2nd order homogeneous linear differential equations with constant coefficients.

Introduction.

The equation of the simplest harmonic motion has the form

$$F = -kx,$$

where F is the force, x is the position function of the object and k is some positive constant. Since $F = ma$ with m being the mass and a the acceleration, and $a = x''$, we get $x'' = -\frac{k}{m}x$ or, equivalently,

$$x'' + \frac{k}{m}x = 0.$$

This is an example of a 2nd order homogeneous linear differential equation with constant coefficients. The general form of such equations is

$$ay'' + by' + cy = 0, \tag{1}$$

where $y = y(t)$ is a function and a, b, c are constants with $a \neq 0$.

General theory.

Lemma 1. If y_1, y_2 are some solutions to the equation (1) then $y = c_1y_1 + c_2y_2$, where c_1, c_2 are any constants, is also a solution.

Proof. This is your Exercise 1.

Definition. y_1, y_2 are called fundamental solutions to the equation (1) if $\dfrac{y_1}{y_2}$ is not a constant function. In other words, $y_1 \neq cy_2$ for any constant c.

Example. $y_1 = e^t, y_2 = e^{3t}$ and $y_3 = 2e^t$ are solutions to the equation $y'' - 4y' + 3y = 0$. Then y_1, y_2 and y_2, y_3 are fundamental solutions but y_1, y_3 are not.

Exercise 2. Verify this example.

The next theorem states the general form of the solutions to the equation.

Theorem 1. If y_1, y_2 are some fundamental solutions to the equation (1) then every solution y has the form $y = c_1y_1 + c_2y_2$, where c_1, c_2 are some constants.

The proof is advanced and requires theory of linear spaces.

Remark. The situations in the space of the solutions to this equation is the same as in the XY-plane. If v_1, v_2 are two vectors from this plane and they are not parallel then every vector in the plane may be represented as $v = c_1v_1 + c_2v_2$ with some constants c_1, c_2. So if you treat the solutions y_1, y_2 as 'vectors' then y_1, y_2 are fundamental solutions if they are not 'parallel'. Hence, any solution has the form that is analogous to the one for vectors in the XY-plane.

Since the general solution involves two constants, the initial condition for particular solutions needs two pieces of information - the values of the function and its derivative at a given point. For example, $y(0) = 1, y'(0) = 2$ etc. This guarantees uniqueness of the solution.

Now we are going to develop techniques that will enable us to determine fundamental solutions to the equation. We start with the definition below.

Definition. The characteristic polynomial for the equation (1) is $P(x) = ax^2 + bx + c$.

Example. $P(x) = 2x^2 - x + 3$ is the characteristic polynomial for $2y'' - y' + 3y = 0$ and $P(x) = x^2 - 2$ is the characteristic polynomial for $y'' - 2y = 0$.

Lemma 2. If P is the characteristic polynomial to the equation (1) and r is its real (complex) root then $y = e^{rt}$ is a real (complex) solution.

Proof. This is your Exercise 3.

We are ready to state the main result of this paper.

Theorem 2.. Let P is the characteristic polynomial to the equation (1). Then there are 3 cases.

Case 1. α, β are two distinct real roots of P. Then $y_1 = e^{\alpha t}$ and $y_2 = e^{\beta t}$ are fundamental solutions so the general solution is

$$y = c_1 e^{\alpha t} + c_2 e^{\beta t}.$$

Case 2. α is a double root of P. Then $y_1 = e^{\alpha t}$ and $y_2 = te^{\alpha t}$ are fundamental solutions so the general solution is

$$y = e^{\alpha t}(c_1 + c_2 t).$$

Case 3. $r = \alpha + \beta i$, $\beta \neq 0$, is a complex root of P. Then $y_1 = \mathrm{Re}(e^{rt}) = e^{\alpha t}\cos(\beta t)$ and $y_2 = \mathrm{Im}(e^{rt}) = e^{\alpha t}\sin(\beta t)$ are fundamental solutions so the general solution is

$$y = e^{\alpha t}(c_1 \cos(\beta t) + c_2 \sin(\beta t)).$$

Equivalently, $y = Ae^{\alpha t}\sin(\beta t + \phi)$ with $A \geq 0$.

Proof. This is your Exercise 4.

Examples.

1. $y'' + y' - 2y = 0$ with $y(0) = 0, y'(0) = 3$.

Solution. The characteristic polynomial is $P(x) = x^2 + x - 2$ and its roots are $\alpha = 1, \beta = -2$. Hence, the general solution is $y = c_1 e^t + c_2 e^{-2t}$. Since $y' = c_1 e^t - 2c_2 e^{-2t}$, the initial condition leads to

$$\begin{cases} c_1 + c_2 = 0 \\ c_1 - 2c_2 = 3, \end{cases}$$

which gives $c_1 = 1, c_2 = -1$ and, hence, $y = e^t - e^{-2t}$.

2. $4y'' - 4y' + y = 0$ with $y(0) = 1, y'(0) = 2$.

Solution. The characteristic polynomial is $P(x) = 4x^2 - 4x + 1$ and its double root is $\alpha = \beta = 0.5$. Hence, the general solution is $y = e^{0.5t}(c_1 + c_2 t)$. The initial condition gives $c_1 = 1$ and $0.5c_1 + c_2 = 2$, which leads to $y = e^{0.5t}(1 + 1.5t)$.

3. $y'' + 2y' + 5y = 0$.

Solution. The characteristic polynomial is $P(x) = x^2 + 2x + 5$ and its complex root is $-1 + 2i$. Hence, the general solution is $y = e^{-t}(c_1 \cos(2t) + c_2 \sin(2t))$ or $y = Ae^{-t} \sin(2t + \phi)$.

Exercises

Solve the equations below.

1. $y'' - 5y' + 6y = 0$.

2. $y' + 4y' + 3y = 0$ with $y(0) = 1, y'(0) = -4$.

3. $9y'' - 6y' + y = 0$.

4. $y'' - 2y' + y = 0$ with $y(0) = -1, y'(0) = 1$.

5. $y'' + y = 0$.

6. $y'' - 4y' + 13y = 0$ with $y(\pi/2) = 0, y'(\pi/2) = 1$.

7. The harmonic motion equation $F = -kx$ with the following initial conditions:

a) $x(0) = A > 0$ (the amplitude) and $v(0) = 0$,

b) $x(0) = 0$ and $v(0) = v_{max}$ (the maximum velocity).

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