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Asymptotic expansion of the nonlocal heat content

by

Tomasz Grzywny and Julia Lenczewska (Wrocław)

Abstract. Let $(p_t)_{t\geq 0}$ be a convolution semigroup of probability measures on $\mathbb{R}^d$ defined by

$$\int_{\mathbb{R}^d} e^{i\langle \xi,x\rangle}\, p_t(\mathrm{d}x) = e^{-t\psi(\xi)}, \quad \xi \in \mathbb{R}^d,$$

and let Ω be an open subset of $\mathbb{R}^d$ with finite Lebesgue measure. We consider the quantity $H_\Omega(t) = \int_\Omega \int_{\Omega - x} p_t(\mathrm{d}y)\,\mathrm{d}x$, called the heat content. We study its asymptotic expansion under mild assumptions on ψ, in particular in the case of the α-stable semigroup.

1. Introduction. Let $d \in \mathbb{N}$. We consider a semigroup $(p_t)_{t\geq 0}$ of probability measures given by

$$\int_{\mathbb{R}^d} e^{i\langle \xi,x\rangle}\, p_t(\mathrm{d}x) = e^{-t\psi(\xi)}, \quad \xi \in \mathbb{R}^d,$$

where ψ is a symbol defined by

$$\psi(\xi) = \int_{\mathbb{R}^d} (1 - e^{i\langle \xi,z\rangle})\, \nu(\mathrm{d}z), \quad \xi \in \mathbb{R}^d,$$

and $\nu(\mathrm{d}z)$ is a Borel measure satisfying

$$\nu(\{0\}) = 0, \qquad \int_{\mathbb{R}^d} (1 \wedge |z|)\, \nu(\mathrm{d}z) < \infty.$$

Let $\{P_t\}_{t\geq 0}$ be the convolution semigroup of operators on $\mathcal{C}_0(\mathbb{R}^d)$ defined by $(p_t)_{t\geq 0}$ and let $\mathcal{L}$ denote its infinitesimal generator, which for $f \in C_c^2(\mathbb{R}^d)$ is given by the formula

$$\mathcal{L}f(x) = \int_{\mathbb{R}^d} (f(x+z) - f(x))\, \nu(\mathrm{d}z). \tag{1}$$

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Let Ω be a nonempty, open subset of $\mathbb{R}^d$ with finite Lebesgue measure $|\Omega|$. We consider the following quantity associated with the semigroup $(p_t)_{t\geq 0}$:

$$H_\Omega(t) = \int_\Omega \int_{\Omega - x} p_t(\mathrm{d}y)\,\mathrm{d}x,$$

which we will call the *heat content*.

We note that the function $u(t,x) = \int_{\Omega - x} p_t(\mathrm{d}y)$ is a weak solution of the initial value problem

$$\frac{\partial}{\partial t} u(t,x) = \mathcal{L}u(t,x), \qquad t>0,\ x\in\mathbb{R}^d,$$
$$u(0,x) = \mathbb{1}_\Omega(x).$$

Therefore, the quantity $H_\Omega(t)$ can be interpreted as the amount of *heat* in Ω if its initial temperature is 1 whereas the initial temperature of Ω^c is zero.

Our main goal is to study the asymptotic expansion of $H_\Omega(t)$ for small t. We observe that

$$H_\Omega(t) = |\Omega| - H(t), \quad \text{where} \quad H(t) = \int_\Omega \int_{\Omega^c - x} p_t(\mathrm{d}y)\,\mathrm{d}x,$$

and hence it suffices to work with the function $H(t)$. One of the main results of [7] states that, for small t,

$$H_\Omega(t) = |\Omega| - t\,\mathrm{Per}_\nu(\Omega) + o(t),$$

where $\mathrm{Per}_\nu(\Omega)$ is the nonlocal perimeter related to the measure ν, defined as

$$\mathrm{Per}_\nu(\Omega) = \int_\Omega \int_{\Omega^c - x} \nu(\mathrm{d}y)\,\mathrm{d}x. \tag{2}$$

For instance, if ν is the α-stable Lévy measure $\nu^{(\alpha)}$ with $\alpha\in(0,1)$, defined by

$$\nu^{(\alpha)}(\mathrm{d}z) = \mathcal{A}_{d,-\alpha}|z|^{-d-\alpha}\,\mathrm{d}z, \quad \text{where} \quad \mathcal{A}_{d,-\alpha} = \frac{2^\alpha \Gamma\left(\frac{d+\alpha}{2}\right)}{\pi^{d/2}\,|\Gamma(-\alpha/2)|},$$

then $\mathrm{Per}_{\nu^{(\alpha)}}(\Omega) = \mathcal{A}_{d,-\alpha}\,\mathrm{Per}_{(\alpha)}(\Omega)$, with $\mathrm{Per}_{(\alpha)}(\Omega)$ being the well-known *α-perimeter* [6], given for $0<\alpha<1$ by

$$\mathrm{Per}_{(\alpha)}(\Omega) = \int_\Omega \int_{\Omega^c} \frac{\mathrm{d}y\,\mathrm{d}x}{|x-y|^{d+\alpha}}.$$

In the present paper, we shall establish the next terms of the asymptotic expansion of the heat content related to convolution semigroups. Our result is new even for the fractional Laplacian $(-\Delta)^{\alpha/2}$ (in our setting we consider $\alpha\in(0,1)$). For instance, if $1/\alpha$ is a natural number, we prove the following expansion of the heat content for the fractional Laplacian:

$$H_\Omega(t) = |\Omega| + \sum_{n=1}^{1/\alpha-1} \frac{(-1)^n}{n!} t^n \operatorname{Per}_{\nu^{(n\alpha)}}(\Omega) + \frac{(-1)^{1/\alpha}}{(1/\alpha-1)!\pi} t^{1/\alpha} \log(1/t) \operatorname{Per}(\Omega) \\ + o(t^{1/\alpha}\log(1/t)),$$

where Per is the classical perimeter of the set (see (8) below). A natural question arises: will the next term be the mean curvature or its nonlocal counterpart?

The key observation to obtain the asymptotic expansion is that the heat content can be expressed as the action of the semigroup on the *covariance function* of a set. We give more general results, concerning the asymptotic expansion of $P_t f$ for Hölder functions f. Our standing assumption is the *weak upper scaling* of the symbol ψ^*, or equivalently, certain scaling properties of the *concentration function* of a Lévy measure (see Theorem 3.3). For a class of convolution semigroups, for instance for the semigroup associated to $\log(1+\Delta)$, we get the full expansion (see Theorem 3.6). We apply these results to obtain the expansion of the heat content (Corollaries 3.2 and 3.4). Using the asymptotic expansion of the heat kernel of the fractional Laplacian, we give a more explicit asymptotic expansion in the case of α-stable semigroups (Theorems 3.10 and 3.11).

The heat content related to the Gaussian semigroup ($\mathcal{L} = \frac{1}{2}\Delta$) of a set at time t was defined by van den Berg [20] by means of the heat semigroup. Van den Berg and Gilkey [21] proved that the heat content, regarded as a function of t, has an asymptotic expansion as t tends to 0. The first three terms in the expansion involve the volume of the set, its perimeter and its mean curvature. The short time behavior of the heat semigroup in connection with the geometry of sets with finite perimeter was also studied by Angiuli, Massari and Miranda [5]. The concept of heat content was extended to the nonlocal setting of α-stable semigroups in 2016 by Acuña Valverde [1], who described the small-time asymptotic behavior of the nonlocal heat content in that case. In the one-dimensional case, the number of known terms of the expansion depends on the parameter α, and in the multidimensional case, the first two terms of the expansion are known. The same author found the first three terms of the asymptotic expansion for the Poisson heat content over the unit ball [2] and over convex bodies [3]. In 2017, Cygan and Grzywny [7] introduced the notion of nonlocal heat content related to general probabilistic convolution semigroups and generalized the above-mentioned results of Acuña Valverde. Later, they proved similar results for the generalized heat content related to convolution semigroups [8]. Mazón, Rossi and Toledo [18] found the full asymptotic expansion of the heat content for nonlocal diffusion with nonsingular kernels. Recently, in a more general setting, the heat content related to the fractional Laplacian in Carnot groups was studied by Ferrari, Miranda, Pallara, Pinamonti, and Sire [9].

2. Preliminaries

2.1. Convolution semigroups. For $f:\mathbb{R}^d\to\mathbb{R}$, let

$$P_t f(x)=\int_{\mathbb{R}^d} f(x+y)\,p_t(\mathrm{d}y),\qquad t\ge 0,\ x\in\mathbb{R}^d.$$

The generator $\mathcal{L}$ of the semigroup $\{P_t\}_{t\ge 0}$ is defined as

$$\mathcal{L}f(x)=\lim_{t\to 0^+}\frac{P_t f(x)-f(x)}{t},$$

for functions for which the above limit exists.

We denote by $\mathcal{C}_0(\mathbb{R}^d)$ the space of continuous functions $f:\mathbb{R}^d\to\mathbb{R}$ vanishing at infinity. For $\beta\in(0,1]$ we define

$$|||f|||_\beta := \sup_{|x-y|\le 1}\frac{|f(x)-f(y)|}{|x-y|^\beta}.$$

We will consider the Hölder space

$$\mathcal{C}_0^\beta=\{f\in\mathcal{C}_0(\mathbb{R}^d): \|f\|_\beta := |||f|||_\beta+\|f\|_\infty<\infty\}.$$

[15, Theorem 3.2] implies that, for a fixed $\beta\in(0,1]$, if $\int_{|y|<1}|y|^\beta\,\nu(\mathrm{d}y)<\infty$, then for $f\in\mathcal{C}_0^\beta$,

$$\mathcal{L}f(x)=\int_{\mathbb{R}^d}(f(x+y)-f(x))\,\nu(\mathrm{d}y). \tag{3}$$

The real part of the symbol ψ equals $\mathrm{Re}[\psi(\xi)]=\int_{\mathbb{R}^d}(1-\cos\langle\xi,z\rangle)\,\nu(\mathrm{d}z)$. We will consider its radial, continuous and nondecreasing majorant defined by

$$\psi^*(r)=\sup_{|\xi|\le r}\mathrm{Re}[\psi(\xi)],\qquad r>0.$$

For $r>0$ we define the *concentration function*

$$h(r)=\int_{\mathbb{R}^d}\left(1\wedge\frac{|x|^2}{r^2}\right)\nu(\mathrm{d}x).$$

By [11, Lemma 4], for all $r>0$,

$$\frac{1}{8(1+2d)}h(1/r)\le\psi^*(r)\le 2h(1/r). \tag{4}$$

Hence h is a more tractable version of ψ^*. By [12, Lemma 2.1],

$$\int_{|z|\ge r}\nu(\mathrm{d}z)\le h(r)\qquad\text{for all } r>0. \tag{5}$$

By [12, Lemma 2.7], if $f\colon [0,\infty) \to [0,\infty)$ is differentiable, $f(0)=0$, $f' \geq 0$ and $f' \in L^1_{\mathrm{loc}}([0,\infty))$, then for all $r>0$,

$$\int\limits_{|z|<r} f(|z|)\,\nu(\mathrm{d}z) = \int\limits_0^r f'(s)\nu(|x|\geq s)\,\mathrm{d}s - f(r)\nu(|x|\geq r). \tag{6}$$

Let $\theta_0 \geq 0$ and $\phi : (\theta_0,\infty) \to [0,\infty]$. We say that ϕ satisfies the *weak upper scaling condition* (at infinity) if there are $\alpha \in \mathbb{R}$ and $\overline{C} \in [1,\infty)$ such that

$$\phi(\lambda\theta) \leq \overline{C}\lambda^{\alpha}\phi(\theta) \quad \text{ for } \lambda \geq 1,\ \theta > \theta_0. \tag{7}$$

For short, $\phi \in \mathrm{WUSC}(\alpha,\theta_0,\overline{C})$. This condition will be our standing assumption on the symbol ψ^* throughout the paper.

The following auxiliary result is a consequence of (4)–(6).

Lemma 2.1. *Let $\alpha \in (0,2]$, $\overline{C} \in [1,\infty)$ and $\theta_0 \in [0,\infty)$. Consider the following conditions:*

(A1) $\psi^* \in \mathrm{WUSC}(\alpha,\theta_0,\overline{C})$,

(A2) *there is $C>0$ such that for all $\lambda \leq 1$ and $r < 1/\theta_0$,*

$$h(\lambda r) \leq C\lambda^{-\alpha}h(r).$$

Then (A1) *implies* (A2) *with $C = c_d\overline{C}$, where $c_d = 16(1+2d)$, while* (A2) *gives* (A1) *with $\overline{C} = c_d C$. If additionally $\theta_0 \in [0,1)$, then* (A1) *and* (A2) *each imply*

(A3) *for all $\varepsilon > 0$,*

$$\int\limits_{|y|<1} |y|^{\alpha+\varepsilon}\,\nu(\mathrm{d}y) < \infty.$$

Proof. We will show that (A1) implies (A2). Using (4), (7) and again (4), we obtain

$$\begin{aligned} h(\lambda r) \leq 8(1+2d)\psi^*((\lambda r)^{-1}) &\leq 8(1+2d)\overline{C}\lambda^{-\alpha}\psi^*(r^{-1}) \\ &\leq 16(1+2d)\overline{C}\lambda^{-\alpha}h(r). \end{aligned}$$

The converse implication can be proved analogously. It remains to show that (A1) implies (A3). By (6) with $f(s) = s^{\alpha+\varepsilon}$ and $r=1$,

$$\begin{aligned} \int\limits_{|y|<1} |y|^{\alpha+\varepsilon}\,\nu(\mathrm{d}y) &= (\alpha+\varepsilon)\int\limits_0^1 s^{\alpha+\varepsilon-1}\nu(|x|\geq s)\,\mathrm{d}s - \nu(|x|\geq 1) \\ &\leq (\alpha+\varepsilon)\int\limits_0^1 s^{\alpha+\varepsilon-1}h(s)\,\mathrm{d}s \\ &\leq (\alpha+\varepsilon)\,Ch(1)\int\limits_0^1 s^{\varepsilon-1}\,\mathrm{d}s = \frac{C(\alpha+\varepsilon)}{\varepsilon}h(1) < \infty. \ \blacksquare \end{aligned}$$

2.2. Heat content. Following [4, Section 3.3], for any measurable set $\Omega \subset \mathbb{R}^d$ we define its *perimeter* $\mathrm{Per}(\Omega)$ as

$$\mathrm{Per}(\Omega) = \sup\Big\{ \int_{\mathbb{R}^d} \mathbb{1}_\Omega(x) \operatorname{div}\phi(x)\,\mathrm{d}x : \phi \in C_c^1(\mathbb{R}^d, \mathbb{R}^d),\ \|\phi\|_\infty \le 1 \Big\}. \tag{8}$$

We mention that, by [4, Proposition 3.62], for any open Ω with Lipschitz boundary $\partial\Omega$ and finite Hausdorff measure $\sigma(\partial\Omega)$ we have

$$\mathrm{Per}(\Omega) = \sigma(\partial\Omega).$$

For any $\Omega \subset \mathbb{R}^d$ with finite Lebesgue measure $|\Omega|$, we define the *covariance function* g_Ω of Ω as follows:

$$g_\Omega(y) = |\Omega \cap (\Omega + y)| = \int_{\mathbb{R}^d} \mathbb{1}_\Omega(x)\, \mathbb{1}_\Omega(x-y)\,\mathrm{d}x, \qquad y \in \mathbb{R}^d. \tag{9}$$

By [10, Proposition 2, Theorem 13 and Theorem 14], g_Ω is symmetric, nonnegative, bounded from above by $|\Omega|$, and $g_\Omega \in \mathcal{C}_0(\mathbb{R}^d)$. Moreover, if $\mathrm{Per}(\Omega) < \infty$, then g_Ω is Lipschitz with

$$2\|g_\Omega\|_{\mathrm{Lip}} \le \mathrm{Per}(\Omega). \tag{10}$$

Moreover, for all $r > 0$ the limit $\lim_{r\to 0^+} \frac{g_\Omega(0) - g_\Omega(ru)}{r}$ exists, is finite and

$$\mathrm{Per}(\Omega) = \frac{\Gamma((d+1)/2)}{\pi^{(d-1)/2}} \int_{\mathbb{S}^{d-1}} \lim_{r\to 0^+} \frac{g_\Omega(0) - g_\Omega(ru)}{r}\,\sigma(\mathrm{d}u). \tag{11}$$

In particular, there is a constant $C = C(\Omega) > 0$ such that

$$0 \le g_\Omega(0) - g_\Omega(y) \le C(1 \wedge |y|). \tag{12}$$

By Cygan and Grzywny [7, Lemma 3], the related function $H(t)$ has the following form:

$$H(t) = \int_{\mathbb{R}^d} (g_\Omega(0) - g_\Omega(y))\, p_t(\mathrm{d}y), \tag{13}$$

and by [7, proof of Lemma 1],

$$\mathrm{Per}_\nu(\Omega) = \int_{\mathbb{R}^d} (g_\Omega(0) - g_\Omega(y))\, \nu(\mathrm{d}y). \tag{14}$$

By [7, Theorem 3], if $\Omega \subset \mathbb{R}^d$ is an open set with $|\Omega| < \infty$ and $\mathrm{Per}(\Omega) < \infty$ (i.e. $\mathbb{1}_\Omega \in \mathrm{BV}(\mathbb{R}^d)$), then

$$t^{-1}H(t) = t^{-1}(g_\Omega(0) - P_t g_\Omega(0)), \tag{15}$$

which converges to $-\mathcal{L}g_\Omega(0) = \mathrm{Per}_\nu(\Omega)$ as t tends to 0.

3. Main results and proofs

3.1. Convolution semigroups for nonlocal operators on $\mathbb{R}^d$

LEMMA 3.1. *Assume that $\psi^* \in \mathrm{WUSC}(\alpha, 1, \overline{C})$ for some $\alpha \in (0,1)$. If $f \in \mathcal{C}_0^\beta$ for some $\beta \in (\alpha, 1]$, then $\mathcal{L}f \in \mathcal{C}_0^{\beta-\alpha}$ and*

$$\|\mathcal{L}f\|_{\beta-\alpha} \le C_1(1-\alpha/\beta)^{-1}h(1)\|f\|_\beta,$$

where $C_1 = C_1(c_d, \overline{C})$. In particular, if $\beta \in (2\alpha, 1]$, then $\mathcal{L}f \in \mathcal{D}(\mathcal{L})$.

Proof. By Lemma 2.1, for all $\lambda \le 1$ and $r \le 1$,

$$h(\lambda r) \le c\lambda^{-\alpha} h(r), \tag{16}$$

where $c = c_d\overline{C}$, and for all $\varepsilon > 0$,

$$\int\limits_{|y|<1} |y|^{\alpha+\varepsilon}\,\nu(\mathrm{d}y) < \infty. \tag{17}$$

First, we will deal with $|||\mathcal{L}f|||_{\beta-\alpha}$. Assume $|x-y| \le 1$. By (3),

$$\begin{aligned} |\mathcal{L}f(x) - \mathcal{L}f(y)| &= \Big|\int\limits_{\mathbb{R}^d} \big(f(x+z) - f(x) - (f(y+z) - f(y))\big)\,\nu(\mathrm{d}z)\Big| \\ &\le \int\limits_{\mathbb{R}^d} |f(x+z) - f(x) - f(y+z) + f(y)|\,\nu(\mathrm{d}z). \end{aligned} \tag{18}$$

We split the integral above as follows:

$$\int\limits_{\mathbb{R}^d} = \int\limits_{|z|\le|x-y|} + \int\limits_{|z|>|x-y|} =: \mathrm{I}_1 + \mathrm{I}_2.$$

We will first deal with I_1. Denote $L = |||f|||_\beta$. We have

$$|f(x) - f(y)| \le L|x-y|^\beta. \tag{19}$$

By (19), the Fubini theorem, (5) and (16) we have

$$\begin{aligned} \mathrm{I}_1 &\le \int\limits_{|z|\le|x-y|} \big(|f(x+z) - f(x)| + |f(y+z) - f(y)|\big)\,\nu(\mathrm{d}z) \\ &\le 2L\int\limits_{|z|\le|x-y|} |z|^\beta\,\nu(\mathrm{d}z) = 2L\int\limits_{|z|\le|x-y|}\int\limits_0^{|z|^\beta} \mathrm{d}s\,\nu(\mathrm{d}z) \\ &= 2L\int\limits_0^{|x-y|^\beta}\int\limits_{s^{1/\beta}\le|z|\le|x-y|} \nu(\mathrm{d}z)\,\mathrm{d}s \le 2L\int\limits_0^{|x-y|^\beta}\int\limits_{|z|\ge s^{1/\beta}} \nu(\mathrm{d}z)\,\mathrm{d}s \\ &\le 2L\int\limits_0^{|x-y|^\beta} h(s^{1/\beta})\,\mathrm{d}s \le 2Lch(1)\int\limits_0^{|x-y|^\beta} s^{-\alpha/\beta}\,\mathrm{d}s \\ &= 2Lch(1)(1-\alpha/\beta)^{-1}|x-y|^{\beta-\alpha}. \end{aligned}$$

Now we will estimate I_2. Using again (19), (5) and (16) we get

$$\begin{aligned}\mathrm{I}_2 &\le \int\limits_{|z|>|x-y|} (|f(x+z)-f(y+z)|+|f(x)-f(y)|)\,\nu(\mathrm{d}z)\\ &\le 2L|x-y|^\beta \int\limits_{|z|>|x-y|} \nu(\mathrm{d}z) \le 2L|x-y|^\beta h(|x-y|)\\ &\le 2Lch(1)|x-y|^{\beta-\alpha}.\end{aligned}$$

Hence

$$\|\!|\mathcal{L}f|\!\|_{\beta-\alpha} \le (1+(1-\alpha/\beta)^{-1})2ch(1)\|\!|f|\!\|_\beta. \tag{20}$$

Futhermore, for $x \in \mathbb{R}^d$,

$$|\mathcal{L}f(x)| \le \int\limits_{\mathbb{R}^d} |f(x+y)-f(x)|\,\nu(\mathrm{d}y).$$

We split the integral above as follows:

$$\int\limits_{\mathbb{R}^d} = \int\limits_{|y|<1} + \int\limits_{|y|\ge 1} =: \mathrm{I}_3 + \mathrm{I}_4.$$

Proceeding as for I_1, we obtain

$$\mathrm{I}_3 \le \|\!|f|\!\|_\beta \int\limits_{|y|<1} |y|^\beta\,\nu(\mathrm{d}y) \le \|\!|f|\!\|_\beta \frac{c}{1-\alpha/\beta} h(1).$$

Next,

$$\mathrm{I}_4 \le 2\|f\|_\infty \int\limits_{|y|\ge 1} \nu(\mathrm{d}y) \le 2\|f\|_\infty h(1).$$

Therefore

$$\begin{aligned}\|\mathcal{L}f\|_\infty &\le \frac{c}{1-\alpha/\beta} h(1)\|\!|f|\!\|_\beta + 2h(1)\|f\|_\infty\\ &\le \left(\frac{c}{1-\alpha/\beta}+2\right)h(1)\|f\|_\beta\\ &\le \frac{3c}{1-\alpha/\beta} h(1)\|f\|_\beta.\end{aligned} \tag{21}$$

By (20) and (21),

$$\begin{aligned}\|\mathcal{L}f\|_{\beta-\alpha} &\le 2h(1)\|f\|_\infty + \left(2+\frac{3}{1-\alpha/\beta}\right)ch(1)\|\!|f|\!\|_\beta\\ &\le \left(2+\left(2+\frac{3}{1-\alpha/\beta}\right)c\right)h(1)\|f\|_\beta\\ &\le \frac{7c}{1-\alpha/\beta} h(1)\|f\|_\beta.\end{aligned} \tag{22}$$

The proof is complete. ∎

Corollary 3.2. *Assume that $\psi^* \in \mathrm{WUSC}(\alpha, 1, \overline{C})$ for some $\alpha \in (0,1)$. If $f \in \mathcal{C}_0^\beta$ for some $\beta \in (n\alpha, 1]$, then $\mathcal{L}^k f \in \mathcal{C}_0^{\beta - k\alpha}$ for $k \in \{1, \ldots, n\}$ and $\mathcal{L}^k f \in \mathcal{D}(\mathcal{L})$ for $k \in \{1, \ldots, n-1\}$.*

It is well-known that for $f \in \mathcal{D}(\mathcal{L})$ and $t \geq 0$, P_t is differentiable and $\frac{\mathrm{d}}{\mathrm{d}t} P_t f = \mathcal{L} P_t f = P_t \mathcal{L} f$; see e.g. Pazy [19, Theorem 1.2.4 c)]. Therefore, if $\mathcal{L}^k f \in \mathcal{D}(\mathcal{L})$ for $k \in \{1, \ldots, n-1\}$, then $\frac{\mathrm{d}^n}{\mathrm{d}t^n} P_t f = \mathcal{L}^n P_t f = P_t \mathcal{L}^n f$. To apply this result, we will use the fact that for $t_0 > 0$, $P_{t_0}(\mathcal{C}_0^\beta) \subset \mathcal{C}_0^\beta$. Indeed,

$$
\begin{aligned}
|P_{t_0} f(x) - P_{t_0} f(y)| &\leq \int_{\mathbb{R}^d} |f(x+z) - f(x) - f(y+z) + f(y)| \, p_{t_0}(\mathrm{d}z) \\
&\leq \int_{\mathbb{R}^d} (|f(x+z) - f(y+z)| + |f(y) - f(x)|) \, p_{t_0}(\mathrm{d}z) \\
&\leq 2L|x-y|^\beta \int_{\mathbb{R}^d} p_{t_0}(\mathrm{d}z) = 2L|x-y|^\beta.
\end{aligned}
$$

Theorem 3.3. *Assume that $\psi^* \in \mathrm{WUSC}(\alpha, 1, \overline{C})$ for some $\alpha \in (0,1)$. If $f \in \mathcal{C}_0^\beta$ for some $\beta \in (n\alpha, 1]$, then*

$$
\lim_{t \to 0^+} t^{-n} \left(P_t f - \sum_{k=0}^{n-1} \frac{t^k}{k!} \mathcal{L}^k f \right) = \frac{1}{n!} \mathcal{L}^n f.
$$

Proof. By Corollary 3.2, $\mathcal{L}^k f \in \mathcal{D}(\mathcal{L})$ for $k \in \{1, \ldots, n-1\}$, hence $P_t f$ is n times differentiable. By Taylor's theorem applied to $t \mapsto P_t f$,

$$
P_t f = \sum_{k=0}^{n-1} \frac{t^k}{k!} \mathcal{L}^k f + \frac{t^n}{n!} P_{\theta_0} \mathcal{L}^n f
$$

for some $\theta_0 \in (0, t)$. The claim follows from the right continuity of P_t at $t = 0$. ■

Theorem 3.3, Lemma 2.1 and (10) give the following result.

Corollary 3.4. *Assume that there exists $C > 0$ such that for all $\lambda \leq 1$ and $r < 1$, $h(\lambda r) \leq C\lambda^{-\alpha} h(r)$ for some $\alpha \in (0,1)$. Let $n \geq 2$. If $n\alpha < 1$, then*

$$
\lim_{t \to 0^+} t^{-n} \left(H(t) - t \operatorname{Per}_\nu(\Omega) + \sum_{k=2}^{n-1} \frac{t^k}{k!} \mathcal{L}^k g_\Omega(0) \right) = -\frac{1}{n!} \mathcal{L}^n g_\Omega(0).
$$

Example 3.5. If $\nu^{(\alpha)}$ is an α-stable Lévy measure, $\alpha \in (0,1)$, then the Hölder space $\mathcal{C}_0^\beta$ is contained in the domain of $\mathcal{L} = -(-\Delta)^{\alpha/2}$ for any $\beta \in (\alpha, 1]$, and we have

$$
\mathcal{L} f(x) = \int_{\mathbb{R}^d \setminus \{0\}} (f(x+y) - f(x)) \, \nu^{(\alpha)}(\mathrm{d}y), \qquad f \in \mathcal{C}_0^\beta, \ x \in \mathbb{R}^d.
$$

The associated semigroup $(p_t)_{t\geq 0}$ is the α-stable semigroup in $\mathbb{R}^d$, determined by $\psi(\xi) = |\xi|^\alpha$. We have $\psi(r\xi) = r^\alpha\psi(\xi)$, so in particular $\psi^* \in \mathrm{WUSC}(\alpha, 0, 1)$. If $f \in \mathcal{C}_0^\beta$ for some $\beta \in (n\alpha, 1]$, then by Corollary 3.2, $\mathcal{L}^k f \in \mathcal{D}(\mathcal{L})$ for $k \in \{1, \ldots, n-1\}$ and $\mathcal{L}^k f \in \mathcal{C}_0^{\beta-k\alpha}$ for $k \in \{1, \ldots, n\}$, and by Theorem 3.3,

(23)

$$\lim_{t\to 0^+} t^{-n}\left(-P_t f + \sum_{k=0}^{n-1} \frac{(-1)^{k-1}t^k}{k!}(-(-\Delta)^{k\alpha/2})f\right) = \frac{(-1)^n}{n!}(-(-\Delta)^{n\alpha/2})f,$$

since $((-\Delta)^{\alpha/2})^n = (-\Delta)^{n\alpha/2}$ for $n\alpha < 2$; see [16, (1.1.12)]. By Corollary 3.4,

$$\lim_{t\to 0^+} t^{-n}\left(H(t) - \sum_{k=1}^{n-1} \frac{(-1)^{k-1}t^k}{k!}\operatorname{Per}_{\nu^{(k\alpha)}}(\Omega)\right) = \frac{(-1)^{n-1}}{n!}\operatorname{Per}_{\nu^{(n\alpha)}}(\Omega).$$

THEOREM 3.6. *Assume there exists $C > 0$ such that for all $\alpha \in (0, 1)$, $\psi^* \in \mathrm{WUSC}(\alpha, 1, C\alpha^{-1})$. If $f \in \mathcal{C}_0^\beta$ for some $\beta \in (0, 1]$, then there exists $t_0 > 0$ such that for all $t \in (0, t_0)$,*

$$P_t f = \sum_{k=0}^{\infty} \frac{t^k}{k!}\mathcal{L}^k f$$

in $\mathcal{C}_0(\mathbb{R}^d)$.

Proof. Without loss of generality, we can assume $h(1) = 1$. For any $N \in \mathbb{N}$, let $\alpha = \alpha(N) = \beta/(2N)$. For $n \in \{1, \ldots, N-1\}$, let $\beta_n = \beta - n\alpha$. Since $N\alpha = \beta/2 < \beta$, by the proof of Theorem 3.3,

$$P_t f(x) = \sum_{n=0}^{N-1} \frac{t^n\mathcal{L}^n f(x)}{n!} + \tilde{R}_N, \quad \text{where} \quad \tilde{R}_N = \frac{t^N P_{\theta_0}\mathcal{L}^N f(x)}{N!}.$$

By (21), for any $N \in \mathbb{N}$,

$$\|\mathcal{L}^N f\|_\infty \leq \frac{7C/\alpha}{1-\alpha/\beta_{N-1}}\|\mathcal{L}^{N-1}f\|_{\beta_{N-1}}. \tag{24}$$

By (22), for $k \in \{1, \ldots, N-1\}$,

$$\|\mathcal{L}f\|_{\beta_k} \leq \frac{7C/\alpha}{1-\alpha/\beta_{k-1}}\|f\|_{\beta_{k-1}}. \tag{25}$$

Using (24) and applying (25) N times we get

$$\begin{aligned}\|\mathcal{L}^N f\|_\infty &\leq \frac{7C/\alpha}{1-\alpha/\beta_{N-1}}\|\mathcal{L}^{N-1}f\|_{\beta_{N-1}} \leq \left(\prod_{k=0}^{N-1}\frac{7C/\alpha}{1-\alpha/\beta_k}\right)\|f\|_\beta \\ &\leq (7C)^N\left(\prod_{k=0}^{N-1}\frac{1/\alpha}{1-2\alpha/\beta}\right)\|f\|_\beta\end{aligned} \tag{26}$$

$$
\begin{aligned}
&= (7C)^N \Big(\prod_{k=0}^{N-1} \frac{2N/\beta}{1-1/N}\Big)\|f\|_\beta \\
&= (14C/\beta)^N N^N \Big(\prod_{k=0}^{N-1} \frac{1}{1-1/N}\Big)\|f\|_\beta \\
&\le (14C/\beta)^N N^N e\|f\|_\beta .
\end{aligned}
$$

By the contractivity of P_{θ_0}, (26) and Stirling's formula,

$$
\begin{aligned}
|\tilde{R}_N| &\le \frac{\|t^N P_{\theta_0}\mathcal{L}^N f\|_\infty}{N!} \le \frac{t^N\|\mathcal{L}^N f\|_\infty}{N!} \\
&\le \frac{cC'^N N^N t^N}{\sqrt{2\pi N}(N/e)^N}\|f\|_\beta \le \frac{c'\tilde{C}^N t^N}{\sqrt{N}}\|f\|_\beta,
\end{aligned}
$$

which tends to 0 as $N\to\infty$. The proof is complete. ∎

Example 3.7. Let $\psi(\xi) = \log(1+|\xi|^2)$, i.e. $\mathcal{L} = -\log(1-\Delta)$. Let $\alpha\in(0,1]$. Since, for $\lambda\ge 1$ and $x\ge 1$,

$$
\log(1+\lambda x) \le \log(\lambda(1+x)) = \frac{1}{\alpha}\log(\lambda^\alpha(1+x)^\alpha) \le \frac{1}{\alpha}\log(\lambda^\alpha(1+x)),
$$

and

$$
\frac{\log(\lambda^\alpha(1+x))}{\log(1+x)} \le \frac{\lambda^\alpha}{\log 2},
$$

we have $\log(1+\cdot)\in \mathrm{WUSC}(\alpha,1,2/\alpha)$. Hence $\psi\in\mathrm{WUSC}(\alpha,1,4/\alpha)$, that is, ψ satisfies the assumptions of Theorem 3.6.

Corollary 3.8. *Assume that there exists $C>0$ such that for all $\alpha\in(0,1)$ and $\lambda\le 1$, $h(\lambda r)\le C\alpha^{-1}\lambda^{-1}h(r)$. Then there exists $t_0>0$ such that for all $t\in(0,t_0)$,*

$$
H(t) = t\,\mathrm{Per}_\nu(\Omega) - \sum_{k=2}^{\infty}\frac{t^k}{k!}\mathcal{L}^k g_\Omega(0).
$$

Example 3.9. Let ν be a finite measure on $\mathbb{R}^d$ and let $(p_t)_{t\ge0}$ be determined by

$$
\psi(\xi) = \int_{\mathbb{R}^d} (1-e^{i\langle\xi,z\rangle})\,\nu(\mathrm{d}z).
$$

In this case

$$
\mathcal{L}f(x) = \int_{\mathbb{R}^d} (f(x+y)-f(x))\,\nu(\mathrm{d}y).
$$

The generator $\mathcal{L}$ can be expressed as a convolution operator

$$
\mathcal{L}f = (\nu-\nu(\mathbb{R}^d)\delta_0)*f,
$$

therefore

$$\mathcal{L}^n f = (\nu - \nu(\mathbb{R}^d)\delta_0)^{*n} * f = \sum_{i=0}^{n} (-1)^{n-i} \binom{n}{i} \nu(\mathbb{R}^d)^{n-i} \nu^{*i} * f,$$

where for $k \in \mathbb{N}$, μ^{*k} denotes the k-fold iteration of the convolution of μ with itself, i.e. $\mu^{*0} = \delta_0$ and $\mu^{*k} = \mu^{*(k-1)} * \mu$ for $k \geq 1$. It is well-known that, since ν is finite and $\mathcal{L}$ is bounded, we have

$$P_t = e^{t\mathcal{L}}.$$

Therefore

$$\begin{aligned} P_t = \sum_{n=0}^{\infty} \frac{t^n}{n!} \mathcal{L}^n &= \sum_{n=0}^{\infty} \sum_{i=0}^{n} (-1)^{n-i} \frac{t^n}{i!(n-i)!} \nu(\mathbb{R}^d)^{n-i} \nu^{*i} \\ &= \sum_{i=0}^{\infty} \sum_{n=i}^{\infty} (-1)^{n-i} \frac{t^n}{i!(n-i)!} \nu(\mathbb{R}^d)^{n-i} \nu^{*i} \\ &= \sum_{j=0}^{\infty} \frac{(-t\nu(\mathbb{R}^d))^j}{j!} \sum_{i=0}^{\infty} \frac{t^i}{i!} \nu^{*i} = e^{-t\nu(\mathbb{R}^d)} \exp(t\nu), \end{aligned}$$

where $\exp(\nu) = \sum_{n=0}^{\infty} \frac{1}{n!} \nu^{*n}$. This expansion follows also from Theorem 3.6. Applying this result to $f = g_{\Omega}$, we extend [17, Theorem 1.2], which holds for compactly supported probabilistic measures with radial density, to general finite measures.

3.2. Heat content for the fractional Laplacian on $\mathbb{R}^d$. Let $(p_t)_{t\geq 0}$ be the α-stable semigroup in $\mathbb{R}^d$, $\alpha \in (0,2)$. We recall that in this case $\psi(\xi) = |\xi|^{\alpha}$ and the corresponding Lévy measure ν is the α-stable Lévy measure $\nu^{(\alpha)}$. The related function h turns into $h(r) = c/r^{\alpha}$, for some $c > 0$.

Let

$$a_n := \frac{1}{\pi^{1+d/2}} \frac{(-1)^{n-1}}{n!} 2^{n\alpha} \Gamma\left(\frac{n\alpha}{2} + 1\right) \Gamma\left(\frac{n\alpha + d}{2}\right) \sin\left(\frac{\pi n\alpha}{2}\right).$$

For $n\alpha/2 \notin \mathbb{N}$,

$$a_n = \frac{(-1)^{n-1}}{n!} \mathcal{A}_{d,-n\alpha}.$$

By Hiraba [14, Remark 2.b)], for $\alpha < 1$ and $x \in \mathbb{R}^d \setminus \{0\}$,

$$p_1(x) = \sum_{n=1}^{\infty} a_n |x|^{-n\alpha - d}. \tag{27}$$

The following two results extend [1, Theorem 1.2]. Note that they provide a more detailed expansion than the one resulting from Corollary 3.4; compare with Example 3.5.

THEOREM 3.10. *Let $\alpha \in (0,1)$ be such that $1/\alpha \notin \mathbb{N}$ and let $\Omega \subset \mathbb{R}^d$ be an open set of finite Lebesgue measure and perimeter. Then*

$$\lim_{t\to 0^+} t^{-1/\alpha}\Bigg(H(t) - \sum_{n=1}^{[1/\alpha]} \frac{(-1)^{n-1}}{n!} t^n \operatorname{Per}_{\nu^{(n\alpha)}}(\Omega)\Bigg) = \frac{\pi^{\frac{d-1}{2}}}{\Gamma\big(\frac{d+1}{2}\big)} \operatorname{Per}(\Omega)\Bigg(\int_0^1 r^d p_1(re_d)\,\mathrm{d}r - \sum_{n=1}^{\infty} \frac{a_n}{1-n\alpha}\Bigg).$$

Proof. Without loss of generality, we can assume that $\operatorname{diam}(\Omega) = 1$. For $1/\alpha \notin \mathbb{N}$, $[1/\alpha] = \lceil 1/\alpha \rceil - 1$, and we will use this formula in order to avoid repeating similar calculations in the next proof. By (13), (14) and the scaling property of p_t,

$$\begin{aligned} H(t) &- \sum_{n=1}^{\lceil 1/\alpha\rceil -1} \frac{(-1)^{n-1}}{n!} t^n \operatorname{Per}_{\nu^{(n\alpha)}}(\Omega) \\ &= \int_{\mathbb{R}^d} (g_\Omega(0) - g_\Omega(x))\Bigg(p_t(x) - \sum_{n=1}^{\lceil 1/\alpha\rceil -1} \frac{(-1)^{n-1}}{n!} t^n \mathcal{A}_{d,-n\alpha}|x|^{-d-n\alpha}\Bigg)\,\mathrm{d}x \\ &= \int_{\mathbb{R}^d} (g_\Omega(0) - g_\Omega(t^{1/\alpha}x))\Bigg(p_1(x) - \sum_{n=1}^{\lceil 1/\alpha\rceil -1} \frac{(-1)^{n-1}}{n!} \mathcal{A}_{d,-n\alpha}|x|^{-d-n\alpha}\Bigg)\,\mathrm{d}x. \end{aligned}$$

We split the above integral into

$$\int_{|x|\le 1} + \int_{1<|x|\le t^{-1/\alpha}} + \int_{|x|>t^{-1/\alpha}} =: \mathrm{I}_1 + \mathrm{I}_2 + \mathrm{I}_3. \tag{28}$$

We have

$$\begin{aligned} \mathrm{I}_1 &= \int_{|x|\le 1} (g_\Omega(0) - g_\Omega(t^{1/\alpha}x))\Bigg(p_1(x) - \sum_{n=1}^{\lceil 1/\alpha\rceil -1} \frac{(-1)^{n-1}}{n!} \mathcal{A}_{d,-n\alpha}|x|^{-d-n\alpha}\Bigg)\,\mathrm{d}x \\ &= t^{1/\alpha}\int_0^1\int_{\mathbb{S}^{d-1}} r^d\, \frac{g_\Omega(0) - g_\Omega(t^{1/\alpha}ru)}{t^{1/\alpha}r} \\ &\qquad \times \Bigg(p_1(re_d) - \sum_{n=1}^{\lceil 1/\alpha\rceil -1} \frac{(-1)^{n-1}}{n!} \mathcal{A}_{d,-n\alpha} r^{-d-n\alpha}\Bigg)\,\sigma(\mathrm{d}u)\,\mathrm{d}r. \end{aligned}$$

By (10), (11) and the Dominated Convergence Theorem,

$$\lim_{t\to 0^+} t^{-1/\alpha}\mathrm{I}_1 = \frac{\pi^{\frac{d-1}{2}}}{\Gamma\big(\frac{d+1}{2}\big)} \operatorname{Per}(\Omega)\Bigg(\int_0^1 r^d p_1(re_d)\,\mathrm{d}r - \sum_{k=1}^{\lceil 1/\alpha\rceil -1} \frac{(-1)^{n-1}}{n!}\, \frac{\mathcal{A}_{d,-n\alpha}}{1-n\alpha}\Bigg).$$

Next,

$$\begin{aligned}\frac{\mathrm{I}_3}{g_\Omega(0)} &= \int\limits_{|x|>t^{-1/\alpha}} \left(p_1(x) - \sum_{n=1}^{\lceil 1/\alpha\rceil-1} \frac{(-1)^{n-1}}{n!}\mathcal{A}_{d,-n\alpha}|x|^{-d-n\alpha}\right)\mathrm{d}x \\ &= \int\limits_{|x|>t^{-1/\alpha}} \sum_{n=\lceil 1/\alpha\rceil}^{\infty} a_n|x|^{-n\alpha-d}\,\mathrm{d}x \\ &= \sum_{n=\lceil 1/\alpha\rceil}^{\infty} a_n \int\limits_{|x|>t^{-1/\alpha}} |x|^{-n\alpha-d}\,\mathrm{d}x = \omega_{d-1}\sum_{n=\lceil 1/\alpha\rceil}^{\infty}\frac{a_n}{n\alpha}t^n.\end{aligned}$$

We have

$$|\mathrm{I}_3| \le g_\Omega(0)\omega_{d-1}\sum_{n=\lceil 1/\alpha\rceil}^{\infty}\frac{|a_n|}{n\alpha}t^n = O(t^{\lceil 1/\alpha\rceil})$$

for $t<1$, thus

$$\lim_{t\to 0^+} t^{-1/\alpha}\mathrm{I}_3 = 0.$$

We have

$$\begin{aligned}\mathrm{I}_2 &= \int\limits_{1<|x|<t^{-1/\alpha}} (g_\Omega(0)-g_\Omega(t^{1/\alpha}x)) \\ &\qquad\times\left(p_1(x) - \sum_{n=1}^{\lceil 1/\alpha\rceil-1} \frac{(-1)^{n-1}}{n!}\mathcal{A}_{d,-n\alpha}|x|^{-d-n\alpha}\right)\mathrm{d}x \\ &= t^{1/\alpha}\int_1^{t^{-1/\alpha}}\int\limits_{\mathbb{S}^{d-1}} r^d\,\frac{g_\Omega(0)-g_\Omega(t^{1/\alpha}ru)}{t^{1/\alpha}r} \\ &\qquad\times\left(p_1(re_d) - \sum_{n=1}^{\lceil 1/\alpha\rceil-1} \frac{(-1)^{n-1}}{n!}\mathcal{A}_{d,-n\alpha}r^{-d-n\alpha}\right)\sigma(\mathrm{d}u)\,\mathrm{d}r \\ &= t^{1/\alpha}\int_1^{t^{-1/\alpha}}\int\limits_{\mathbb{S}^{d-1}} \frac{g_\Omega(0)-g_\Omega(t^{1/\alpha}ru)}{t^{1/\alpha}r}\sum_{n=\lceil 1/\alpha\rceil}^{\infty} a_n\,\sigma(\mathrm{d}u)\,r^{-n\alpha}\,\mathrm{d}r.\end{aligned}$$

By (10),

$$\begin{aligned}|\mathrm{I}_2| &\le \frac{\mathrm{Per}(\Omega)}{2}t^{1/\alpha}\sum_{n=\lceil 1/\alpha\rceil}^{\infty}|a_n|\int\limits_{1<|x|\le t^{-1/\alpha}}|x|^{1-n\alpha-d}\,\mathrm{d}x \\ &\le \frac{\mathrm{Per}(\Omega)}{2}t^{1/\alpha}\sum_{n=\lceil 1/\alpha\rceil}^{\infty}|a_n|\int\limits_{|x|>1}|x|^{1-n\alpha-d}\,\mathrm{d}x \\ &= \frac{\mathrm{Per}(\Omega)}{2}\omega_{d-1}t^{1/\alpha}\sum_{n=\lceil 1/\alpha\rceil}^{\infty}\frac{|a_n|}{n\alpha-1},\end{aligned}$$

hence

$$\mathrm{I}_2 = \sum_{n=\lceil 1/\alpha\rceil}^{\infty} a_n \int\limits_{1<|x|\le t^{-1/\alpha}} \frac{g_\Omega(0) - g_\Omega(t^{1/\alpha}x)}{|x|^{d+n\alpha}}\,\mathrm{d}x$$

and

$$\lim_{t\to 0^+} t^{-1/\alpha}\mathrm{I}_2 = \sum_{n=\lceil 1/\alpha\rceil}^{\infty} a_n \lim_{t\to 0^+} t^{-1/\alpha} \int\limits_{1<|x|\le t^{-1/\alpha}} \frac{g_\Omega(0) - g_\Omega(t^{1/\alpha}x)}{|x|^{d+n\alpha}}\,\mathrm{d}x.$$

We get

$$t^{-1/\alpha}\mathrm{I}_2 = \sum_{n=\lceil 1/\alpha\rceil}^{\infty} a_n \int_1^{t^{-1/\alpha}} \mathcal{M}_\Omega(t,r) r^{-n\alpha}\,\mathrm{d}r,$$

where

$$\mathcal{M}_\Omega(t,r) = \int\limits_{\mathbb{S}^{d-1}} \frac{g_\Omega(0) - g_\Omega(rt^{1/\alpha}u)}{rt^{1/\alpha}}\,\sigma(\mathrm{d}u). \tag{29}$$

We claim that

$$\lim_{t\to 0^+} \int_1^{t^{-1/\alpha}} \mathcal{M}_\Omega(t,r) r^{-n\alpha}\,\mathrm{d}r = \frac{\pi^{(d-1)/2}}{\Gamma((d+1)/2)}\,\mathrm{Per}(\Omega)\frac{1}{n\alpha - 1}. \tag{30}$$

Indeed, by (10) and (11),

$$0 \le \mathcal{M}_\Omega(t,r) \le \tfrac{1}{2}\,\mathrm{Per}(\Omega)\sigma(\mathbb{S}^{d-1}) \tag{31}$$

and, for any $r > 0$,

$$\lim_{t\to 0^+} \mathcal{M}_\Omega(t,r) = \frac{\pi^{(d-1)/2}}{\Gamma((d+1)/2)}\,\mathrm{Per}(\Omega). \tag{32}$$

Moreover,

$$\int_1^{t^{-1/\alpha}} r^{-n\alpha}\,\mathrm{d}r \le \int_1^{\infty} r^{-n\alpha}\,\mathrm{d}r = \frac{1}{n\alpha - 1}$$

and hence (30) follows by the Dominated Convergence Theorem. ■

THEOREM 3.11. *Let $\alpha \in (0,1)$ be such that $1/\alpha \in \mathbb{N}$ and let $\Omega \subset \mathbb{R}^d$ be an open set of finite Lebesgue measure and perimeter. Then*

$$\lim_{t\to 0^+} (t^{1/\alpha}\log(1/t))^{-1}\Big(H(t) - \sum_{n=1}^{1/\alpha - 1} \frac{(-1)^{n-1}}{n!} t^n\,\mathrm{Per}_{\nu^{(n\alpha)}}(\Omega)\Big) = \frac{(-1)^{1/\alpha - 1}}{(1/\alpha - 1)!\pi}\,\mathrm{Per}(\Omega).$$

Proof. By the proof of Theorem 3.10,

$$H(t)-\sum_{n=1}^{1/\alpha-1}\frac{(-1)^{n-1}}{n!}t^n\operatorname{Per}_{\nu^{(n\alpha)}}(\Omega)=\mathrm{I}_1+\mathrm{I}_2+\mathrm{I}_3,$$

where

$$\mathrm{I}_1=t^{1/\alpha}\int_0^1\int_{\mathbb{S}^{d-1}}r^d\,\frac{g_\Omega(0)-g_\Omega(t^{1/\alpha}ru)}{t^{1/\alpha}r}\times\Big(p_1(re_d)-\sum_{n=1}^{1/\alpha-1}\frac{(-1)^{n-1}}{n!}\mathcal{A}_{d,-n\alpha}r^{-d-n\alpha}\Big)\,\sigma(\mathrm{d}u)\,\mathrm{d}r,$$

$$\mathrm{I}_2=\int_{1<|x|\le t^{-1/\alpha}}(g_\Omega(0)-g_\Omega(t^{1/\alpha}x))\sum_{n=1/\alpha}^{\infty}a_n|x|^{-n\alpha-d}\,\mathrm{d}x,$$

$$\mathrm{I}_3=|g_\Omega(0)|\omega_{d-1}\sum_{n=1/\alpha}^{\infty}\frac{a_n}{n\alpha}t^n.$$

By (10), (11) and the Dominated Convergence Theorem,

$$\lim_{t\to0^+}(t^{1/\alpha}\log(1/t))^{-1}\mathrm{I}_1=0.$$

Next,

$$\lim_{t\to0^+}(t^{1/\alpha}\log(1/t))^{-1}\mathrm{I}_3=0.$$

By (10),

$$\begin{aligned}|\mathrm{I}_2|&\le\frac{\operatorname{Per}(\Omega)}{2}t^{1/\alpha}\sum_{n=1/\alpha}^{\infty}|a_n|\int_{1<|x|\le t^{-1/\alpha}}|x|^{1-n\alpha-d}\,\mathrm{d}x\\&=\frac{\operatorname{Per}(\Omega)}{2}\Big(\sum_{n=1/\alpha+1}^{\infty}|a_n|t^{1/\alpha}\frac{t^{n-1/\alpha}-1}{1-n\alpha}+\frac{a_{1/\alpha}}{\alpha}t^{1/\alpha}\log(1/t)\Big)\\&\le\frac{\operatorname{Per}(\Omega)}{2}\Big(\sum_{n=1/\alpha+1}^{\infty}|a_n|\frac{t^{1/\alpha}}{n\alpha-1}+\frac{a_{1/\alpha}}{\alpha}t^{1/\alpha}\log(1/t)\Big).\end{aligned}$$

Therefore

$$\mathrm{I}_2=\sum_{n=1/\alpha}^{\infty}a_n\int_{1<|x|\le t^{-1/\alpha}}(g_\Omega(0)-g_\Omega(t^{1/\alpha}x))|x|^{-n\alpha-d}\,\mathrm{d}x.$$

We have

$$
\begin{aligned}
&(t^{1/\alpha}\log(1/t))^{-1}\int\limits_{1<|x|\le t^{-1/\alpha}}(g_\Omega(0)-g_\Omega(t^{1/\alpha}x))|x|^{-n\alpha-d}\,\mathrm{d}x\\
&\qquad=\log(1/t)^{-1}\int\limits_1^{t^{-1/\alpha}}\int\limits_{\mathbb{S}^{d-1}}\frac{g_\Omega(0)-g_\Omega(t^{1/\alpha}ru)}{t^{1/\alpha}r}\,\sigma(\mathrm{d}u)\,r^{-n\alpha}\,\mathrm{d}r\\
&\qquad=\log(1/t)^{-1}\int\limits_1^{t^{-1/\alpha}}\mathcal{M}_\Omega(t,r)r^{-n\alpha}\,\mathrm{d}r.
\end{aligned}
$$

We claim that

$$
\lim_{t\to 0^+}\log(1/t)^{-1}\int\limits_1^{t^{-1/\alpha}}\mathcal{M}_\Omega(t,r)r^{-1}\,\mathrm{d}r=\frac{\pi^{\frac{d-1}{2}}}{\alpha\Gamma\left(\frac{d+1}{2}\right)}\operatorname{Per}(\Omega),\tag{33}
$$

where

$$
\mathcal{M}_\Omega(t,r)=\int\limits_{\mathbb{S}^{d-1}}\frac{g_\Omega(0)-g_\Omega(rt^{1/\alpha}u)}{rt^{1/\alpha}}\,\sigma(\mathrm{d}u).
$$

Indeed, by substitution,

$$
\begin{aligned}
\log(1/t)^{-1}\int\limits_1^{t^{-1/\alpha}}\mathcal{M}_\Omega(t,r)r^{-1}\,\mathrm{d}r&=\log(1/t)^{-1}\int\limits_0^{\log(1/t)/\alpha}\mathcal{M}_\Omega(t,e^r)\,\mathrm{d}r\\
&=\int\limits_0^{1/\alpha}\mathcal{M}_\Omega(t,t^{-r})\,\mathrm{d}r,
\end{aligned}
$$

and by (10), (11) and the Dominated Convergence Theorem,

$$
\begin{aligned}
\lim_{t\to 0^+}\int\limits_0^{1/\alpha}\mathcal{M}_\Omega(t,t^{-r})\,\mathrm{d}r&=\int\limits_0^{1/\alpha}\int\limits_{\mathbb{S}^{d-1}}\lim_{t\to 0^+}\frac{g_\Omega(0)-g_\Omega(t^{1/\alpha-r}u)}{t^{1/\alpha-r}}\,\sigma(\mathrm{d}u)\,\mathrm{d}r\\
&=\frac{\pi^{(d-1)/2}}{\alpha\Gamma((d+1)/2)}\operatorname{Per}(\Omega).
\end{aligned}
$$

We claim that for $n\ge 1/\alpha+1$ we have

$$
\lim_{t\to 0^+}\log(1/t)^{-1}\int\limits_1^{t^{-1/\alpha}}\mathcal{M}_\Omega(t,r)r^{-n\alpha}\,\mathrm{d}r=0.\tag{34}
$$

Indeed, by (31), (32) and since we have

$$
\int\limits_1^{t^{-1/\alpha}}\log(1/t)^{-1}r^{-n\alpha}\,\mathrm{d}r\le\int\limits_1^{\infty}r^{-n\alpha}\,\mathrm{d}r=\frac{1}{n\alpha-1}
$$

for $t < 1/e$, (34) follows by the Dominated Convergence Theorem. Formulas (33) and (34) yield the claim of the theorem. ∎

3.3. Heat content for general stable operators on $\mathbb{R}$. Let $\alpha \in (0,1) \cup (1,2)$, $\beta \in [-1,1]$ and $\gamma > 0$. We consider the convolution semigroup $(p_t)_{t \geq 0}$ on $\mathbb{R}$ such that

$$\psi(\xi) = \gamma|\xi|^\alpha \left(1 - i\beta \tan\left(\frac{\pi\alpha}{2}\right) \operatorname{sgn}(\xi)\right), \quad \xi \in \mathbb{R}.$$

The corresponding Lévy measure on $\mathbb{R}$ is given by

$$\nu(\mathrm{d}x) = \frac{c_+ \mathbb{1}_{x \geq 0} + c_- \mathbb{1}_{x<0}}{|x|^{1+\alpha}}\, \mathrm{d}x,$$

where

$$c_+ = -\frac{1+\beta}{2\Gamma(-\alpha)\cos(\pi\alpha/2)} \quad \text{and} \quad c_- = -\frac{1-\beta}{2\Gamma(-\alpha)\cos(\pi\alpha/2)}.$$

Let $\Omega = (a,b) \subset \mathbb{R}$. We have $g_\Omega(x) = (b-a-|x|)\mathbb{1}_{[0,b-a)}(|x|)$. For $\alpha \in (0,1)$, $\mathrm{Per}_\nu(\Omega) = (c_+ + c_-)\alpha^{-1}(1-\alpha)^{-1}(b-a)^{1-\alpha}$.

We can fix the parameter γ without loss of generality. Assume that $\gamma = \cos(\pi\beta\alpha/2)$ if $\alpha < 1$ and $\gamma = \cos\big(\pi\beta\frac{2-\alpha}{2}\big)$ if $\alpha > 1$.

Let

$$b_n := \frac{(-1)^{n-1}}{\pi} \frac{\Gamma(n\alpha+1)}{n!} \sin(\pi n\alpha\rho),$$

where $\rho = \frac{1+\beta}{2}$ if $\alpha < 1$, and $\rho = \frac{1-\beta(2-\alpha)/\alpha}{2}$ if $\alpha > 1$. By [22, (2.4.8)], for $\alpha < 1$ and $x > 0$,

$$p_1(x) = \sum_{n=1}^{\infty} b_n x^{-n\alpha-1}. \tag{35}$$

By [22, (2.5.4)], for $\alpha > 1$ and $\beta \neq -1$, and any $N \in \mathbb{N}$,

$$p_1(x) = \sum_{n=1}^{N} b_n x^{-n\alpha-1} + O_\alpha(x^{-(N+1)\alpha-1}) \tag{36}$$

as $x \to \infty$. Let

$$d_n = \frac{(-1)^{n-1}}{\pi} \frac{2\Gamma(n\alpha)}{n!} \sin\left(\frac{\pi n\alpha}{2}\right) \cos\left(\frac{\pi n\beta\big(\alpha \wedge \frac{2-\alpha}{\alpha}\big)}{2}\right)$$

This constant will appear in the following proposition, which complements [1, Theorem 1.1]. We generalize the results for $\alpha < 1$ to the nonsymmetric case. The last result is new even for the symmetric case, since previously only the first two terms of the expansion were known.

PROPOSITION 3.12. *Let $\Omega = (a,b)$, $|\Omega| = b - a$.*

(i) *Let $0 < \alpha < 1$ and $0 < t < \min\{|\Omega|^\alpha, e^{-1}\}$.*

(a) *If $1/\alpha \notin \mathbb{N}$, then there is a constant C_α independent of Ω such that*

$$H(t) = \sum_{n=1}^{[1/\alpha]} \frac{d_n |\Omega|^{1-n\alpha}}{1-n\alpha} t^n + C_\alpha\, t^{1/\alpha} + O_\alpha(t^{[1/\alpha]+1})$$

as $t \to 0^+$, where

$$C_\alpha = \int_{\mathbb{R}} (1 \wedge |x|) p_1(x)\,\mathrm{d}x - \frac{1}{\sqrt{\pi}} \sum_{n=1}^{\infty} \frac{d_n}{1-n\alpha}.$$

(b) *If $\alpha = 1/N$ for some $N \in \mathbb{N}$, then there is a constant $C_N = C_N(\Omega)$ such that*

$$H(t) = \sum_{n=1}^{N-1} \frac{d_n |\Omega|^{1-n/N}}{1-n\alpha} t^n + C_N t^N + N d_N t^N \log\left(\frac{1}{t}\right) + O_\alpha(t^{N+1})$$

as $t \to 0^+$, where

$$C_N = \int_{\mathbb{R}} (1 \wedge |x|) p_1(x)\,\mathrm{d}x + \frac{\log(|\Omega|)}{\sqrt{\pi}} d_N - \frac{1}{\sqrt{\pi}} \sum_{n \neq N} \frac{d_n}{1-n\alpha}.$$

(ii) *If $1 < \alpha < 2$, $|\beta| \neq 1$, then, for any $N \in \mathbb{N}$,*

$$H(t) = t^{1/\alpha} \int_{\mathbb{R}} |x| p_1(x)\,\mathrm{d}x + \sum_{n=1}^{N} \frac{d_n |\Omega|^{1-n\alpha}}{1-n\alpha} t^n + O_\alpha(t^{N+1})$$

as $t \to 0^+$.

Note that by [13, Proposition 1.4],

$$\int_{\mathbb{R}} |x| p_1(x)\,\mathrm{d}x = \frac{2}{\pi} \Gamma\left(1 - \frac{1}{\alpha}\right) \mathrm{Re}\left(1 + i\beta \tan\left(\frac{\pi\alpha}{2}\right)\right)^{1/\alpha}.$$

Proof. (i) can be proved analogously to [1, Theorem 1.1], using (35). We will prove (ii). We have

$$\begin{aligned} H(t) &= \int_{\mathbb{R}} (|\Omega| \wedge |x|) p_t(x)\,\mathrm{d}x = \int_{\mathbb{R}} (|\Omega| \wedge t^{1/\alpha}|x|) p_1(x)\,\mathrm{d}x \\ &= t^{1/\alpha} \int_{|x| < |\Omega| t^{-1/\alpha}} |x| p_1(x)\,\mathrm{d}x + \int_{|x| \geq |\Omega| t^{-1/\alpha}} (|\Omega| - t^{1/\alpha}|x|) p_1(x)\,\mathrm{d}x \\ &= t^{1/\alpha} \int_{\mathbb{R}} |x| p_1(x)\,\mathrm{d}x + \int_{|x| \geq |\Omega| t^{-1/\alpha}} (|\Omega| - t^{1/\alpha}|x|) p_1(x)\,\mathrm{d}x. \end{aligned}$$

By (36), for any $N \in \mathbb{N}$,

$$p_1(x) = \sum_{n=1}^{N} b_n x^{-n\alpha-1} + O_\alpha(x^{-(N+1)\alpha-1})$$

as $x \to \infty$. Therefore

$$\begin{aligned}
&\int_{|\Omega| t^{-1/\alpha}}^{\infty} (|\Omega| - t^{1/\alpha} x) p_1(x)\,\mathrm{d}x \\
&\quad = \int_{|\Omega| t^{-1/\alpha}}^{\infty} \Big(\sum_{n=1}^{N} (-1)^{n-1} b_n x^{-n\alpha-1} + O_\alpha(x^{-(N+1)\alpha-1}) \Big) (|\Omega| - t^{1/\alpha} x)\,\mathrm{d}x \\
&\quad = \sum_{n=1}^{N} \frac{b_n}{n\alpha(1-n\alpha)} |\Omega|^{1-n\alpha} t^n + \int_{|\Omega| t^{-1/\alpha}}^{\infty} O_\alpha(x^{-(N+1)\alpha-1})(|\Omega| - t^{1/\alpha} x)\,\mathrm{d}x.
\end{aligned}$$

We also have

$$\begin{aligned}
\Big| \int_{|\Omega| t^{-1/\alpha}}^{\infty} & O_\alpha(x^{-(N+1)\alpha-1})(|\Omega| - t^{1/\alpha} x)\,\mathrm{d}x \Big| \\
&\le C \int_{|\Omega| t^{-1/\alpha}}^{\infty} (t^{1/\alpha} x - |\Omega|) x^{-(N+1)\alpha-1}\,\mathrm{d}x \\
&= C \frac{1}{(N+1)\alpha((N+1)\alpha - 1)} |\Omega|^{1-(N+1)\alpha}\, t^{N+1}.
\end{aligned}$$

The calculations for $\int_{-\infty}^{-\Omega| t^{-1/\alpha}} (|\Omega| + t^{1/\alpha} x) p_1(x)\,\mathrm{d}x$ are similar since $p_1(-x)$ corresponds to $p_1(x)$ with parameters $(\alpha, -\beta, \gamma)$. ∎

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Tomasz Grzywny, Julia Lenczewska
Faculty of Pure and Applied Mathematics
Wrocław University of Science and Technology
50-370 Wrocław, Poland
E-mail: tomasz.grzywny@pwr.edu.pl
julia.lenczewska@pwr.edu.pl