. Using the definition of the improper integral of the first kind, check if the integrals given below converge, diverge to ∞, $-\infty$, or diverge. If possible, evaluate the integrals.

$$\begin{aligned}\int_0^{\infty} \frac{dx}{x^2+1} &= \lim_{T\to\infty} \int_0^T \frac{dx}{x^2+1} \\ &= \lim_{T\to\infty} (\text{arctg}T - \text{arctg}0) \\ &= \frac{\pi}{2}\end{aligned}$$

$$\begin{aligned}\int_{\pi}^{\infty} \cos x dx &= \lim_{T\to\infty} \int_{\pi}^{T} \cos x dx \\ &= \lim_{T\to\infty} (\sin T - \sin\pi) \\ &= \lim_{T\to\infty} (\sin T)\end{aligned}$$

Note, however, that the above limit does not exist, and therefore that the above integral diverges. In turn,

$$\begin{aligned}\int_{-\infty}^{-1} \frac{dx}{x} &= \lim_{S\to-\infty} \int_{S}^{-1} \frac{dx}{x} \\ &= \lim_{S\to-\infty} (\ln|x|)_{x=S}^{x=-1} \\ &= \lim_{S\to-\infty} (-\ln|S|) \\ &= -\infty\end{aligned}$$

and therefore this integral diverges to $-\infty$.

DEFINITION 1.2. Let f be a function which is continuous on the real line, i.e. on the interval $(-\infty, \infty)$. The improper integral of the first kind of the function f over the real line is given by

$$\int_{-\infty}^{\infty} f(x)dx := \int_{-\infty}^{0} f(x)dx + \int_{0}^{\infty} f(x)dx$$

thus it is defined in terms of two improper integrals of the first kind defined previously. We will say that this integral *converges* if both integrals on the right-hand side converge. If both integrals on the right-hand side diverge to ∞, we will say that this integral *diverges to* ∞. In turn, if both integrals on the right-hand side diverge to $-\infty$, we will say that the considered integral *diverges to* $-\infty$. In the remaining cases we will say that the integral *diverges*.

We can replace 0 in Definition 1.2 by any finite number a without changing the outcome. However, in most cases it seems convenient to split up the infinite interval into $(-\infty, 0]$ and $[0, \infty)$. Now, let us consider an easy example.

EXAMPLE 2.

$$\begin{aligned}
\int_{-\infty}^{\infty} xe^{-x^2}dx &= \lim_{S\to-\infty}\int_S^0 xe^{-x^2}dx + \lim_{T\to\infty}\int_0^T xe^{-x^2}dx \\
&= \lim_{S\to\infty}\left(-\frac{1}{2}e^{-x^2}\right)_{x=S}^{x=0} + \lim_{T\to\infty}\left(-\frac{1}{2}e^{-x^2}\right)_{x=0}^{x=T} \\
&= \lim_{S\to\infty}\left(\frac{1}{2}e^{-S^2}-\frac{1}{2}\right) + \lim_{T\to\infty}\left(\frac{1}{2}-\frac{1}{2}e^{-T^2}\right) \\
&= 0-\frac{1}{2}+\frac{1}{2}-0 \\
&= 0
\end{aligned}$$

EXAMPLE 3. Let us consider an important class of integrals of the form

$$\int_a^{\infty}\frac{dx}{x^p}$$

where $a > 0$ and $p \in \mathbf{R}$. One can show that the above integrals converge for $p > 1$ and diverge to ∞ for $p \le 1$. In fact, take $p \neq 1$. Then

$$\begin{aligned}
\int_a^{\infty} x^{-p}dx &= \lim_{T\to\infty}\left(\frac{1}{-p+1}x^{-p+1}\right)_{x=a}^{x=T} \\
&= \lim_{T\to\infty}\frac{1}{-p+1}(T^{-p+1}-a^{-p+1})
\end{aligned}$$

However, it is easy to see that for $p > 1$ the limit in the last formula is finite and equals $-\frac{1}{-p+1}a^{-p+1}$, whereas for $p < 1$ the limit is equal to ∞. The case $p = 1$ has to be treated separately since then

$$\begin{aligned}
\int_a^{\infty} x^{-1}dx &= \lim_{T\to\infty}(\ln|x|)_{x=a}^{x=T} \\
&= \lim_{T\to\infty}(\ln T-\ln a) \\
&= \infty
\end{aligned}$$

Using these facts, we conclude that, for instance, the integral

$$\int_2^{\infty}\frac{dx}{\sqrt{x^3}}$$

converges since $p = 3/2 > 1$. Even slightly more complicated examples can be solved.

For instance,

$$\int_0^{\infty}\frac{dx}{\sqrt{(2x+1)^3}}$$

is also convergent since one can perform a substitution $t = 2x + 1$ which reduces the above integral to the integral

$$\int_1^{\infty}\frac{dt}{t^{3/2}}$$

which is convergent (the details are left to the reader).

Of course, the ideal situation is when we can evaluate the improper integrals directly using the definitions. Unfortunately, it is not always possible, as it is also the case with proper (i.e. usual) integrals. Nevertheless, it is important to know at least whether an improper integral converges, diverges to ∞ or $-\infty$, or diverges. Even if we do not know the exact value of a convergent integral, we can then always use approximations and computer methods to get pretty good estimates. But is is essential to know that looking for an estimate is meaningful. For that purpose, we need some simple criteria which would tell us when an improper integral converges.

THEOREM 1. (COMPARISON TEST). *Suppose that* $0 \leq f(x) \leq g(x)$ *for all* $x \in [a, \infty)$*, where* a *is a (finite) real number and* f *and* g *are continuous on* $[a, \infty)$*. Then*
(a) *if* $\int_a^\infty g(x)dx$ *converges then* $\int_a^\infty f(x)dx$ *converges*
(b) *if* $\int_a^\infty f(x)dx$ *diverges to* ∞*, then* $\int_a^\infty g(x)dx$ *diverges to* ∞.

The intuitive sense of this test is clear. In (a), the area under the graph of a "smaller" function is finite if the area under the graph of a "bigger" function is finite. A similar reasoning can be used in (b). Note also that a similar test can be phrased for integrals over te interval $(-\infty, b]$.

EXAMPLE 4. Let us look at the integral

$$\int_1^\infty \frac{\sqrt{x}dx}{x+1}$$

and study its convergence or divergence to ∞. First, we have to look for a nonnegative function f or g and set the above integrand to be g or f, respectively. This is probably the hardest part of the whole test since it is not clear what we should expect. Therefore, let us look at the function

$$\frac{\sqrt{x}}{x+1}$$

and its behavior at infinity. We can see that 1 in the denominator is irrelevant for large x and therefore we can expect that this function behaves like $1\sqrt{x}$ for large x, of which the improper integral diverges. This means that we can expect the above integral to diverge. To use (b) of Theorem 1, we set

$$f(x) = \frac{\sqrt{x}}{x+x} \leq \frac{\sqrt{x}}{x+1} = g(x)$$

(the inequality holds for $x \geq 1$) and evaluate

$$\int_1^\infty f(x) = \int_1^\infty \frac{dx}{2\sqrt{x}} = \infty$$

(here $p = 1/2 < 1$). By the comparison test (b), the integral of the function g also diverges.

EXAMPLE 5. On the other hand, the integral

$$\int_2^\infty \frac{1+\sin x}{x^2}dx$$

converges since

$$f(x) = \frac{1+\sin x}{x^2} \le \frac{1+1}{x^2} = g(x),$$

both functions f and g are nonnegative and

$$\int_2^\infty g(x)dx < \infty$$

which follows from convergence of the integral $\int_2^\infty x^{-2}dx$.

THEOREM 2. (LIMIT RATIO TEST) *Suppose that f and g are both positive (or, both negative) continuous functions on $[a,\infty)$ and that*

$$\lim_{x\to\infty} \frac{f(x)}{g(x)} = K$$

where $0 < K < \infty$. Then

(a) $\int_a^\infty f(x)dx$ *converges if and only if* $\int_a^\infty g(x)dx$ converges

(b) $\int_a^\infty f(x)dx$ *diverges to ∞ if and only if* $\int_a^\infty g(x)dx$ *diverges to ∞.*

(c) $\int_a^\infty f(x)dx$ *diverges to $-\infty$ if and only if* $\int_a^\infty g(x)dx$ *diverges to $-\infty$.*

A similar test holds for improper integrals over $(-\infty, b]$.

The ratio test says that if the asymptotic behavior of functions f, g is the same, then they either both converge or diverge to ∞ or $-\infty$. Clearly, they can diverge to ∞ only if they are both positive, and they can diverge to $-\infty$ only if they are both negative. It is very important that $K \neq 0$ and $K \neq \infty$. If $K = 0$ or $K = \infty$, the test does not hold.

EXAMPLE 6. Let us check for convergence the integral

$$\int_\pi^\infty \frac{dx}{x^2+\sin x}$$

We can see that the function $f(x) = 1/(x^2+\sin x)$ is positive on $[\pi,\infty)$. Let us take the positive function

$$g(x) = \frac{1}{x^2}$$

which is a natural choice for the ratio test since $\sin x$ is irrelevant as compared with x^2 at infinity. Note that

$$\lim_{x\to\infty} \frac{f(x)}{g(x)} = \lim_{x\to\infty} \frac{x^2}{x^2+\sin x} = 1 = K$$

and that $\int_\pi^\infty x^{-2} < \infty$, which implies that the investigated integral also converges by the ratio test (a).