International Baccalaureate

World School no. 000971

Extended Essay

MATHEMATICS

From Mice Pursuit Problem to Laplace transform.

SESSION: May 2014

Author: Aleksander Tadeusz Janczewski

Candidate number: 000971-0007

Supervisor: Mr. Krzysztof Michalik

[Word count: 3477]

Abstract

Cyclic pursuit problems' internal symmetry enables statement of problem conditions in a very compact form. Laplace transform allows for solving linear differential equations in a short and 'elegant' way. This essay combines both of them by answering the research question: **What are the trajectories of mice pursuing one another in a regular polygon and how does properties of Laplace transform allow one to solve this problem?**

In the beginning the mice problem is introduced in the simplest regular polygon (equilateral triangle). Parametrization and initial analysis is performed – this leads to formulation of a system of two linear differential equations.

The need of method of solving the system of differential equations introduces the reader to the next part of the essay, where the Laplace transform of real variable functions is defined and its properties are investigated – linearity, transforms of derivatives (complete proof included) and Lerch theorem (uniqueness of transforms) among others. A table of Laplace transforms of a number of useful functions is presented (some of the results are derived in given examples). Instances of application of the transform to solving differential equations are also given.

Subsequently, already introduced methods are applied to solving the previously stated system of linear differential equations – the trajectories of mice in an equilateral triangle are found explicitly. The generalization of the solution is presented afterwards. Solution of a general system of equations results in explicit parametric equations of mice trajectories in a regular n-polygon, examples of solutions are presented in form of graphs.

Analysis of usefulness of Laplace transform in solving the problem is presented – the key properties identified are: mapping linear differential equations of order one into linear algebraic equations and uniqueness of transforms (guaranteeing uniqueness of solutions). Remarks on potential further research and possible alternative parametrization are added.

Word count: 295

Table of content

I. Introduction

Pursuit problems, where pursued moved along a straight line, were studied by Leonardo da Vinci and the general case was analysed by P. Bouger in 1732. Cyclic pursuit problems (also known as mice, bugs, missiles, dogs, ships problems) have an appealing internal symmetry. The mice problem can be traced back to H. Brocard and was popularized by H. Steinhaus. In the form of the problem adapted in this paper, mice are placed in the vertices of a regular polygon and each mouse chases its nearest right-hand neighbour. Analysis of the positions of pursuer and pursued subject leads to formulation of a system of differential equations.

Laplace transform is an example of the integral transform, which has been developed by a mathematician and astronomer Pierre-Simon Laplace[1] in his work on probability theory. Laplace transform has various applications – it is used in probability, signal analysis, solving electrical circuits, solving linear differential equations and their systems among others - it is recognized as a useful tool in physics and engineering. See [1].

This essay aim is a short introduction to Laplace transforms method of solving linear differential equations and its application to solving mice pursuit problem. My research question is: **What are the trajectories of mice pursuing one another in a regular polygon and how does properties of the Laplace transform allow one to solve this problem?**

[1] Pierre-Simon Laplace was a french mathematician, astronomer and physicist, who is one of authors of Probability theory. Laplace also formulated Laplace equation and developed Laplace transform. He lived from 23rd March 1749 to 5th March 1827.

II. Introduction of the mice problem

For initial analysis let's chose a particular regular polygon – e.g. an equilateral triangle (the 'simplest' regular polygon). Mice are placed in vertices of the triangle (facing interior of the polygon). Let's assume that the mice speeds are equal – we assume that the mice behave in a 'symmetrical' way, i.e. in the same way with respect to their (symmetrical) position in the polygon and each mouse chases its nearest right-hand neighbour.

Let $M_1(t) = (x_1(t), y_1(t))$ represent the position of mouse no. 1 at time $t \geq 0$, where $M_1(0) = P_1$ and we choose $P_1(0,0)$. In a similar way $M_2(t)$ is a position of mouse no. 2 with initial condition $M_2(0) = P_2$ and we choose $P_2(1,0)$. Similarly, $M_3(t)$ is a position of mouse no. 3 with initial condition $M_3(0) = P_3$ where $P_3\left(\frac{1}{2}, \frac{\sqrt{3}}{2}\right)$ as we choose the length of triangle's side as our unit. Mice positions (after some time) are symmetrical. See the figure below.

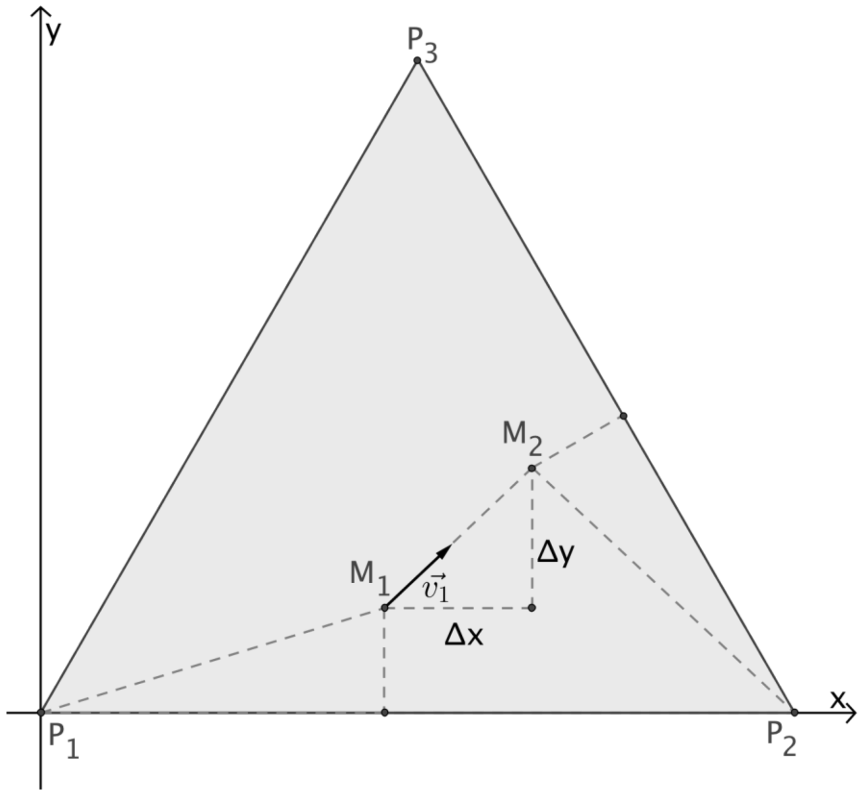


Figure 1. Triangle in which mice chase each other.[2]

[2] The figures are obtained by means of free software *GeoGebra 4.2.19.0* .

From the symmetry of the system it may be concluded that the relation between the position of mouse no. 1 and mouse no. 2 may be represented by means of the following transformations:

$M_1(t)$ rotated $\frac{2\pi}{3}$ around $(0,0)$ and translated by $\overrightarrow{P_1P_2}$ gives $M_2(t)$,

so

$$(x_2, y_2) = T_{\binom{1}{0}}\left(R_{\frac{2}{3}\pi,}(x_1, y_1)\right) =$$

$$= \left(x_1 \cos\tfrac{2}{3}\pi - y_1 \sin\tfrac{2}{3}\pi, x_1 \sin\tfrac{2}{3}\pi + y_1 \cos\tfrac{2}{3}\pi\right) =$$

$$= \left(-\tfrac{1}{2}x_1 + \tfrac{\sqrt{3}}{2}y_1 + 1, \tfrac{\sqrt{3}}{2}x_1 - \tfrac{1}{2}y_1\right),$$

where R_φ represents rotation[3] by angle φ around origin and $T_{\vec{v}}$ stands for translation[4] by vector $\vec{v}$.

Vector $\overrightarrow{v_1}(t) = \overrightarrow{M_1M_2}(t)$ (see Figure 1) represents the direction of the movement of mouse no. 1 (as it chases mouse no. 2) at time $t \geq 0$, hence it is tangent to the path of mouse no. 1 at $M_1(t)$ so its gradient is the gradient of the path in $M_1(t)$:

$$\frac{dy_1}{dx_1} = \frac{\Delta y_1}{\Delta x_1} = \frac{y_2(t) - y_1(t)}{x_2(t) - x_1(t)} = \frac{\frac{\sqrt{3}}{2}x_1(t) - \frac{3}{2}y_1(t)}{1 - \frac{3}{2}x_1(t) - \frac{\sqrt{3}}{2}y_1(t)} \quad . \tag{1}$$

Moreover we can notice that

$$\frac{dy_1}{dx_1} = \frac{\frac{dy_1}{dt}}{\frac{dx_1}{dt}} = \frac{\frac{\sqrt{3}}{2}x_1(t) - \frac{3}{2}y_1(t)}{1 - \frac{3}{2}x_1(t) - \frac{\sqrt{3}}{2}y_1(t)} \ ,$$

so a[5] solution may be obtained by solving the following system of equations,

$$\begin{cases} \dfrac{dy_1}{dt} = \frac{\sqrt{3}}{2}x_1(t) - \frac{3}{2}y_1(t) \ , \\ \dfrac{dx_1}{dt} = 1 - \frac{3}{2}x_1(t) - \frac{\sqrt{3}}{2}y_1(t) \ , \end{cases} \tag{2}$$

which is a system of linear differential equations of order one.

[3] See Appendix A1 for description of rotations in number-plane.

[4] See Appendix A2 for description of translations in number-plane.

[5] The 'a' means a possible solution but from the statement of the problem we deduce defined behaviour of the mice (at each moment mouse 'knows what to do' which leads to a conclusion that a solution will be the unique one) i.e. we postulate existence of unique solution to this problem.

In order to solve the problem one may use the Laplace transform, which (among other applications) allows solving differential equations. Therefore the purpose of this essay is also to introduce the reader to the Laplace transform and show him or her how the use of this mathematical tool allows solving the mice problem.

III. The Laplace transform and its properties

Definition 1.

In this paper we will consider the Laplace transform that maps a real[6] function $f(t)$ of real variable t into a corresponding real function $\Phi(s)$ of real variable s defined by

$$\mathcal{L}\big(f(t)\big) = \Phi(s) \overset{\text{def}}{=} \int_0^\infty f(t)e^{-st}\,dt,$$

so it is an example of a integral transform; $\Phi(s)$ is appropriately defined if and only if the integral is convergent. When the integral is divergent and then there is no Laplace transform defined for a function. See [7].

The notation $\mathcal{L}\big(f(t)\big)$ is used to represent the Laplace transform of function $f(t)$, the resulting function being $\Phi(s)$. The domain of the resulting function's independent variable s is chosen appropriately so that the integral of Laplace transform is convergent (if possible). The usual use of t as an independent variable of function $f(t)$ is due to the transform's roots from physics – t denoting time. See [8].

Example 1.

Let's consider a very simple example of Laplace transform of $f(t) = 1$:

$$\mathcal{L}\big(f(t)\big) = \mathcal{L}(1) = \Phi(s) \overset{\text{def}}{=} \int_0^\infty 1 \cdot e^{-st}dt = \lim_{\tau\to\infty} \int_0^\tau e^{-st}dt\ .$$

Now let's consider following cases.

i) If $s = 0$

$$\lim_{\tau\to\infty} \int_0^\tau e^{-st}dt\ = \lim_{\tau\to\infty} \int_0^\tau 1dt\ = \infty\,.$$

So, for $s = 0$ integral is divergent and thus the Laplace transform is undefined.

ii) If $s \neq 0$

$$\lim_{\tau\to\infty} \int_0^\tau e^{-st}dt = \lim_{\tau\to\infty} \left[-\tfrac{1}{s} \cdot e^{-st}\right]\Big|_0^\tau =$$

$$= -\tfrac{1}{s} \cdot \lim_{\tau\to\infty} [e^{-s\tau} - e^0] = \begin{cases} -\tfrac{1}{s} \cdot (e^\infty - 1) = \infty, for\ \ s < 0 \\ -\tfrac{1}{s} \cdot (e^{-\infty} - 1) = \dfrac{1}{s}, for\ \ s > 0\ . \end{cases}$$

[6] We will focus on real functions, however, in general, the Laplace transform maps functions of complex variables.

So, for $s < 0$ the integral defining Laplace transform is divergent, so the transform is undefined there; for $s > 0$ Laplace transform of $f(t) = 1$ exists and is equal to $\frac{1}{s}$. Hence, Laplace transform of $f(t) = 1$ is well defined for $s\epsilon(0, \infty)$: $\Phi(s) = \frac{1}{s}$.

Example 2.

Let's consider Laplace transform of a $f(t) = e^{t^3}$.

We begin with

$$\mathcal{L}(e^{t^3}) = \int_0^\infty e^{t^3} \cdot e^{-st}dt = \int_0^\infty e^{t^3-st}dt = \infty\,,$$

as $(t^3 - st) \to \infty$ when $t \to \infty$ for any real s,

so Laplace transform of $f(t) = e^{t^3}$ is undefined.

The two examples presented show that Laplace transform is not always (not for any $f(t)$) well defined. That implies a need of specifying some restrictions to $f(t)$ to guarantee existence of its Laplace transform. An example of such a set of restrictions is:

f(t) together with it's $f'(t)$ is continuous for $t\ \epsilon\ [0, \infty)$; (O1)

f(t) $= 0$ for $t\ \epsilon\ (-\infty, 0)$; (O2)

f(t) is a function of exponential order of s_0 i.e.:

$$\exists s_0 \geq 0, \exists M > 0, \forall t \in [0, +\infty)\ \ |f(t)| < Me^{s_0 t} \qquad (O3)$$

Function $f(t)$ of real variable t is called *original* when it satisfies the above conditions. See [2].

In Example 1 we considered $f(t) = 1$ which satisfies (O1); (O2)- if truncated appropriately and (O3):

$$|f(t)| = 1 < Me^{s_0 t}\,,\text{where } M = 2, s_0 = 0\,,\text{for } t\ \epsilon[0, +\infty)$$

and its transform was well defined for $s > 0 = s_0$.

The function considered in Example 2 does not satisfy (O3) as t^3 exceeds $s_0 t$ for large t, for any given s_0 if t is large enough.

Conditions (O1), (O2) and (O3) sufficiency is guaranteed by the following theorem.

Theorem 1 (Convergence of the Laplace transform of originals).

If $f(t)$ satisfies (O1), (O2) and (O3) then $\mathcal{L}(f(t)) = \Phi(s)$ is well defined (the improper integral defining the transform is convergent) for $s > s_0$.

Proof of Theorem 1.

Let's consider

$$\int_0^\infty |f(t)| \cdot e^{-st}\, dt = \int_0^\infty |f(t) \cdot e^{-st}|\, dt .$$

Taking $a(t) = |f(t) \cdot e^{-st}|$ and using (O3)we have

$$0 \le a(t) = |f(t) \cdot e^{-st}| = |f(t)| \cdot e^{-st} \le Me^{s_0 t} \cdot e^{-st} = Me^{t(s_0-s)} = b(t).$$

Moreover, for $s_0 < s$, we have $s_0 - s < 0$ and we obtain

$$\int_0^\infty b(t)dt = \int_0^\infty Me^{t(s_0-s)}dt = \lim_{\tau\to\infty} \int_0^\tau Me^{t(s_0-s)}dt = \lim_{\tau\to\infty} \frac{M}{s_0-s}\left[e^{t(s_0-s)}\right]\Big|_0^\tau =$$

$$= \frac{M}{s_0-s} \cdot \lim_{\tau\to\infty}\left[e^{\tau(s_0-s)} - e^0\right] = \frac{M}{s_0-s}[e^{-\infty} - 1] = \frac{M}{s-s_0},$$

so it is a convergent integral for $s_0 < s$.

Hence by Comparison Test for improper integrals

$$\int_0^\infty |f(t) \cdot e^{-st}| dt \text{ is convergent,}$$

thus

$$\int_0^\infty f(t) \cdot e^{-st}\, dt \text{ is absolutely convergent ,}$$

so it is also convergent from Absolute Convergence Theorem. ■

Example 3.

Let's consider one more simple example to demonstrate Theorem 1 in action: $f(t) = \begin{cases} 0, & t < 0 \\ e^{4t}, & t \geq 0 \end{cases}$, truncated version of $f(t) = e^{4t}$. This function satisfies (O1), (O2) and (O3) with e.g.: $M = 2, s_0 = 4$, so $\Phi(s)$ is well defined for $s > s_0 = 4$.

Indeed,

$$\mathcal{L}(e^{4t}) = \int_0^{\infty} e^{4t} \cdot e^{-st} dt = \lim_{\tau \to \infty} \int_0^{\tau} e^{t(4\text{-}s)} dt = \lim_{\tau \to \infty} \left[e^{t(4\text{-}s)} \cdot \frac{1}{4\text{-}s} \right] \Big|_0^{\tau} =$$

$$= \frac{1}{4-s} \cdot \lim_{\tau \to \infty} \left[e^{\tau(4\text{-}s)} - e^{0} \right] = \frac{1}{4-s} \cdot \lim_{\tau \to \infty} \left(e^{\tau(4-s)} \right) + \frac{1}{s-4} = \frac{1}{4-s} \cdot 0 + \frac{1}{s-4} = \frac{1}{s-4},$$

for $s > 4$.

Example 4.

Let's consider a more complicated example $f(t) = sin(\beta t)$, where $\beta > 0$, truncated according to (O2). This function also satisfies (O1)-(O3) with e.g.: $M = 2, s_0 = 0$, so $\Phi(s)$ is well defined for $s > s_0 = 0$, although the derivation of its formula is quite complicated.

First, the indefinite integral $f(t) = sin(\beta t) \cdot \mathrm{e}^{-\mathrm{st}}$ will be calculated:

$$I = \int sin(\beta t) \cdot e^{-st} dt \left| \begin{matrix} u_1 = e^{-st} & v'_1 = sin\, \beta t \\ u'_1 = -se^{-st} & v_1 = -\frac{1}{\beta} cos\, \beta t \end{matrix} \right| =$$

$$= -e^{-st} \cdot \frac{1}{\beta} cos\, \beta t - s \cdot \frac{1}{\beta} \int e^{-st} \cdot cos\, \beta t\ \ dt \left| \begin{matrix} u_2 = e^{-st} & v'_2 = cos\, \beta t \\ u'_2 = -se^{-st} & v_2 = \frac{1}{\beta} sin\, \beta t \end{matrix} \right| =$$

$$= -e^{-st} \cdot \frac{1}{\beta} cos\, \beta t - \frac{s}{\beta} \left(e^{-st} \cdot \frac{1}{\beta} sin\, \beta t + \frac{s}{\beta} \int e^{-st} \cdot sin\, \beta t\ \ dt \right) =$$

$$= -e^{-st} \cdot \frac{1}{\beta} cos\, \beta t - \frac{s}{\beta^2} \cdot e^{-st} \cdot sin\, \beta t - \frac{s^2}{\beta^2} \int e^{-st} \cdot sin\, \beta t\ \ dt =$$

$$= -e^{-st} \left(\frac{1}{\beta} cos\, \beta t + \frac{s}{\beta^2} sin \beta t \right) - \frac{s^2}{\beta^2} \cdot I \quad ,$$

so

$$I \left(1 + \frac{s^2}{\beta^2} \right) = -e^{-st} \left(\frac{1}{\beta} cos\, \beta t + \frac{s}{\beta^2} sin\, \beta t \right) \ ,$$

which yields

$$I = \frac{-e^{-st} \left(\frac{1}{\beta} cos\, \beta t + \frac{s}{\beta^2} sin \beta t \right)}{\left(1 + \frac{s^2}{\beta^2} \right)},$$

and finally

$$\int sin(\beta t) \cdot e^{-st} dt = \frac{-e^{-st}(\beta \cdot cos\, \beta t + s \cdot sin \beta t)}{\beta^2 + s^2} + C \ .$$

To calculate $\mathcal{L}(t)$ we will use a 'trick':

$$\mathcal{L}(sin\beta t) = \int_0^\infty sin(\beta t)\cdot e^{-st}dt = \sum_{k=1}^{\infty} I_k \text{, where } I_k = \int_{(k-1)\cdot\frac{2\pi}{\beta}}^{k\cdot\frac{2\pi}{\beta}} sin(\beta t)\cdot e^{-st}dt\,.$$

We get

$$I_1 = \int_0^{\frac{2\pi}{\beta}} sin(\beta t)\cdot e^{-st}dt = \left[-\tfrac{1}{s^2+\beta^2}\cdot e^{-st}(\beta\cos\beta t + s\sin\beta t)\right]\Big|_0^{\frac{2\pi}{\beta}} =$$

$$= -\tfrac{1}{s^2+\beta^2}\left[e^{-\frac{2\pi s}{\beta}}\cdot\beta - e^0\cdot\beta\right] = \tfrac{\beta}{s^2+\beta^2}\left(e^0 - e^{-\frac{2\pi s}{\beta}}\right)\ ,$$

$$I_2 = \int_{\frac{2\pi}{\beta}}^{\frac{4\pi}{\beta}} sin(\beta t)\cdot e^{-st}dt = \left[-\tfrac{1}{s^2+\beta^2}\cdot e^{-st}(\beta\cos\beta t + s\sin\beta t)\right]\Big|_{\frac{2\pi}{\beta}}^{\frac{4\pi}{\beta}} =$$

$$= -\tfrac{1}{s^2+\beta^2}\left[e^{-\frac{4\pi s}{\beta}}\cdot\beta - e^{-\frac{2\pi s}{\beta}}\cdot\beta\right] = \tfrac{\beta}{s^2+\beta^2}\left(e^{-\frac{2\pi s}{\beta}} - e^{-\frac{4\pi s}{\beta}}\right),$$

and, generally

$$I_k = \int_{(k-1)\cdot\frac{2\pi}{\beta}}^{k\cdot\frac{2\pi}{\beta}} sin(\beta t)\cdot e^{-st}dt = \left[-\tfrac{1}{s^2+\beta^2}\cdot e^{-st}(\beta\cos\beta t + s\sin\beta t)\right]\Big|_{(k-1)\cdot\frac{2\pi}{\beta}}^{k\cdot\frac{2\pi}{\beta}} =$$

$$= -\tfrac{1}{s^2+\beta^2}\left[e^{-k\cdot\frac{2\pi}{\beta}s}\cdot\beta - e^{-(k-1)\cdot\frac{2\pi}{\beta}s}\cdot\beta\right] = \tfrac{\beta}{s^2+\beta^2}\left(e^{-\frac{2\pi s(k-1)}{\beta}} - e^{-\frac{2\pi sk}{\beta}}\right).$$

Having obtained general formula of I_k, we are able to evaluate sum of the following series:

$$\sum_{k=1}^{\infty} I_k = \lim_{n\to\infty}\sum_{k=1}^{n} I_k = \lim_{n\to\infty}(I_1 + I_2 + I_3 + \cdots + I_{n-1} + I_n) =$$

$$= \tfrac{\beta}{s^2+\beta^2}\cdot\lim_{n\to\infty}\left(\left(e^0 - e^{\frac{-2\pi s}{\beta}}\right) + \left(e^{\frac{-2\pi s}{\beta}} - e^{\frac{-4\pi s}{\beta}}\right) + \left(e^{\frac{-4\pi s}{\beta}} - e^{\frac{-6\pi s}{\beta}}\right) + \cdots\right.$$

$$\left. + \left(e^{\frac{-(n-1)2\pi s}{\beta}} - e^{\frac{-2\pi ns}{\beta}}\right)\right) =$$

$$= \tfrac{\beta}{s^2+\beta^2}\cdot\lim_{n\to\infty}\left(1 - e^{\frac{-2\pi ns}{\beta}}\right) = \tfrac{\beta}{s^2+\beta^2}(1-0) = \tfrac{\beta}{s^2+\beta^2}\,,$$

for $s > 0$.

Finally we arrive with

$$\mathcal{L}(sin\beta t) = \Phi(s) = \frac{\beta}{s^2+\beta^2}\ for\, s > s_0 = 0.$$

More results of a Laplace transforms of some basic functions are shown in a table below. See [2]

Function	Transform
1	$\frac{1}{s}$
t^n	$\frac{n!}{s^{n+1}}$
$e^{\alpha t}$	$\frac{1}{s-\alpha}$
$\sin \beta t$	$\frac{\beta}{s^2+\beta^2}$
$\cos \beta t$	$\frac{s}{s^2+\beta^2}$
$t^n e^{\alpha t}$	$\frac{n!}{(s-\alpha)^{n+1}}$
$e^{\alpha t} \sin \beta t$	$\frac{\beta}{(s-\alpha)^2+\beta^2}$
$e^{\alpha t} \cos \beta t$	$\frac{s-\alpha}{(s-\alpha)^2+\beta^2}$
$\sqrt{t}$	$\frac{\sqrt{\pi}}{2s^{\frac{2}{3}}}$

Table 1. Functions and their Laplace transforms.

Using the above table one can easily transform functions as it is presented in the next example.

Example 5.

Let's consider

$$\mathcal{L}\big(f(t)\big) = \mathcal{L}\big(e^{5t} \cos 6t\big) = \frac{s-5}{(s-5)^2+6^2},$$ as stated in the Table 1.

A direct consequence of the definition of the Laplace transform as integral is its linearity[7].

For $f(t)$ and $g(t)$ satisfying O1, O2 and O3, then:

$$\mathcal{L}(f(t)+g(t)) = \mathcal{L}\big(f(t)\big) + \mathcal{L}\big(g(t)\big) \qquad \textit{(additivity)}$$

$$\mathcal{L}(af(t)) = a \cdot \mathcal{L}\big(f(t)\big) \qquad \textit{(homogeneity)}$$

[7] Due to linearity of integrals.

Basing on linearity of the Laplace transform we can more effectively derive the Laplace transforms of linear combinations of basic functions. This is demonstrated in the following example.

Example 6.

Let's consider $f(t) = 2\,sin(3t) + 5e^{4t} + 7$, so

$$\mathcal{L}\big(f(t)\big) = \mathcal{L}(2\,sin(3t) + 5e^{4t} + 7) = 2\mathcal{L}(sin(3t)) + 5\mathcal{L}(e^{4t}) + 7\mathcal{L}(1) =$$

$$= 2\cdot\frac{3}{s^2+3^2} + 5\cdot\frac{1}{s-4} + 7\cdot\frac{1}{s} = \frac{6}{s^2+9} + \frac{5}{s-4} + \frac{7}{s}\,.$$

We end this chapter with a theorem, which plays a crucial role in applying Laplace transform techniques in solving differential equation. Theorem resolves the question of the uniqueness of Laplace transforms.

Theorem 2 (Uniqueness of Laplace transform).

Distinct continuous functions on $[0,\infty)$ have distinct Laplace transforms. See [6].

The result is known as Lerch's theorem, it means

$$\mathcal{L}\big(f(t)\big) = \mathcal{L}\big(g(t)\big) \Rightarrow f(t) = g(t)\,.$$

IV. The Laplace transform of derivatives.

The Laplace transform of derivative plays a key role, when it comes to differential equations of any kind. In order to proceed with Laplace transform of derivatives some crucial lemma is needed.

Lemma 1.

If $f(t)$ satisfies (O1), (O2) and (O3) and $s > s_0$ then $\lim_{t\to\infty}(f(t)e^{-st}) = 0$.

Proof of Lemma 1.

From (O3)

$$|f(t)| < Me^{s_0 t} ,$$

so

$$-Me^{s_0 t} < f(t) < Me^{s_0 t},$$

which gives

$$-Me^{s_0 t}e^{-st} < f(t)e^{-st} < Me^{s_0 t}e^{-st} ,$$

and

$$-Me^{t(s_0-s)} < f(t)e^{-st} < Me^{t(s_0-s)}.$$

We know that $s_0 - s < 0$ (since $s > s_0$) so, $\lim_{t\to\infty} t(s_0 - s) \to -\infty$.

Therefore $\lim_{t\to\infty} e^{t(s_0-s)} \to 0$, therefore both boundaries (lower and upper) converge:

$$\lim_{t\to\infty}\left(-Me^{t(s_0-s)}\right) = -M \cdot 0 = 0 ,$$

$$\lim_{t\to\infty}\left(Me^{t(s_0-s)}\right) = M \cdot 0 = 0 ,$$

so, by squeeze theorem, $\lim_{t\to\infty} f(t)e^{-st} = 0$.

Now let us use Lemma 1 to derive the Laplace transform of first-order derivative of an original.

Lemma 2 (Laplace transform of first derivative).

Let $f(t)$ be differentiable and satisfy (O1), (O2) and (O3) and $f'(t)$ exist, then

$$\mathcal{L}\left(f'(t)\right) = s\mathcal{L}\left(f\ (t)\right) - f(0) .$$

Proof of Lemma 2.

$$\mathcal{L}\big(f'(t)\big) = \int_0^\infty f'(t)e^{-st}dt = \lim_{\tau\to\infty}\int_0^\tau f'(t)e^{-st}dt \begin{vmatrix} u' = f'(t) & v = e^{-st} \\ u = f\,(t) & v' = -se^{-st} \end{vmatrix} =$$

$$= \lim_{\tau\to\infty}\left([f\,(t)e^{-st}]|_0^\tau + s\int_0^\tau f(t)e^{-st}dt\right) =$$

$$= \lim_{\tau\to\infty}\left([f(\tau)e^{-s\tau} - f(0)\cdot e^0]\; + s\int_0^\tau f(t)e^{-st}dt\right) =$$

$$= 0 - f(0)t + s\int_0^\infty f(t)e^{-st}dt = \; s\mathcal{L}\big(f\,(t)\big) - f(0)\,.$$

Example 7.

Let $y = f(t)$ and $\mathcal{L}\big(f(t)\big) = \; Y(s)$, then

$$\mathcal{L}(3f'(t) + f(t) - 4t) =$$

$$= 3\mathcal{L}\big(f'(t)\big) + \;\mathcal{L}\big(f(t)\big) - 4\mathcal{L}(t) =$$

$$= 3s\,Y(s) - f(0) - \;Y(s) - 4\cdot\tfrac{1}{s^2} =$$

$$= (3s-1)\,Y(s) - f(0) - \tfrac{4}{s^2}\,.$$

Having calculated the Laplace transform of the first derivative we may use it to derive Laplace transform of the second derivative.

Lemma 3 (Laplace transform of second derivative).

Let $f(t)$ satisfy (O1), (O2) and (O3) and $f(t)$ be differentiable twice, then

$$\mathcal{L}\big(f''(t)\big) = s^2\mathcal{L}\big(f(t)\big) - sf(0) - f'(0)\,.$$

Proof of Lemma 3.

By Lemma 2 we get

$$\mathcal{L}\big(f''(t)\big) = \;\mathcal{L}\left(\big(f'(t)\big)'\right) = s\mathcal{L}\big(f'(t)\big) - f'(0) =$$

$$s\left(s\mathcal{L}\big(f(t)\big) - f(0)\right) - f'(0) = s^2\mathcal{L}\big(f(t)\big) - sf(0) - f'(0)\,.$$

Lemmas 2 and 3 reveal a pattern and one could conjecture general formula for the Laplace transform of n^{th}-order derivative that we will prove by means of mathematical induction.

Theorem 3 (Laplace transform of the nth-order derivative).

If $f(t)$ is differentiable n times and satisfies (O1), (O2) and (O3), then

$$\mathcal{L}\left(f^{(n)}(t)\right) = s^n\mathcal{L}\big(f(t)\big) - \sum_{i=1}^{i=n} s^{n-k} f^{(k-1)}(0)\,. \tag{3}$$

Proof of Theorem 3.

Mathematical induction over n will be used. Let's introduce proposition $P(n)$

$$P(n):\quad \mathcal{L}\left(f^{(n)}(t)\right) = s^n\mathcal{L}\big(f(t)\big) - \sum_{i=1}^{i=n} s^{n-i} f^{(i-1)}(0)\,.$$

I. For $n = 1$

$$LHS_{P(1)} = \mathcal{L}\left(f^{(1)}(t)\right) = \mathcal{L}\big(f'(t)\big) = s\mathcal{L}\big(f\ (t)\big) - f(0) = RHS_{P(1)}\,,$$

by Lemma 3 $P(1)$ is true.

II. We will prove that for any given $k\epsilon Z^+$

$$P(k):\qquad \mathcal{L}\left(f^{(k)}(t)\right) = s^k\mathcal{L}\big(f(t)\big) - \sum_{i=1}^{i=k} s^{k-i} f^{(i-1)}(0)\,,$$

$$\Downarrow$$

$$P(k+1):\qquad \mathcal{L}\left(f^{(k+1)}(t)\right) = s^{k+1}\mathcal{L}\big(f(t)\big) - \sum_{i=1}^{i=k+1} s^{k+1-i} f^{(i-1)}(0)\ .$$

Proof of II:

$$LHS_{P(k+1)} = \mathcal{L}\left(f^{(k+1)}(t)\right) = \mathcal{L}\big([f'(t)]^{(k)}\big) =$$

$$= s^k\mathcal{L}\big(f'(t)\big) - \sum_{i=1}^{i=k} s^{k-i}\,(f')^{(i-1)}(0) =$$

$$= s^k\left(s\mathcal{L}\big(f(t)\big) - f(0)\right) - \sum_{i=1}^{i=k} s^{k-i} f^{(i)}(0) =$$

$$= s^{k+1}\mathcal{L}\big(f(t)\big) - s^k f(0) - \sum_{i=1}^{i=k} s^{k-i} f^{(i)}(0) =$$

$$= s^{k+1}\mathcal{L}\big(f(t)\big) - \sum_{i=0}^{i=k} s^{k-i} f^{(i)}(0) = \qquad |\text{changing counter } i = j-1|$$

$$= s^{k+1}\mathcal{L}\big(f(t)\big) - \sum_{j=1}^{j=k+1} s^{k-(j-1)} f^{(j-1)}(0) = RHS_{P(k+1)}$$

so $P(k)$ implies $P(k+1)$ is true .

From I, II and Mathematical Induction Principle $P(n)$ is true for any given $k\epsilon Z^+$. ■

V. The Laplace transform and its application to solving linear differential equations.

Theorem 2 and Theorem 3 open up the possibility to deal with the Laplace transform as a tool for solving linear differential equations.

General form of a linear differential equation of the n-th order is

$$f^{(n)}(t) + p_{n-1}(t) \cdot f^{(n-1)} + \cdots + p_2(t) \cdot f''(t) + p_1(t) \cdot f'(t) + p_0(t) \cdot f(t) = h(t),$$

where $p_k(t), h(t)$ are functions of t. See [2].

An example of differential equation is

$$f'''(t) + \sin t\, f''(t) - \sqrt{t} f'(t) + 2t f(t) + \ 3 = t^4 + 2t^2\ ,$$

which is an example of a linear differential equation of third order.

Example 9.

This example demonstrates how the Laplace transform is used to solve differential equations.

For instance, let's consider the following equation

$$f'(t) - 2f(t) = e^{2t} \cos t\ \ ,\ \text{knowing that } f(0) = 0.$$

As Laplace transforms of equal functions are equal

$$\mathcal{L}\big(f'(t) - 2f(t)\big) = \mathcal{L}(e^{2t} \cos t).$$

From linearity we get

$$\mathcal{L}\big(f'(t)\big) - \ 2\mathcal{L}\big(f(t)\big) = \mathcal{L}(e^{2t} \cos t).$$

From Table 1

$$\mathcal{L}\big(f'(t)\big) - \ 2\mathcal{L}\big(f(t)\big) = \frac{s-2}{(s-2)^2+1^2}\,.$$

Now, by means of Theorem 3

$$s\Phi(s) - f(0) - 2\Phi(s) = \frac{s-2}{(s-2)^2+1^2}\,.$$

We know that $\mathrm{f}(0) = 0$, so

$$\Phi(s)(s-2) = \frac{s-2}{(s-2)^2+1^2}\,,$$

hence

$$\Phi(s) = \frac{1}{(s-2)^2+1^2}\,,$$

and again from Table 1

$$\mathcal{L}(f(t)) = \mathcal{L}(e^{2t}\cos t)\,,$$

by Theorem 2 we get

$$f(t) = e^{2t}\cos t\;\;.$$

The previously mentioned Theorem 2 guarantees that the solution is unique on $[0, \infty)$.

Example 10

Let's consider following equation

$$f''(t) - 2f'(t) + f(t) = 1 + t,$$

with initial conditions $f(0) = 0$, and $f'(0) = 0$.

We have

$$\mathcal{L}\big(f''(t) - 2f'(t) + f(t)\big) = \mathcal{L}(1 + t\,),$$

so

$$\mathcal{L}(f''(t)) - 2\mathcal{L}(f'(t)) + \mathcal{L}(f(t)) = \mathcal{L}(1) + \mathcal{L}(t),$$

then

$$\mathcal{L}(f''(t)) - 2\mathcal{L}(f'(t)) + \mathcal{L}(f(t)) = \frac{1}{s} + \frac{1}{s^2}\,,$$

and we get

$$s^2\mathcal{L}(f(t)) - f'(0) - sf(0) - 2s\mathcal{L}(f(t)) + 2f(0) + \mathcal{L}(f(t)) = \frac{1}{s} + \frac{1}{s^2}.$$

We know that $f(0) = 0$, and $f'(0) = 0$, so

$$s^2\mathcal{L}(f(t)) - 2s\mathcal{L}(f(t)) + \mathcal{L}(f(t)) = \frac{1}{s} + \frac{1}{s^2}\,,$$

then

$$\mathcal{L}(f(t)) \cdot [s^2 - 2s + 1] = \frac{s+1}{s^2}\,,$$

which yields

$$\mathcal{L}(f(t)) = \frac{s+1}{s^2\cdot(s^2-2s+1)}\,.$$

The result needs to be simplified into partial fractions in order to identify the original functions. So

$$\frac{s+1}{s^2\cdot(s^2-2s+1)} = \frac{s+1}{s^2\cdot(s-1)^2} = \frac{A}{s} + \frac{B}{s^2} + \frac{C}{s-1} + \frac{D}{(s-1)^2} = \frac{As+B}{s^2} + \frac{Cs+D-C}{(s-1)^2} =$$

$$= \frac{(As+B)\cdot(s-1)^2 + (Cs+D-C)\cdot s^2}{s^2\cdot(s-1)^2} = \frac{As^3 - 2As^2 + As + Bs^2 - 2Bs + B + Cs^3 - Cs^2 + Ds^2}{s^2\cdot(s-1)^2} =$$

$$= \frac{(A+C)\cdot s^3 + (-2A+B-C+D)s^2 + (A-2B)s + B}{s^2\cdot(s-1)^2}\;\;,$$

which leads to

$$\begin{cases} A + C = 0\ , \\ -2A + B - C + D = 0\ , \\ A - 2B = 1\ , \\ B = 1\ \ . \end{cases}$$

Solving the above we get

A=3, B=1, C=-3, D=2.

Therefore the fraction can be written as follows

$$\frac{s+1}{s^2\ \cdot (s^2 - 2s + 1)} = \frac{3}{s} + \frac{1}{s^2} - \frac{3}{s-1} + \frac{2}{(s-1)^2}\ ,$$

hence

$$\mathcal{L}(f(t)) = \frac{s+1}{s^2\ \cdot (s^2-2s+1)} = \mathcal{L}\left(\frac{3}{s} + \frac{1}{s^2} - \frac{3}{s-1} + \frac{2}{(s-1)^2}\right) =$$

$$= 3\mathcal{L}\left(\frac{1}{s}\right) + \mathcal{L}\left(\frac{1}{s^2}\right) - 3\mathcal{L}\left(\frac{1}{s-1}\right) + 2\mathcal{L}\left(\frac{1}{(s-1)^2}\right) =$$

$$= \mathcal{L}(3 + t - 3e^t + 2te^t)\ ,$$

so, by Theorem 2

$$f(t) = 3 + t - 3e^t + 2te^t.$$

VI. Laplace transform and mice problem

Getting back to the concept of pursuit curve and three-mouse problem, now, thanks to Laplace transform we may solve the system of differential equations and derive formulas depicting position and movement of mice on a xy-plane in time. See Figure 1.

Getting back to (2)

$$\begin{cases} \dfrac{dy_1}{dt} = \frac{\sqrt{3}}{2}x_1(t) - \frac{3}{2}y_1(t), \\ \dfrac{dx_1}{dt} = 1 - \frac{3}{2}x_1(t) - \frac{\sqrt{3}}{2}y_1(t)\ . \end{cases}$$

These conditions reflect instantaneous checking 'targets' positions and adjusting mice's own positions instantly while running.

For simplicity and legibility of notation let's put $A = \frac{\sqrt{3}}{2},\ B = \frac{-3}{2}$ and $x(t), y(t)$ for $x_1(t), y_1(t)$. This gives

$$\begin{cases} \dfrac{dy}{dt} = Ax(t) + By(t), & (3) \\ \dfrac{dx}{dt} = 1 + Bx(t) - Ay(t), & (4) \end{cases}$$

with initial conditions $y(0) = 0$ and $x(0) = 0$.

After taking the Laplace transform of both sides of the equations we arrive with

$$sY(s) = A \cdot \mathcal{L}\big(x(t)\big) + B \cdot \mathcal{L}\big(y(t)\big) = AX(s) + BY(s)\ ,$$

$$sX(s) = 1 \cdot \mathcal{L}(1) + B \cdot \mathcal{L}\big(x(t)\big) - A \cdot \mathcal{L}\big(y(t)\big) = \frac{1}{s} + BX(s) - AY(s)\ .$$

Also since they are interrelated they create a system of equations:

$$\begin{cases} X(s) \cdot (-A) + Y(s) \cdot (s - B) = 0\ , \\ X(s) \cdot (s - B) + Y(s) \cdot (A) = \frac{1}{s}, \end{cases}$$

and also

$$\begin{cases} X(s) \cdot (-As + AB) + Y(s) \cdot (s - B)^2 = 0\ , \\ X(s) \cdot (-As + AB) + Y(s) \cdot (-A^2) = -\frac{A}{s}. \end{cases}$$

Adding the equations eliminates $X(s)$ and we get

$$Y(s) \cdot (s - B)^2 + Y(s) \cdot (A^2) = \frac{A}{s}\ ,$$

which gives

$$Y(s) = \frac{A}{s(A^2 + (s-B)^2)} \ .$$

Using $X(s) \cdot (-A) + Y(s) \cdot (s-B) = 0$ and the above solution we can derive

$$X(s) = \frac{Y(s) \cdot (s-B)}{A} = \frac{s-B}{s \cdot (A^2 + (s-B)^2)} \ .$$

Now

$$\begin{cases} X(s) = \frac{s-B}{s \cdot (A^2+(s-B)^2)} \ , \\ Y(s) = \frac{A}{s(A^2+(s-B)^2)} \ . \end{cases}$$

Partial fractions

$$X(s) = \frac{s-B}{s \cdot (A^2 + (s-B)^2)} = \frac{P}{s} + \frac{Qs+R}{(s-B)^2 + A^2} =$$

$$= \frac{P \cdot ((s-B)^2 + A^2) + Qs^2 + Rs}{s \cdot (A^2 + (s-B)^2)} =$$

$$= \frac{(P+Q)s^2 + (R-2BP)s + (PB^2 + PA^2)}{s \cdot (A^2 + (s-B)^2)} \ .$$

It leads to

$$\begin{cases} P + Q = 0 \ , \\ R - 2BP = 1 \ , \\ P(A^2 + B^2) = -B \ , \end{cases}$$

and

$$\begin{cases} Q = -P = \dfrac{B}{AB} \ , \\ R = 1 - 2BP = \dfrac{(A^2 - B^2)}{A^2 + B^2} \ , \\ P = \dfrac{-B}{A^2 + B^2} \ . \end{cases}$$

Hence

$$\boldsymbol{X(s)} = \frac{-B}{A^2+B^2} \cdot \frac{1}{s} + \frac{B}{A^2+B^2} \cdot \frac{s}{(s-B)^2+A^2} + \frac{(A^2-B^2)}{A^2+B^2} \cdot \frac{1}{(s-B)^2+A^2} =$$

$$= \frac{-B}{A^2+B^2} \cdot \frac{1}{s} + \frac{B}{A^2+B^2} \cdot \frac{s-B}{(s-B)^2+A^2} + \frac{-B}{A^2+B^2} \cdot \frac{-B}{(s-B)^2+A^2} + \frac{(A^2-B^2)}{A^2+B^2} \cdot \frac{1}{(s-B)^2+A^2} =$$

$$= \frac{-B}{A^2+B^2} \cdot \frac{1}{s} + \frac{B}{A^2+B^2} \cdot \frac{s-B}{(s-B)^2+A^2} + \frac{-B}{A^2+B^2} \cdot \frac{-B}{(s-B)^2+A^2} + \frac{(A^2-B^2)}{A^2+B^2} \cdot \frac{1}{(s+B)^2+A^2} =$$

$$= \frac{-B}{A^2+B^2} \cdot \mathcal{L}(1) + \frac{B}{A^2+B^2} \cdot \mathcal{L}(e^{Bt} \cos At) + \left(\frac{B^2}{A^2+B^2} + \frac{(A^2-B^2)}{A^2+B^2} \right) \cdot \frac{1}{(s-B)^2+A^2} =$$

$$= \frac{-B}{A^2+B^2} \cdot \mathcal{L}(1) + \frac{B}{A^2+B^2} \cdot \mathcal{L}(e^{Bt} \cos At) + \frac{A}{A^2+B^2} \cdot \mathcal{L}(e^{Bt} \sin At) =$$

$$= \mathcal{L}\left(\frac{-B}{A^2+B^2} \cdot 1 + \frac{B}{A^2+B^2} \cdot e^{Bt} \cos At + \frac{A}{A^2+B^2} \cdot e^{Bt} \sin At\right),$$

so

$$x(t) = \frac{-B}{A^2+B^2} \cdot 1 + \frac{B}{A^2+B^2} \cdot e^{Bt} \cos At + \frac{A}{A^2+B^2} \cdot e^{Bt} \sin At \ . \qquad (5)$$

For given A and B

$$x(t) = -\tfrac{1}{2} e^{-\frac{3}{2}t} \cos \frac{\sqrt{3}}{2} t \ + \tfrac{\sqrt{3}}{6} e^{-\frac{3}{2}t} \sin \frac{\sqrt{3}}{2} t + \tfrac{1}{2}.$$

The solution for $y(t)$ may be derived from the previous equation

$$\frac{dx}{dt} = 1 + Bx(t) - Ay(t),$$

which is equivalent to

$$y(t) = \frac{1 + Bx(t) - \frac{dx}{dt}}{A}.$$

Derivative of $x(t)$ equals as follows

$$\frac{dx}{dt} = e^{Bt}(\cos At + \sin At) \ ,$$

hence

$$y(t) = \frac{1 + B \cdot \left(\frac{-B}{A^2+B^2} \cdot 1 + \frac{B}{A^2+B^2} \cdot e^{Bt} \cos At + \frac{A}{A^2+B^2} \cdot e^{Bt} \sin At\right) - e^{Bt}(\cos At + \sin At)}{A} =$$

$$= \frac{1 + \frac{-B^2}{A^2+B^2} \cdot 1 + \frac{B^2}{A^2+B^2} \cdot e^{Bt} \cos At + \frac{AB}{A^2+B^2} \cdot e^{Bt} \sin At - e^{Bt}(\cos At + \sin At)}{A} =$$

$$= \frac{\frac{A^2+B^2}{A^2+B^2} + \frac{-B^2}{A^2+B^2} \cdot 1 + \left(\frac{B^2}{A^2+B^2} - \frac{A^2+B^2}{A^2+B^2}\right) \cdot e^{Bt} \cos At + \left(\frac{AB}{A^2+B^2} - \frac{A^2+B^2}{A^2+B^2}\right) \cdot e^{Bt} \sin At}{A} \ ,$$

so

$$y(t) = \frac{A}{A^2+B^2} \cdot 1 - \frac{A}{A^2+B^2} \cdot e^{Bt} \cos At - \frac{B}{A^2+B^2} \cdot e^{Bt} \sin At. \qquad (6)$$

For given A and B

$$y(t) = -\tfrac{\sqrt{3}}{6} e^{-\frac{3}{2}t} \cos \tfrac{\sqrt{3}}{2} t - \tfrac{1}{2} e^{-\frac{3}{2}t} \sin \tfrac{\sqrt{3}}{2} t + \tfrac{\sqrt{3}}{6}.$$

The trajectory of the movement of the first mouse starting from $P_1(0,0)$ of a equilateral triangle with a side length 1 at time $t \geq 0$ is determined by the given formula

$$M_1(t)=\big(x(t),y(t)\big)=\left(\frac{1}{2}+e^{-\frac{3}{2}t}\left(-\frac{1}{2}cos\frac{\sqrt{3}}{2}t+\frac{\sqrt{3}}{6}sin\frac{\sqrt{3}}{2}t\right),\frac{\sqrt{3}}{6}+e^{-\frac{3}{2}t}\left(-\frac{\sqrt{3}}{6}cos\frac{\sqrt{3}}{2}t-\frac{1}{2}sin\frac{\sqrt{3}}{2}t\right)\right).$$

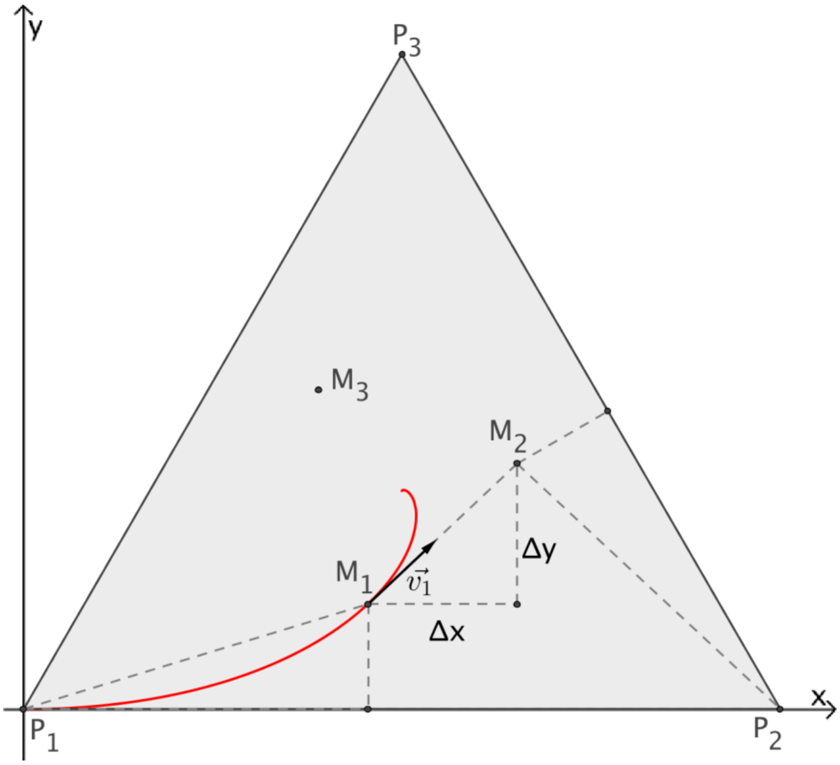


Figure 2 – The trajectory of movement of mouse no. 1.

The trajectory of movement of the other two mice can be derived by rotating the trajectory of movement of first mouse (x_1, y_1) as follows

$M_1(t)$ rotated $\frac{2\pi}{3}$ around (0,0) and translated by $\overrightarrow{P_1P_2}$ gives $M_2(t)$,

$M_1(t)$ rotated $2\cdot\frac{2\pi}{3}$ around (0,0) and translated by $\overrightarrow{P_1P_3}$ gives $M_3(t)$,

this leads to

$$\overrightarrow{OM_2}=\begin{pmatrix}x_2\\y_2\end{pmatrix}=\begin{pmatrix}cos\frac{2}{3}\pi & -sin\frac{2}{3}\pi\\ sin\frac{2}{3}\pi & cos\frac{2}{3}\pi\end{pmatrix}\begin{pmatrix}x_1\\y_1\end{pmatrix}+\begin{pmatrix}1\\0\end{pmatrix}=$$

$$=\begin{pmatrix}-\frac{1}{2} & -\frac{\sqrt{3}}{2}\\ \frac{\sqrt{3}}{2} & -\frac{1}{2}\end{pmatrix}\begin{pmatrix}\frac{1}{2}+e^{-\frac{3}{2}t}\left(-\frac{1}{2}cos\frac{\sqrt{3}}{2}t+\frac{\sqrt{3}}{6}sin\frac{\sqrt{3}}{2}t\right)\\ \frac{\sqrt{3}}{6}+e^{-\frac{3}{2}t}\left(-\frac{\sqrt{3}}{6}cos\frac{\sqrt{3}}{2}t-\frac{1}{2}sin\frac{\sqrt{3}}{2}t\right)\end{pmatrix}+\begin{pmatrix}1\\0\end{pmatrix}=$$

$$=\begin{pmatrix}\frac{1}{2}+e^{-\frac{3}{2}t}\left(\frac{1}{2}cos\frac{\sqrt{3}}{2}t+\frac{\sqrt{3}}{6}sin\frac{\sqrt{3}}{2}t\right)\\ \frac{\sqrt{3}}{6}+e^{-\frac{3}{2}t}\left(-\frac{\sqrt{3}}{6}cos\frac{\sqrt{3}}{2}t+\frac{1}{2}sin\frac{\sqrt{3}}{2}t\right)\end{pmatrix}.$$

We get

$$M_2(t)=\big(x(t),y(t)\big)=\left(\frac{1}{2}+e^{-\frac{3}{2}t}\left(\frac{1}{2}cos\frac{\sqrt{3}}{2}t+\frac{\sqrt{3}}{6}sin\frac{\sqrt{3}}{2}t\right),\frac{\sqrt{3}}{6}+e^{-\frac{3}{2}t}\left(-\frac{\sqrt{3}}{6}cos\frac{\sqrt{3}}{2}t+\frac{1}{2}sin\frac{\sqrt{3}}{2}t\right)\right).$$

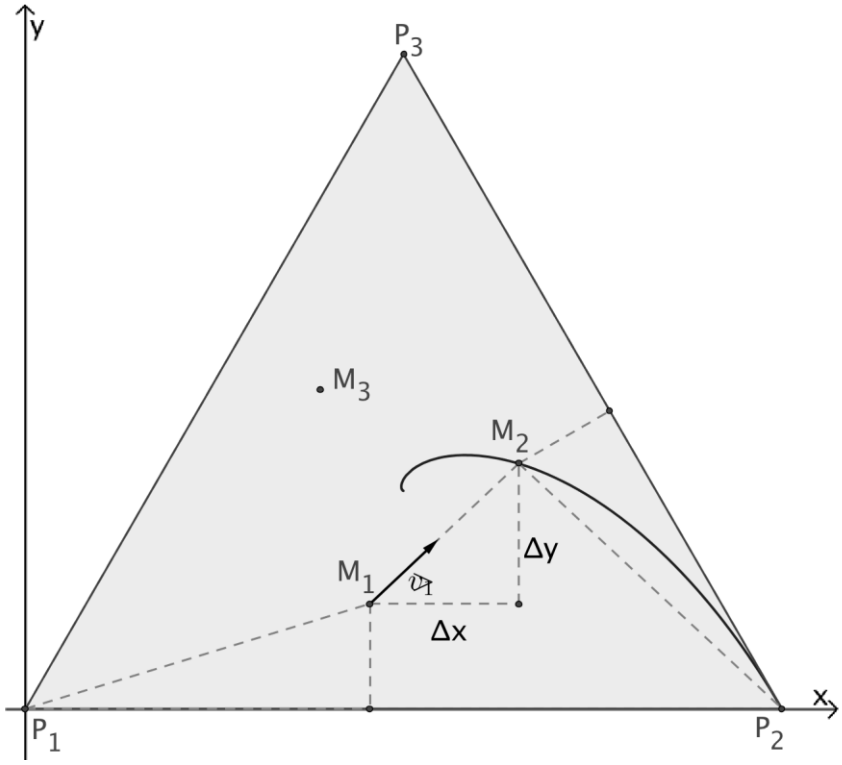


Figure 3 – The trajectory of movement of mouse no. 2.

And the third mouse M_3

$$\begin{pmatrix} x_3 \\ y_3 \end{pmatrix} = \begin{pmatrix} x_2 \\ y_2 \end{pmatrix} \begin{pmatrix} cos\,\frac{2}{3}\pi & -\,sin\,\frac{2}{3}\pi \\ sin\,\frac{2}{3}\pi & cos\,\frac{2}{3}\pi \end{pmatrix} + \begin{pmatrix} 1 \\ 0 \end{pmatrix} =$$

$$= \begin{pmatrix} \frac{1}{2} + e^{-\frac{3}{2}t}\left(-\frac{1}{2}cos\,\frac{\sqrt{3}}{2}t + \frac{\sqrt{3}}{6}sin\,\frac{\sqrt{3}}{2}t\right) \\ \frac{\sqrt{3}}{6} + e^{-\frac{3}{2}t}\left(-\frac{\sqrt{3}}{6}cos\,\frac{\sqrt{3}}{2}t - \frac{1}{2}sin\,\frac{\sqrt{3}}{2}t\right) \end{pmatrix} \begin{pmatrix} -\frac{1}{2} & -\frac{\sqrt{3}}{2} \\ \frac{\sqrt{3}}{2} & -\frac{1}{2} \end{pmatrix} + \begin{pmatrix} 1 \\ 0 \end{pmatrix} =$$

$$= \begin{pmatrix} \frac{1}{2} + e^{-\frac{3}{2}t}\left(-\frac{\sqrt{3}}{3}sin\,\frac{\sqrt{3}}{2}t\right) \\ \frac{\sqrt{3}}{6} + e^{-\frac{3}{2}t}\left(\frac{\sqrt{3}}{3}cos\,\frac{\sqrt{3}}{2}t\right) \end{pmatrix},$$

and finally

$$M_3(t) = \big(x(t), y(t)\big) = \left(\frac{1}{2} + e^{-\frac{3}{2}t}\left(-\frac{\sqrt{3}}{3}sin\,\frac{\sqrt{3}}{2}t\right), \frac{\sqrt{3}}{6} + e^{-\frac{3}{2}t}\left(\frac{\sqrt{3}}{3}cos\,\frac{\sqrt{3}}{2}t\right)\right).$$

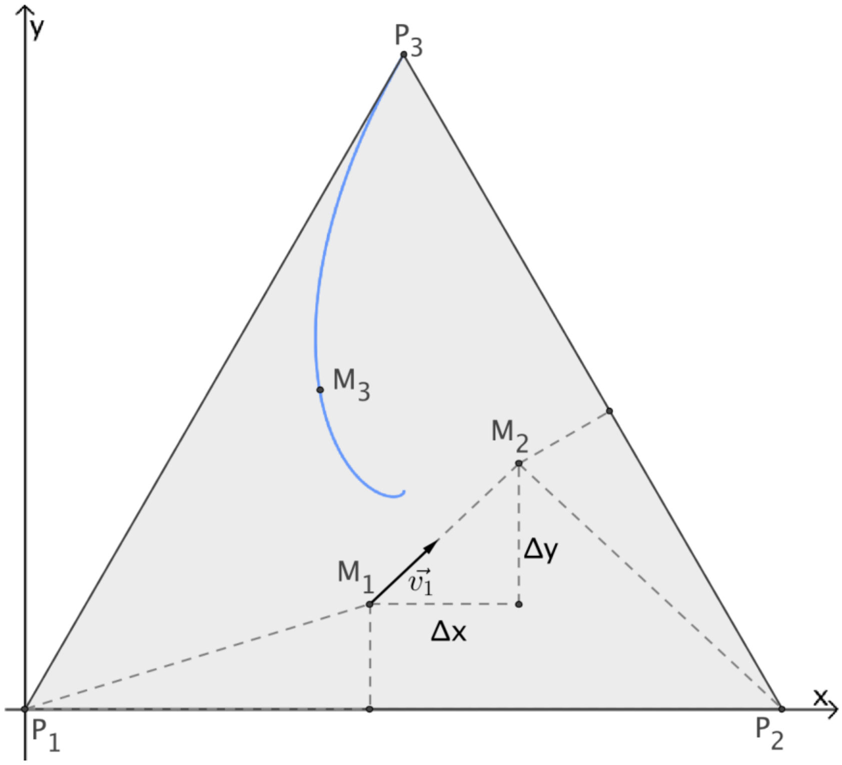


Figure 4 – The trajectory of movement of mouse no. 3.

We have derived the parametric formulas that determine position of each mice in time:

$$M_1(t) = \big(x(t), y(t)\big) = \left(\frac{1}{2} + e^{-\frac{3}{2}t}\left(-\frac{1}{2}cos\frac{\sqrt{3}}{2}t + \frac{\sqrt{3}}{6}sin\frac{\sqrt{3}}{2}t\right), \frac{\sqrt{3}}{6} + e^{-\frac{3}{2}t}\left(-\frac{\sqrt{3}}{6}cos\frac{\sqrt{3}}{2}t - \frac{1}{2}sin\frac{\sqrt{3}}{2}t\right)\right),$$

$$M_2(t) = \big(x(t), y(t)\big) = \left(\frac{1}{2} + e^{-\frac{3}{2}t}\left(\frac{1}{2}cos\frac{\sqrt{3}}{2}t + \frac{\sqrt{3}}{6}sin\frac{\sqrt{3}}{2}t\right), \frac{\sqrt{3}}{6} + e^{-\frac{3}{2}t}\left(-\frac{\sqrt{3}}{6}cos\frac{\sqrt{3}}{2}t + \frac{1}{2}sin\frac{\sqrt{3}}{2}t\right)\right),$$

$$M_3(t) = \big(x(t), y(t)\big) = \left(\frac{1}{2} + e^{-\frac{3}{2}t}\left(-\frac{\sqrt{3}}{3}sin\frac{\sqrt{3}}{2}t\right), \frac{\sqrt{3}}{6} + e^{-\frac{3}{2}t}\left(\frac{\sqrt{3}}{3}cos\frac{\sqrt{3}}{2}t\right)\right).$$

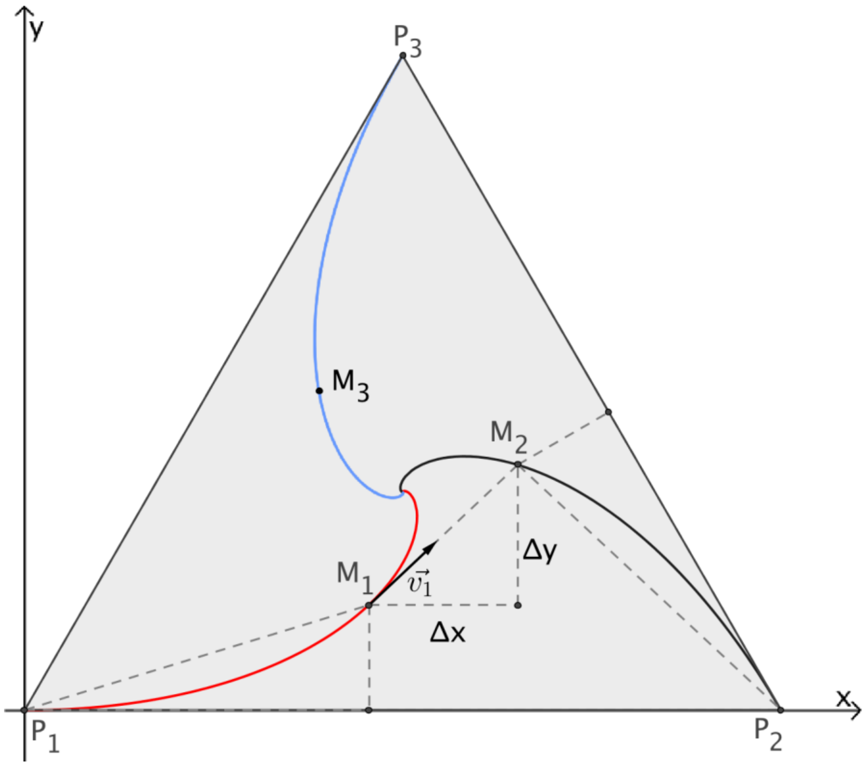


Figure 5 – The trajectories of all three mice.

To generalize solution, let's consider regular n-polygon $P_1P_2P_3 \dots P_n$ where $P_1(0,0)$ and $P_2(1,0)$ and let P be the centre of a circumscribed circle (see Appendix B). Using notation introduced previously we can observe that

$M_1(t)$ rotated $\frac{2\pi}{n}$ around $(0,0)$ and translated by $\overrightarrow{P_1P_2}$ gives $M_2(t)$,

which leads to

$$\overrightarrow{OM_2} = \begin{pmatrix} x_2 \\ y_2 \end{pmatrix} = \begin{pmatrix} cos\frac{2\pi}{n} & -sin\frac{2\pi}{n} \\ sin\frac{2\pi}{n} & cos\frac{2\pi}{n} \end{pmatrix} \begin{pmatrix} x_1 \\ y_1 \end{pmatrix} + \overrightarrow{P_1P_2} =$$

$$= \begin{pmatrix} x_1 cos\frac{2\pi}{n} - y_1 sin\frac{2\pi}{n} \\ x_1 sin\frac{2\pi}{n} + y_1 cos\frac{2\pi}{n} \end{pmatrix} + \begin{pmatrix} 1 \\ 0 \end{pmatrix} =$$

$$= \begin{pmatrix} x_1 cos\frac{2\pi}{n} - y_1 sin\frac{2\pi}{n} + 1 \\ x_1 sin\frac{2\pi}{n} + y_1 cos\frac{2\pi}{n} \end{pmatrix}.$$

so

$$M_2(t) = \left(x_1(t) cos\frac{2\pi}{n} - y_1(t) sin\frac{2\pi}{n} + 1,\ \ x_1(t) sin\frac{2\pi}{n} + y_1(t) cos\frac{2\pi}{n}\right).$$

Therefore

$$\frac{dy_1}{dx_1} = \frac{\Delta y_1}{\Delta x_1} = \frac{y_2(t) - y_1(t)}{x_2(t) - x_1(t)} = \frac{x_1(t) sin\frac{2\pi}{n} + y_1(t) cos\frac{2\pi}{n} - y_1(t)}{x_1(t) cos\frac{2\pi}{n} - y_1(t) sin\frac{2\pi}{n} + 1 - x_1(t)}.$$

Moreover we can notice that

$$\frac{dy_1}{dx_1} = \frac{\frac{dy_1}{dt}}{\frac{dx_1}{dt}} = \frac{x_1(t) sin\frac{2\pi}{n} + y_1(t)\left(cos\frac{2\pi}{n} - 1\right)}{x_1(t)\left(cos\frac{2\pi}{n} - 1\right) - y_1(t)\, sin\frac{2\pi}{n} + 1},$$

so, we get

$$\begin{cases} \frac{dy_1}{dt} = x_1(t) sin\frac{2\pi}{n} + y_1(t)\left(cos\frac{2\pi}{n} - 1\right), & (7) \\ \frac{dx_1}{dt} = x_1(t)\left(cos\frac{2\pi}{n} - 1\right) - y_1(t)\, sin\frac{2\pi}{n} + 1 . & (8) \end{cases}$$

For simplicity and legibility of notation let's put $A = sin\frac{2\pi}{n}$, $B = cos\frac{2\pi}{n} - 1$ and $x(t), y(t)$ for $x_1(t), y_1(t)$. This gives

$$\begin{cases} \frac{dy}{dt} = Ax(t) + By(t), \\ \frac{dx}{dt} = 1 + Bx(t) - Ay(t), \end{cases}$$

with initial conditions $y(0) = 0$ and $x(0) = 0$, which is the same as (3), (4). So we have (5) and (6) again:

$$x(t) = \frac{-B}{A^2 + B^2} \cdot 1 + \frac{B}{A^2 + B^2} \cdot e^{Bt} \cos At + \frac{A}{A^2 + B^2} \cdot e^{Bt} \sin At,$$

$$y(t) = \frac{A}{A^2 + B^2} \cdot 1 - \frac{A}{A^2 + B^2} \cdot e^{Bt}\, cos\, At - \frac{B}{A^2 + B^2} \cdot e^{Bt}\, sin\, At.$$

But for given A and B these simplify now to

$$x(t) = \frac{1}{2} + \frac{1}{2} e^{\left(-2\,sin^2\frac{\pi}{n}\right)t}\left(-cos\left(\left(sin\frac{2\pi}{n}\right)t\right) + cot\frac{\pi}{n} \cdot sin\left(\left(sin\frac{2\pi}{n}\right)t\right)\right),$$

$$y(t) = \frac{1}{2} cot\frac{\pi}{n} + \frac{1}{2} e^{\left(-2\,sin^2\frac{\pi}{n}\right)t}\left(-cot\frac{\pi}{n} \cdot cos\left(\left(sin\frac{2\pi}{n}\right)t\right) - sin\left(\left(sin\frac{2\pi}{n}\right)t\right)\right),$$

in a more compact form

$$x(t) = \frac{1}{2} + \frac{1}{2} e^{\left(-2\,sin^2\frac{\pi}{n}\right)t} csc\frac{\pi}{n}\, sin\left(t\, sin\frac{2\pi}{n} - \frac{\pi}{n}\right),$$

$$y(t) = \frac{1}{2} cot\frac{\pi}{n} - \frac{1}{2} e^{\left(-2\,sin^2\frac{\pi}{n}\right)t} csc\frac{\pi}{n}\, cos\left(t\, sin\frac{2\pi}{n} - \frac{\pi}{n}\right).$$

Reversing our notation simplification we get parametric equations of the position of mouse no. 1 for $t \geq 0$ in regular n-polygon

$$x_1(t) = \frac{1}{2} + \frac{1}{2} e^{\left(-2\,sin^2\frac{\pi}{n}\right)t} csc\frac{\pi}{n}\, sin\left(t\, sin\frac{2\pi}{n} - \frac{\pi}{n}\right),$$

$$y_1(t) = \frac{1}{2} cot\frac{\pi}{n} - \frac{1}{2} e^{\left(-2\,sin^2\frac{\pi}{n}\right)t} csc\frac{\pi}{n}\, cos\left(t\, sin\frac{2\pi}{n} - \frac{\pi}{n}\right).$$

Now, we will derive equations of the position of the remaining mice. Let's focus on mouse no. k, ($1 \leq k \leq n$). We know that

$M_1(t)$ rotated $\frac{2\pi}{n}$ around $(0,0)$ and translated by $\overrightarrow{P_1P_2}$ gives $M_2(t)$,

$M_1(t)$ rotated $2 \cdot \frac{2\pi}{n}$ around $(0,0)$ and translated by $\overrightarrow{P_1P_3}$ gives $M_3(t)$,

and so on, generally

$M_1(t)$ rotated $(k-1) \cdot \frac{2\pi}{n}$ around $(0,0)$ and translated by $\overrightarrow{P_1P_k}$ gives $M_k(t)$.

Therefore (in matrix-vector notation - see Appendix)

$$\overrightarrow{OM_k} = \begin{pmatrix} x_k \\ y_k \end{pmatrix} = \begin{pmatrix} \cos\frac{(k-1)2\pi}{n} & -\sin\frac{(k-1)2\pi}{n} \\ \sin\frac{(k-1)2\pi}{n} & \cos\frac{(k-1)2\pi}{n} \end{pmatrix} \begin{pmatrix} x_1 \\ y_1 \end{pmatrix} + \overrightarrow{P_1P_k} =$$

$$= \begin{pmatrix} x_1 \cos\frac{(k-1)2\pi}{n} - y_1 \sin\frac{(k-1)2\pi}{n} \\ x_1 \sin\frac{(k-1)2\pi}{n} + y_1 \cos\frac{(k-1)2\pi}{n} \end{pmatrix} + \overrightarrow{P_1P_k}\,.$$

Since $P_1(0,0)$ is O and $\overrightarrow{P_1P_k} = \overrightarrow{OP_k}$, we know that (see Appendix part 2)

$$\overrightarrow{P_1P_k} = \begin{pmatrix} \frac{1}{2} - \frac{1}{2}\cos\frac{(k-1)\cdot 2\pi}{n} + \frac{1}{2}\sin\frac{(k-1)\cdot 2\pi}{n}\cot\frac{\pi}{n} \\ \frac{1}{2}\cot\frac{\pi}{n} - \frac{1}{2}\sin\frac{(k-1)\cdot 2\pi}{n} - \frac{1}{2}\cos\frac{(k-1)\cdot 2\pi}{n}\cot\frac{\pi}{n} \end{pmatrix},$$

so we get

$$\overrightarrow{OM_k} = \begin{pmatrix} x_1 \cos\frac{(k-1)2\pi}{n} - y_1 \sin\frac{(k-1)2\pi}{n} + \frac{1}{2} - \frac{1}{2}\cos\frac{(k-1)\cdot 2\pi}{n} + \frac{1}{2}\sin\frac{(k-1)\cdot 2\pi}{n}\cot\frac{\pi}{n} \\ x_1 \sin\frac{(k-1)2\pi}{n} + y_1 \cos\frac{(k-1)2\pi}{n} + \frac{1}{2}\cot\frac{\pi}{n} - \frac{1}{2}\sin\frac{(k-1)\cdot 2\pi}{n} - \frac{1}{2}\cos\frac{(k-1)\cdot 2\pi}{n}\cot\frac{\pi}{n} \end{pmatrix}.$$

Substitution of $x_1(t)$ and $y_1(t)$ (and simplification[8]) leads to

$$x_k(t) = e^{-2t\sin^2\left(\frac{\pi}{n}\right)}\left(\sin\left(t\sin\frac{2\pi}{n} - \frac{\pi}{n}\right)\frac{\cos\frac{2(k-1)\pi}{n}}{2\sin\frac{\pi}{n}} + \cos\left(t\sin\frac{2\pi}{n} - \frac{\pi}{n}\right)\frac{\sin\frac{2(k-1)\pi}{n}}{2\sin\frac{\pi}{n}}\right) + \frac{1}{2},$$

$$y_k(t) = e^{-2t\sin^2\left(\frac{\pi}{n}\right)}\left(\sin\left(t\sin\frac{2\pi}{n} - \frac{\pi}{n}\right)\frac{\sin\frac{2(k-1)\pi}{n}}{2\sin\frac{\pi}{n}} - \cos\left(t\sin\frac{2\pi}{n} - \frac{\pi}{n}\right)\frac{\cos\frac{2(k-1)\pi}{n}}{2\sin\frac{\pi}{n}}\right) + \frac{\cot\frac{\pi}{n}}{2},$$

which may be expressed in more compact form

$$x_k(t) = \frac{1}{2} + \frac{1}{2}e^{-2t\sin^2\left(\frac{\pi}{n}\right)}\csc\frac{\pi}{n}\sin\left(t\sin\frac{2\pi}{n} + \frac{(2k-3)\pi}{n}\right),$$

$$y_k(t) = \frac{1}{2}\cot\frac{\pi}{n} - \frac{1}{2}e^{-2t\sin^2\left(\frac{\pi}{n}\right)}\csc\frac{\pi}{n}\cos\left(t\sin\frac{2\pi}{n} + \frac{(2k-3)\pi}{n}\right).$$

The above equations describe position of mouse no. k in a regular n-polygon at time $t \geq 0$, if we place the n-polygon $P_1P_2P_3 \dots P_n$ in such a way that $P_1(0,0)$ and $P_2(1,0)$.

Graphs of solution of other cases n=5 and n=10 are presented below.

[8] Simplified with a use of CAS (Computer Algebra System) software (*Derive 5.04*).

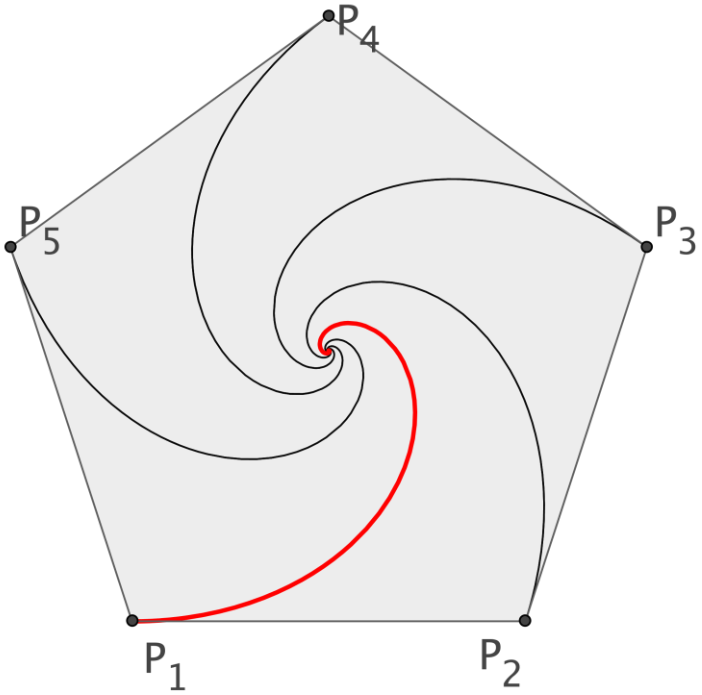


Figure 6 – Five mice pursuit problem trajectories.

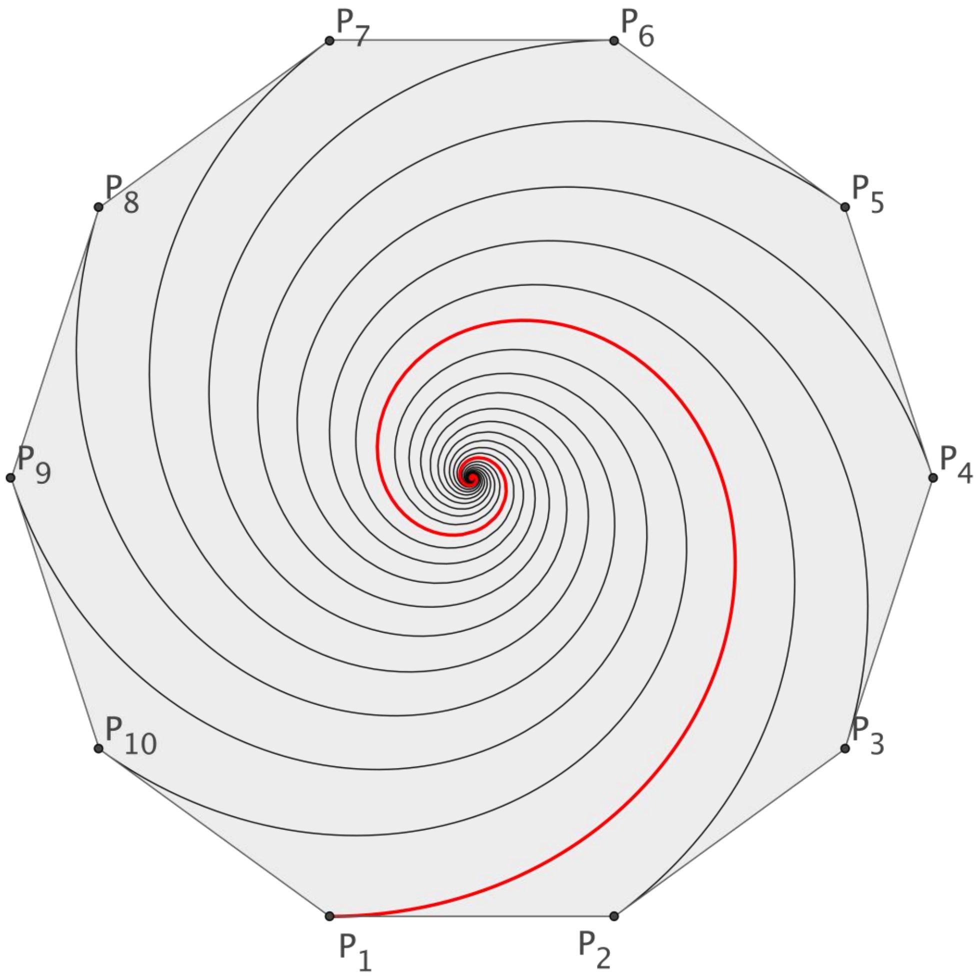


Figure 7 – Ten mice pursuit problem trajectories.

VII. Conclusion

Mice pursuing problem, considered firstly in a regular triangle (as a simplest case of regular polygon), led us to a system of simultaneous differential equations. Thanks to a symmetry of the mice behaviour these were first order linear differential equations for $x_1(t)$and $y_1(t)$ – parametric equations of the mouse no. 1 trajectory.

An introduction to Laplace transform and some of its properties allowed us to solve the system of equations in this case. The key property was presented and proved in a Theorem 3 which, when combined with linearity of Laplace transform guarantees that Laplace transform maps linear differential equations (systems of linear differential equations) into algebraic equations (systems of algebraic linear equations, respectively). Moreover equations of first order are mapped into linear algebraic equations, so in our case we obtained a system of two linear algebraic equations, solution of which gave us Laplace transforms of our unknown functions $x_1(t)$ and $y_1(t)$.

Partial fraction decomposition of the results confronted with the table of Laplace transforms (Table 1) enabled us to derive explicit formulas for $x_1(t)$ and $y_1(t)$. The (Theorem 2) Lerch's theorem (another key property for us) guaranteed uniqueness of the solutions. The symmetry of the problem allowed us to produce formulas for $x_2(t)$, $y_2(t)$ and $x_3(t)$, $y_3(t)$.

Restating the problem in a general form (regular n-polygon case) did not change the general pattern of solving process (thanks to "elegance" of Laplace transform) although the calculations involved were more complicated.

The mice trajectories turned out to be logarithmic spirals beginning at the polygon's vertices and converging to its centre[9]. A number of interesting questions arises now (for example what are the lengths of the trajectories, what happens when n goes to infinity) and some possible improvements are apparent - the centre of polygon could be placed in O and one could use constant size polygon (e.g. R=1=radius of a circumscribed circle) and complex numbers for parameterization (vertices would be roots of 1 and rotation would be just multiplication).

[9] A centre of an inscribed and circumscribed circle, mass centre.

VIII. Bibliography

1. Beerends, R. 2003. *Fourier and Laplace transforms*. Cambridge: Cambridge university press.

2. *Gewert, M. and Skoczylas Z. (2001). Równanie różniczkowe zwyczajne: Teoria, przykłady, zadania,*Wrocław: Oficyna Wydawnicza GiS

3. Larson, R. and Edwards, B. 2014. *Calculus*. Boston, MA: Brooks/Cole, Cengage Learning.

4. Liu, V. 2005. *The four mice pursuit curve solution using Laplace Transforms*. [e-book] http://www.stanford.edu/~vkl/documents/4mice.pdf [Accessed: 30 September 2013].

5. *Kącki, E. and Siewierski L. (1993). Wybrane działy matematyki wyższej z ćwiczenimi,* Warszawa: Wydawnictwi Naukowe PWN

6. Orloff, J. 2013. *18.03 Uniqueness of Laplace Transform*. [e-book] B: Available through: http://web.mit.edu/jorloff/www/18.03-esg/notes/laplaceuniqueness.pdf [Accessed: 20 October 2013].

7. *Stewart, J., (2012). Calculus.* 7th ed. Belmonts: Brooks/Cole Pub Co.

8. *Shiff, L. J. (1999). The Laplace Transform: Theory and Applications.* [e-book] New York: Springer-Verlag *Available at:* Google Books <http://books.google.com>[Accessed: 20 September 2013].

9. Strang, G. n.d. *Calculus*. [e-book] Available through: http://ocw.mit.edu/ans7870/resources/Strang/Edited/Calculus/Calculus.pdf [Accessed: 10 September 2013].

IX. Appendix

Analytical forms of geometrical transforms used (in this paper):

A1

- rotation about (0,0) by directed angle φ:

$$R_\varphi(x, y) = (x', y') = (x \cos \varphi - y \sin \varphi, x \sin \varphi + y \cos \varphi)$$

in position rotation:

$$\begin{pmatrix} x' \\ y' \end{pmatrix} = M_\varphi \cdot \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} \cos \varphi & -\sin \varphi \\ \sin \varphi & \cos \varphi \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} x\cos \varphi - y \sin \varphi \\ x \sin \varphi + y \cos \varphi \end{pmatrix}$$

where M_φ is the rotation matrix.

A2

- translation by vector $\vec{v} = \begin{pmatrix} v_x \\ v_1 \end{pmatrix}$

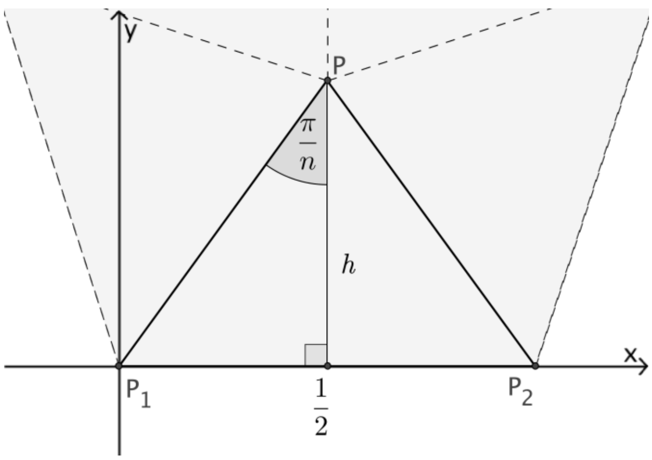
$$T_{\vec{v}}(x, y) = (x', y') = (x + v_x, y + v_1)$$

in position vector rotation:

$$\begin{pmatrix} x' \\ y' \end{pmatrix} = \begin{pmatrix} x \\ y \end{pmatrix} + \vec{v} = \begin{pmatrix} x + v_x \\ y + v_1 \end{pmatrix}$$

B

Derivation of the coordinates of P – centre of circumscribed circle of the regular n-polygon. Let's consider following regular polygon



From the above figure it may be concluded that

$$\frac{\frac{1}{2}}{h} = \tan\frac{\pi}{n} \text{, which leads to } h = \frac{1}{2}\cot\frac{\pi}{n}.$$

Therefore, $P\left(\frac{1}{2}, \frac{1}{2}\cot\frac{\pi}{n}\right)$ is a centre of an inscribed and circumscribed circle in every regular n-polygon, if we place the n-polygon $P_1P_2P_3 \dots P_n$ in such a way that $P_1(0,0)$ and $P_2(1,0)$.