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WROCLAW, POLAND

Mathematical Exploration

Internal Assesment

Game of 1000.

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Introduction.

Game of 1000 is a popular card game which I often play since I was 7. Lately it also became favorite game of my younger brother, who gets better and better in this quite complex game. In my opinion this game is especially fascinating as it involves a combination of planning, predicting moves of opponents and luck. My personal involvement connected with this game made me wish to master my skills on a field of this game.

As moves of the opponents can be well predicted, especially with an assumption that they are not making any mistakes (which I am going to consistently use throughout my exploration) it occurred to me that this gives basis to model the other two elements essential for winning: luck and strategy. While studying probability during classes, it became clear to me, that to some extent it allows to measure luck, which is important in many games. This gives basis to planning and allows more accurate decision-making processes.

The aim of this exploration is thus to give such an insight into the rules of game of 1000 that would allow a player make more conscious decisions about his or her choices. I decided to try to use Mathematics as a tool to derive improved strategies of playing the game of 1000.

Rules of Game of 1000.

Game of 1000 is a series of short hands between 3 players. Throughout my exploration I will be deriving the strategy from a point of view of player A, who tries to win with opponents B and C. To win the whole game, a player needs to collect 1000 points, which are counted at the end of each short hand. Points are calculated on the basis of cards collected by each of the players in tricks. To play the game of 1000, standard deck of 24 cards, with a 9 being the lowest one and an Ace the highest, in 4 suits: hearts, diamonds, clubs or spades is needed. Each card has a different strength and different value in case of points. In the order from the strongest to the weakest:
Ace: 11 points,
Ten: 10 points,
King: 4 points,
Queen: 3 points,
Jack: 2 points,
Nine: 0 points.

All together cards have 120 points. A King and a Queen in the same suit together create a Report, which gives a player having this two cards together additional points (if he or she manages to report such situation by placing a Queen in front of other players while being the leading player). Reports have different values:
Report of Hearts: 100 points,
Report of Diamonds: 80 points,
Report of Clubs: 60 points,
Report of Spades: 40 points.

One hand in game of 1000 has two stages: the auction and the game. At the beginning cards need to be dealt: each of the 3 players get 7 cards and 3 cards, not seen by any of the players, are left in the middle to create a set called “musik” which is a subject of the auction. Player on the left hand of the dealer (the positions change clock wisely with every hand) starts the auction with declaring that he will gather 100 points in this hand (this player is obligated to do so, no matter what cards he/she has). Two remaining players may join the auction by declaring that they will gather more points (multiples of 10 are allowed only) and they do so in the clockwise direction. The auction finishes when none of the players wishes to submit a higher declaration then the last one. The winner of the auction becomes the leading player and picks up “musik”. Now, seeing 10 cards, the player has to decide how many points will he or she in fact gather during this hand (only multiples of 10 are allowed and declaration may be only higher or equal one stated during the Auction) and also needs to give each of his opponents one card (so that all players have set of 8 cards).

Then the stage of the game begins. Leading player starts it by placing one card in front of his opponents, who then give away one of their cards. It needs to match the suit of the card leading player gave (as long as players have such suit in their sets). Player with the strongest card collects weaker ones (if given card does not much in suit with one given by leading player, its strength does not matter). Each set of such collected cards is called a trick. Player who collected cards automatically becomes the leading player. If a leading player gives a Queen of certain suit declaring a Report (the player needs to have a King in this suit to do so) suit of this Report becomes dominant over other suits – they become trumps. The rule that players need to give the cards of matching suits still holds, but if one of the players gives a trump because he does not have a card in matching suit, no matter what strength is the card itself, player collects cards given by his opponents.

At the end of each hand, players are calculating the points they gathered. Leading player may either collect as many points as he declared or more, and in this case declared at the beginning number of points is recorded, or he may not gather sufficient amount of points and these are then subtracted from his overall score. Points two other players gathered simply add to their overall score.

The scope of analysis.

Firstly, I wanted to focus on the second stage of the game of 1000 to see what configurations of cards give certain probabilities of winning. Only basing on this knowledge it is possible to assess whether it is worth to enter the auction.

It is practically impossible to analyze the possibility of winning with all combinations of sets of cards as there are

$$\binom{24}{7} = 346104$$

different possible arrangements of cards in a set given to one player at the beginning of the game. Because of that, more general approach which would allow drawing certain conclusions for many card arrangements needs to be developed.

It is important to note that strategies of playing game of 1000 are various and moves of opponents may be unpredictable and thus anything can be analyzed from a point of a leading player only. Only in this situation player has sufficient control over the situation to make any relevant predictions.

There are plenty of things that can be calculated in relation to the Game of 1000, but I decided to focus on 3 issues which are in my opinion most important and relevant in the strategy of the game. First of all I wish to focus on analysis of probability of winning taking into account one suit only, as this allows to analyze all possible arrangements and thanks to this analysis some conclusions can be drawn which are relevant to set of cards in few suits mixed together. Application of this analysis is visible even in second area of my focus, so problems with deciding if rising the declaration after winning the Auction will pay off and to what level should it be raised. Third thing which always bothered me while playing the Game of 1000 is if I can find a Report or card fitting part of Report I already have in ‘musik”.

Analysis of probabilities of winning with one suit only.

This analysis refers to the Game stage of the Game of 1000. At this stage of the game, it is important to take as many cards as possible, what is equal to not loosing as many cards as possible from the set this is given to each player. If a player does not lose his cards, it means that he collected cards of his opponents.

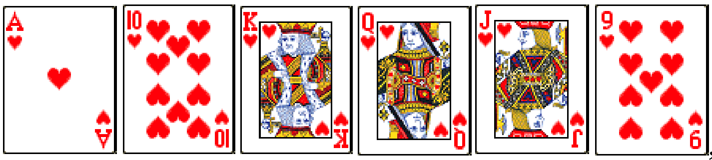
Thus,
success: not giving any of cards player A has to his opponents B or C.

Taking into account cards of one suit (hearts, diamonds, clubs or spades) only, player A may see different situations at the beginning of the game. Situation which is taken into account is the beginning of the Game stage, where player A being a leading player has already given 1 card to each of the opponents B and C. This cards were not in the considered suit and thus they have no influence on calculated probabilities (player A has no knowledge about the distribution of cards in considered suit among opponents). Here each of the players has 7 cards and thus player A may have:

- 6 cards in one suit,
- 5 cards in one suit,
- 4 cards in one suit,
- 3 cards in one suit,
- 2 cards in one suit,
- 1 card in one suit.

Player A may be sure of his success, so that he will not lose any of his cards in considered suit, if and only if the distribution of cards in this one suit among his opponents plays no role. Rules of the game, mainly fact that opponents need to give him cards in the suit he chooses if they have such, determine that he has probability of winning *P (success) = 1*. Such sets of cards are:

- 6 cards in one suit: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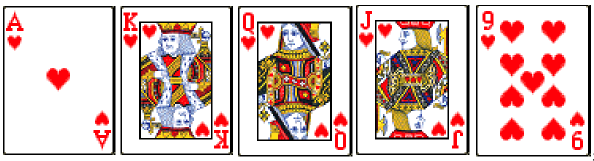
- 5 cards in one suit and an Ace is not missing: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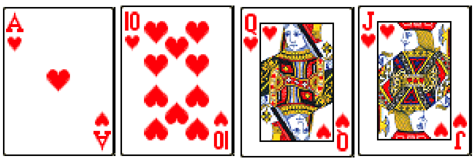
- 4 cards in one suit, with an Ace and a Ten included: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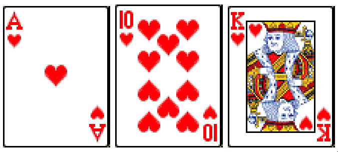
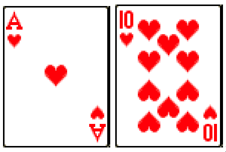
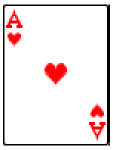
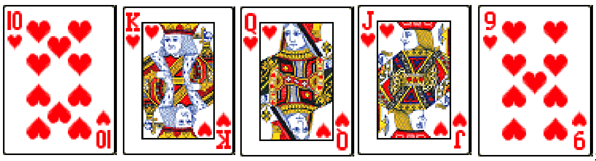
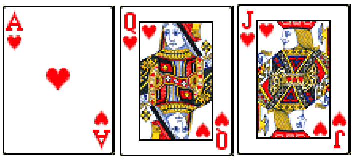
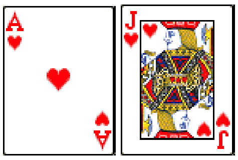
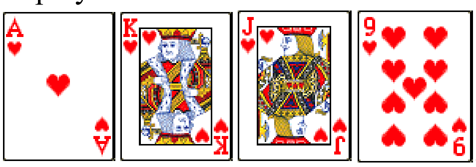
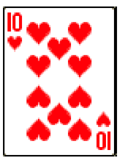
- 3 cards in one suit and this cards are an Ace, a Ten and a King:
,
- 2 cards in one suit and this cards are an Ace and a Ten:
,
- 1 card in one suit and it is an Ace:
.

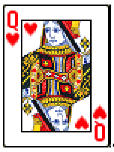
There are also many sets of cards, in which player A may be sure of failure (assuming that his opponents will not make a mistake), so *P(success) = 0*. Such sets of cards are:

- an Ace missing:
,
- 2, 3 cards in one suit, an Ace is included, but a Ten and King are missing:
 or .

In case of some sets of cards probability of winning is not as obvious as previously. From my experience they appear quite often during the game. In such cases player may win or lose, depending on the distribution of the remaining cards among his opponents. Such sets of cards are:

- 4 cards in one suit, missing cards are Ten and one other card, but not an Ace, so player's A set of cards in one suit may be for example:
.
Cards set in the suit which is taken into consideration, which are missing in player's A must then be separated in order to give success to player A. So opponent B needs to have only one card in considered suit, for example:
,

while second card in considered suit needs to be in opponent's C set of cards:

.

Here,

P(success) = P (opponent B has exactly 1 card in given suit) = $\frac{\binom{2}{1}\times\binom{14}{7}}{\binom{16}{8}} = \frac{8}{15} \approx 0.533,$

where:

$\binom{2}{1}$ – 1 card in chosen suit out of 2 missing in player's A set,

$\binom{14}{7}$ – remaining 7 cards opponent B needs to have out of remaining 14 cards which are not in a chosen suit,

$\binom{16}{8}$ – total of 8 cards given to opponent B out of 16 cards available.

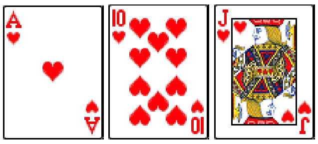
It is interesting to note that this case is an example of hypergeometric distribution. Thus, its general formula is:

$$P(X = x) = \frac{\binom{M}{x}\binom{N-M}{n-x}}{\binom{N}{n}}, \qquad (1)$$

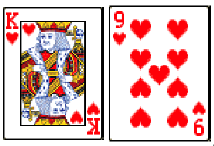
where:

X – random variable of hypergeometric distribution with parameters *N, M, n,*
x – desired number of cards in considered suit, which a player should have,
N – set of all available cards,
M – set of all cards in considered suit available to the player,
n – set of cards given to one player.

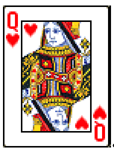
- 3 cards in one suit and these cards are an Ace, a Ten and one other card, but not a King nor Queen (these 2 cards are missing), so player's A set of cards in one suit may be for example:

  .

  Player A will be successful if and only if one of the opponents will have 1 card in considered suit while other will have 2 cards in considered suit. It means that one of the opponents, B for example, may have the following cards in considered suit:

  ,

  while the other opponent, here C, may have:

  .

Here,

P(success) = P (opponent B or C has exactly 1 card in given suit) = $\frac{\binom{3}{1}\times\binom{13}{7}}{\binom{16}{8}} \times 2 = \frac{4}{5} = 0.8$,

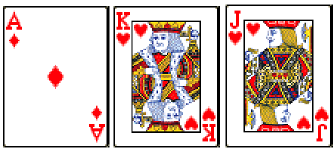
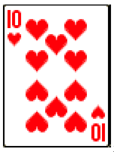
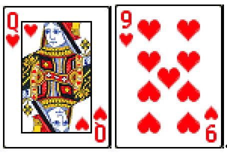
where:

$\binom{3}{1}$ – 1 card in given suit chosen out of set of 3 such cards,

$\binom{13}{7}$ - remaining 7 cards opponent B needs to have out of remaining 13 cards which are not in a chosen suit,

$\binom{16}{8}$ - total of 8 cards given to opponent B out of 16 cards available,

$\times$ 2 – there are 2 opponents.

- 3 cards in one suit and these cards are an Ace, a King and one other card, but not a Ten (a Ten is missing),
  so player's A set of cards in one suit may be for example:
  .
  Player A will be successful if and only if one of the opponents will have only a Ten in a given suit. It means that one of the opponents, B for example, must have the following cards in considered suit:
  ,
  while the other opponent, here C, must have:
  .

Here,

P(success) = P (opponent B or C has exactly one card in given suit and this card is a Ten) =
$\frac{\binom{1}{1}\times\binom{13}{7}}{\binom{16}{8}} \times 2 = \frac{2}{15} \times 2 = \frac{4}{15} \approx 0.267$,

where:

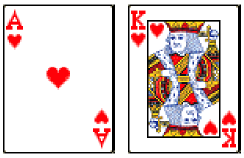
$\binom{1}{1}$ – a Ten in given suit chosen out of 1 element set of such Tens,

$\binom{13}{7}$ – remaining 7 cards opponent B needs to have out of remaining 13 cards which are not in a chosen suit,

$\binom{16}{8}$ – total of 8 cards given to opponent B out of 16 cards available,

$\times$ 2 – there are 2 opponents.

- 2 cards in one suit and these cards are an Ace and a King:



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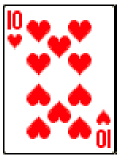
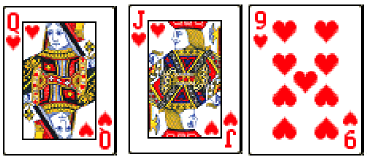
Player A will be successful if and only if one of the opponents will have only a Ten in a given suit. So set of cards in considered suit opponent B has is:



,

while opponent C has:



.

Here,

P(success) = P(opponent B or C has exactly one card in given suit and this card is a Ten) =

$\frac{\binom{1}{1}\times\binom{12}{7}}{\binom{16}{8}} \times 2 = \frac{4}{65} \times 2 = \frac{8}{65} \approx 0.123,$

where:

$\binom{1}{1}$ – a Ten in given suit chosen out of 1 element set of such Tens,

$\binom{12}{7}$ – remaining 7 cards opponent B needs to have out of remaining 12 cards which are not in a chosen suit,

$\binom{16}{8}$ – total of 8 cards given to opponent B out of 16 cards available,

$\times\ 2$ – there are 2 opponents.

These 3 situations, where probability of winning is not that easy to be determined compared to other distributions of cards considered previously, can be also tackled from another point during the game. As player A is here the leading player, he or she picks up “musik” and for a moment holds 10 cards, while opponents B and C have 7 cards each. As player A aims at keeping the greatest possible number of cards in one suit, he or she will not give away any of this cards. So in fact the probability of certain distribution of missing cards in considered suit can be also calculated at this point.

Thus, player A holds:

- 4 cards in one suit, missing cards are Ten and one other card, but not an Ace.

  Here,

  P(success) = P (opponent B has exactly 1 card in given suit) $= \frac{\binom{2}{1}\times\binom{12}{6}}{\binom{14}{7}} = \frac{7}{13} \approx 0.538,$ (2)

where:

$\binom{2}{1}$ – 1 card in chosen suit out of 2 missing in player's A set,

$\binom{12}{6}$ - remaining 6 cards opponent B needs to have out of remaining 12 cards which are not in a chosen suit,

$\binom{14}{7}$- total of 7 cards given to opponent B out of 14 cards available. Player A holds 10 cards.

- 3 cards in one suit and these cards are an Ace, a Ten and one other card, but not a King nor Queen (these 2 cards are missing).

  Here,

  P(success) = P (opponent B or C has exactly 1 card in given suit) = $\frac{\binom{3}{1}\times\binom{11}{6}}{\binom{14}{7}} \times 2$ = $\frac{21}{52} \times 2 = \frac{42}{52} \approx 0.808,$ (3)

  where:

  $\binom{3}{1}$ – 1 card in given suit chosen out of set of 3 such cards,

  $\binom{11}{6}$ – remaining 6 cards opponent B needs to have out of remaining 11 cards which are not in a chosen suit,

  $\binom{14}{7}$ – total of 7 cards given to opponent B out of 14 cards available, as player A holds 10 cards,

  $\times$ 2 – there are 2 opponents.

- 3 cards in one suit and these cards are an Ace, a King and one other card, but not a Ten (a Ten is missing).

  Here,

  P(success) = P (opponent B or C has exactly one card in given suit and this card is a Ten) = $\frac{\binom{1}{1}\times\binom{11}{6}}{\binom{14}{7}} \times 2$ = $\frac{7}{52} \times 2 = \frac{14}{52} \approx 0.269,$ (4)

  where:

  $\binom{1}{1}$ – a Ten in given suit chosen out of 1 element set of such Tens,

  $\binom{11}{6}$ – remaining 6 cards opponent B needs to have out of 14 cards total – 3 cards in chosen suit player A doesn't have,

  $\binom{14}{7}$ – total of 7 cards given to opponent B out of 14 cards available, as player A holds 10 cards,

  $\times$ 2 – opponent C may have the configuration which is considered for opponent B.

- 2 cards in one suit and these cards are an Ace and a King.
  Here,
  P(success) = = P(opponent B or C has exactly one card in given suit and this card is a Ten)
  $= \frac{\binom{1}{1}\times\binom{10}{6}}{\binom{14}{7}} \times 2 = \frac{35}{572} \times 2 = \frac{35}{286} \approx 0.122,$ (5)
  where:
  $\binom{1}{1}$ – a Ten in given suit chosen out of 1 element set of such Tens,
  $\binom{10}{6}$ – remaining 6 cards opponent B needs to have out of remaining 10 cards which are not in a chosen suit,
  $\binom{14}{7}$ – total of 7 cards given to opponent B out of 14 cards available, as player A holds 10 cards,
  $\times 2$ – opponent C may have the configuration which is considered for opponent B.

Analysis of probabilities of success with one suit gives a general image of which strategies are yielding most probable success. Most of the cases are clear, there is only a chance of winning or losing, independently from the distribution of cards among players. In few situations it is important to realize which distribution is more likely going to occur, because it gives a sense of making a considerate decision. To me, surprising is that holding 4 cards in one suit and having a missing Ten gives a probability of success greater than 0.5. Before I calculated this probability I was absolutely sure that none of the sides has a greater chance of winning in this distribution of cards and it appears that this set of cards actually gives a greater possibility of winning. It is even more surprising that while holding 3 cards in one suit it is highly probable to keep them all, given of course that player hold an Ace and a Ten. In case of 2 other doubtful sets of cards, the probabilities of winning are rather low, so player should not relay on luck in this cases, or at least I will not while playing Game of 1000.

If the doubtful cases are tackled from the perspective of a player holding 10 cards, the probabilities of occurrence of distributions of cards which are favorable to player A, surprisingly to me, change. This change is quite small, reaches 3^{rd} decimal place, so from the point of view of a player it is not very meaningful. But it is worth noticing, that in 3 cases (when player holds 3 or 4 cards in considered suit) the probability increases. It is especially important in case of set with 4 cards in one suit, as probability of distribution of cards allowing player to succeed is even greater. In case of player A holding 2 cards in considered suit only, with change of the perspective, probability of success decreases. It only shows that it is not wise to relay on this set of cards, as probability of success was quite low even in the first considered situation.

Analysis if it is worth to raise the declaration using full sets of cards.

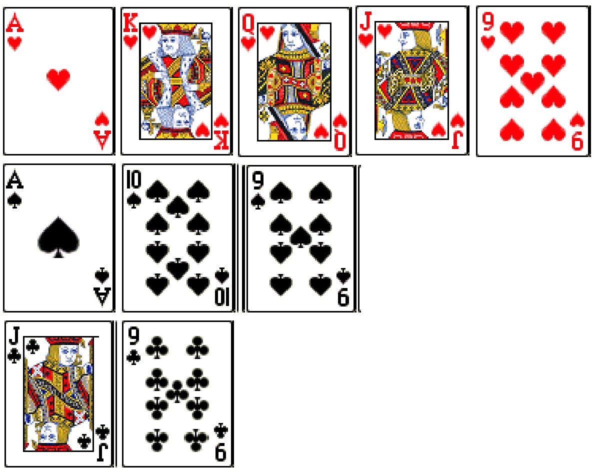
At the beginning of the Game stage, player who won the auction needs to declare how many points will he or she gather. It is especially important to be able to estimate the possible amount of points gathered as precisely as possible, because in case of declaring too many points, player will lose declared amount of points from the overall score and in case of declaring too little points, points player gathered which exceed the declaration will not be added to the overall score (so player will lose them in fact). Because of this rule, it is often very tempting to declare gathering more points. From my experience, in case some sets of cards which seem to be very favorable, it is popular to declare gathering a great number of points. It is often combined then with losing, due to too high expectations. I would like to check what is the real possibility of winning certain amount of points with 2 sets of cards which always seemed most problematic to me. Number of all possible arrangements is too high to be able to analyze all arrangements.

To conduct this analysis, I will use the most valuable Reports, however the rules will apply to Reports of each suit in exactly the same way. It is only important to keep certain cards in the same suit. As numbers of points possible to be gathered will be analyzed, this score should be adjusted depending on the Report. Similarly, the Nine and the Jack can be exchanged in each set with no influence on the probability (strength of both this cards is comparable in the flow of game).

Here,
success – collecting as many points as declared,
failure – not collecting as many points as declared.

Situation which is here taken under consideration is that player A won the Auction and holds 10 cards, with 2 of them being irrelevant as they will be soon given to the opponents. But at this very moment, opponents B and C hold 7 cards each, and player A does not know the distribution of 14 cards only.

First set of cards:



With this set, declaration which would be the safest, so which would give player A *P(success) = 1* is 190 points. 100 points is given by the Report of Hearts, and following 90 by suit of Hearts, an Ace and a Ten of spades which player is sure to collect.

The highest possible declaration with this set of cards is 220 point, 100 points constitutes the Report of Hearts and 120 points may be gathered by not giving away any of the cards and collecting all trick.

To collect all cards, the following condition must be met:

- remaining 3 spades must be distributed among opponents B and C in such way that one of them will have 1 card and second one 2 cards.

In the previous part, analysis of probability of winning with one suit only, such situation was analyzed. Situation considered in this part is while player A holds 10 cards and opponents B and C have 7 cards each and thus second value calculated in the previous part needs to be taken into account, which is from the equation (3) *P(success) =0.808*.

Here, X is a random variable that represents gain or loss in a game.

Thus, as the highest possible declaration is being considered:

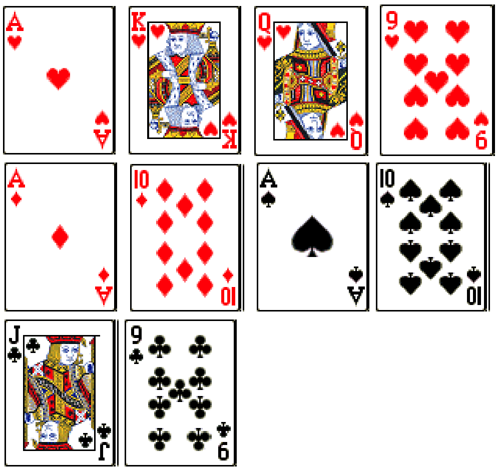
X	-220	220
$P(X=x)$	1-0.808 = 0.192	0.808

Here, the expected value of X is:

$EX = -220 \times 0.192 + 220 \times 0.808 = -42.24 + 177.76 = 135.52.$

On the other hand the safe strategy assuming the loss of the last trick is sure to bring 190 points. So in the long run the safer strategy seems to be a better choice, because in case of risky one, the expected return is smaller than 190 points. In this case greed is rather not paying-off.

<u>Second set of cards:</u>



This arrangement is extremely lucky, declaration which has *P(success) = 1* is 150 points (the Report of Hearts, 3 Aces and 2 Tens). The highest possible declaration is however is again 220 points.

For player A to succeed, one condition must be met:

- the Ten and the Jack in suit of Hearts must be separated.

In this arrangement, only one suit gives any risk of failure. Thus, previously derived probabilities may be used.
Situation under consideration here is: 4 cards in one suit and a Ten is missing.

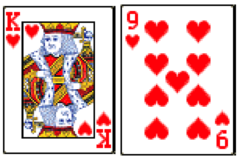
As X is a random variable that represents gain or loss in a game, taking into account value calculated in equation (2) with the highest possible declaration is being considered:

X	-220	220
$P(X=x)$	1-0.538 = 0.462	0.538

and the expected value of X is:
$EX = -220 \times 0.462 + 220 \times 0.538 = -101.64 + 118.36 = 16.72.$

Comparison of gain in case of safe strategy and the expected value obtained for the risky one it can be stated that the safe strategy is better. It offers smaller reward, but it still gives a possibility of gathering many points. As here expected value is much smaller than 150 points ensured by safe strategy it would be unwise to risk losing 220 points.

In order to increase the probability of success even more, player A may decide to allow losing one trick. If a Ten of Hearts happen to be grouped together in set of one of the opponents, then this opponent will pick up a Queen of Hearts (assuming he or she makes no mistakes during the game and will save this Ten till this moment). In this case, player A has:



And as the Report of Hearts was declared (in this move player A lost the Queen of Hearts) both of this cards are trumps and have a greater strength than any card in other suit. So in order to win, opponent picking up a Queen of Hearts must have another Report as well.

Thus, with the decision of player A that he or she is willing to give up Queen of Hearts, giving up one trick which value should be estimated as 30 points (10 points for a Ten using which player picks up Queen of Hearts worth 3 points and between 2-11 points because of the 3rd unknown card placed by second opponent – as this is the end of the game, more valuable cards are usually given), in order to lose one of the opponents must have at the beginning of game stage:

- Ten of Hearts,
- Jack of Hearts,
- Report in another suit (Diamonds, Clubs or Spades).

Situation which is being here considered still involves opponents having 7 cards out of 14 not known to the player A.

As there are 3 Reports available to opponents, I will use a well-known formula for union of three events

$$P(A_1 \cup A_2 \cup A_3) = P(A_1) + P(A_2) + P(A_3) - P(A_1 \cap A_2) - P(A_1 \cap A_3) - P(A_2 \cap A_3) + P(A_1 \cap A_2 \cap A_3). \qquad (6)$$

Let:

$A_1 = R_1$ – Report of Diamonds and 2 cards in suit of Hearts which player A does not have (Ten and Jack),

$A_2 = R_2$ – Report of Clubs and 2 cards in suit of Hearts which player A does not have (Ten and Jack),

$A_3 = R_3$ – Report of Spades and 2 cards in suit of Hearts which player A does not have (Ten and Jack).

Since 3 Reports and two cards in suit of Hearts (a Ten and a Jack) make 8 cards and each opponent holds 7 cards,

$$R_1 \cap R_2 \cap R_3 = \emptyset$$

and

$$P(R_1 \cap R_2 \cap R_3) = 0. \qquad (7)$$

Next,

$$P(opponent\ B\ or\ C\ having\ Ten\ of\ Hearts, Jack\ of\ Hearts\ and\ a\ Report) = P(R_1) = P(R_2) = P(R_3) = \frac{\binom{1}{1}\times\binom{1}{1}\times\binom{2}{2}\times\binom{10}{3}}{\binom{14}{7}} = \frac{5}{143}, \qquad (8)$$

where:

$\binom{1}{1}, \binom{2}{2}$ – a Ten of Hearts, Jack of Hearts or a Report are chosen out of set of only one such cards,

$\binom{10}{3}$ – remaining 3 cards opponent needs to have for 7 card set taken out of 10 cards set available to opponents and not being any of the particularly distinguished cards,

$\binom{14}{7}$ – whole set of 7 cards opponent Has out of set of 14 cards available to opponents,

and

$$\mathrm{P}(pponent\ B\ or\ C\ having\ Ten\ of\ Hearts, Jack\ of\ Hearts\ and\ 2\ Reports) = P(R_1 \cap R_2) = P(R_1 \cap R_3) = P(R_2 \cap R_3) = \frac{\binom{1}{1}\times\binom{1}{1}\times\binom{2}{2}\times\binom{2}{2}\times\binom{8}{1}}{\binom{14}{7}} = \frac{1}{429}, \qquad (9)$$

where:

$\binom{1}{1}, \binom{2}{2}$ – a Ten of Hearts, Jack of Hearts or a Report are chosen out of set of only one such cards,

$\binom{8}{1}$ – remaining 1 card opponent needs to have for 7 card set taken out of 8 cards set available to opponents and not being any of the particularly distinguished cards,

$\binom{14}{7}$ – whole set of 7 cards opponent Has out of set of 14 cards available to opponents.

From the formula (6) and with the values calculated in equations (7), (8) and (9):
$P(failure) = \frac{5}{143} + \frac{5}{143} + \frac{5}{143} - \frac{1}{429} - \frac{1}{429} - \frac{1}{429} + 0 = \frac{14}{143}$.

As any of the 2 opponents may have the required set of cards, but they cannot have same sets of cards at the same time (events are disjoint), value needs to be multiplied by 2.
Thus,
P(failure) $\frac{14}{143} \times 2 = \frac{28}{143} \approx 0.196$.

As X is a random variable that represents gain or loss in a game, with the declaration of 190 point (220 – 30 for 1 trick) being considered:

X	-190	190
P(X=x)	0.196	1- 0.196 = 0.804

Expected value of X is:
$EX = -190 \times 0.196 + 190 \times 0.804 = -37.24 + 152.76 = 115.52$.

Comparing 2 strategies referring to the second set of cards shows that in case of second strategy average gain is much higher than with higher declaration. It shows that this strategy may pay-off much better. The expected value is still smaller then the safe declaration but high probability of success shows that on a short run there is a high possibility of winning. Using safe strategy, player is sure of winning, but also does not have a chance of getting many points. In this case, where chance of winning with a bit riskier declaration is that high I think that it is worth a try. Success is then not sure, but it allows to collect more points, so gives a greater chance of winning whole game, especially if this strategy will be used from time to time, because on a long run, as expected value is smaller than declaration in safe strategy, it may lead to the loss of points. Choice of strategy may also depend on the stage of game. If the game has just started and all players have little points, it is better to chose the safe declaration not to get negative value of points. On the other hand, having high score it is most important to quickly gather as many points as possible to beat the opponents. Then strategy of declaring 190 points would be the most successful.

Analysis of the strategies involving raising of the declaration shows application of probabilities of success calculated for one set only. It also illustrates which points should be taken into account while assessing probability of winning. In many cases it can be seed that greed does not pay – off as there is a lot to be lost. But with sufficient care I think it is worth to take a risk as the whole game requires collecting of many points – 1000 to be precise, and this is in fact quite a lot. Without a small risk it may take too long to collect that many points.

The Auction stage.

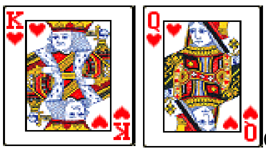
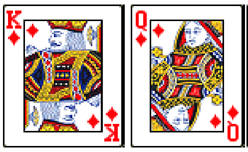
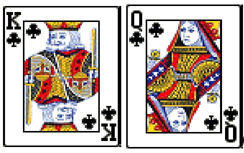
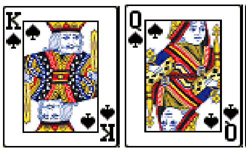
On this stage of the game of 1000 players need to assess their ability to collect certain amount of points basing on their 7 cards. If they win the auction, 3 unknown at the beginning cards will be available to them. This 3 cards play an important role, as they can dramatically change the set of cards one player has.

Probability of obtaining a Report in "musik".

If player A, at the beginning of the game, does not find a Report among his or her cards, there still is a chance of getting such set of cards. It all depends on the other cards player hold, especially parts of the Reports.

In this part of the analysis, for the sake of clarity let:
$R_1 - R_4$ denote whole Report in "musik", either Report of Hearts, Diamonds, Clubs or Spades (suits are enumerated according to their strength), and these events are mutually disjoint because each Report has 2 cards and whole set of cards in "musik" is 3,
$Q_1 - Q_4$ denote a Queen, either in suit of Hearts, Diamonds, Clubs or Spades (suits are enumerated according to their strength), which needs to be found in "musik" to create a whole Report with a King present in set of cards player A holds. It is connected with an assumption that player A in his or her set has a King, not a Queen but this cards may be exchanged with no influence on the calculated probabilities as always simply a card missing to create a whole Report is needed. This assumption gives a possibility of more clear reference.

To calculate a probability of finding a Report or a Queen in desired suit in "musik" formula for hypergeometric distribution (1) introduced in the first part of the analysis will be often used.

If player A has no parts of the Reports.
So player A is searching for:
 or  or  or  in "musik".
In this case, player A may get a Report if and only if it is found in "musik".

Probability of finding one particular Report in "musik" is:

$$P(R_1) = P(R_2) = P(R_3) = P(R_4) = \frac{\binom{2}{2}\binom{17-2}{3-2}}{\binom{17}{3}} = \frac{3}{136}, \qquad (10)$$

where:
$\binom{2}{2}$ – 2 cards creating a particular Report and there are only 2 such cards,
$\binom{17-2}{3-2}$ – remaining 1 card in "musik" which is chosen out of set of cards which player A does not have and which are not belonging to this particular Report.

In this part, let R denote any Report found in "musik", thus $R = R_1 \cup R_2 \cup R_3 \cup R_4$, and events are mutually disjoint.

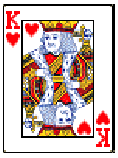
Thus, probability of finding at least one Report in "musik" is:

$$P(R) = P(R_1) + P(R_2) + P(R_3) + P(R_4) = \frac{3}{136} + \frac{3}{136} + \frac{3}{136} + \frac{3}{136} = \frac{12}{136} \approx 0.0882.$$

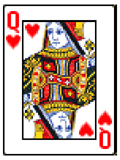
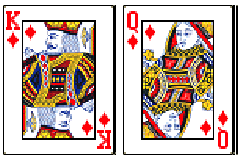
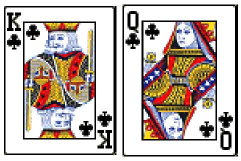
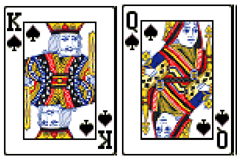
From performed calculation it can be seen that the probability of having a Report thanks to finding it in "musik" is extremely low. It shows that players should not count on such occurrence and thus should not declare collecting many points in the Auction stage if they have no parts of Reports as I expected.

<u>If player A has 1 part of the Report.</u>

According to made assumptions, player A has:



And is searching for:

 or  or  or  in "musik".

In this case player A may get a Report by finding one of available 3 in "musik".

Next, let R denote any Report found in "musik", thus $R = R_2 \cup R_3 \cup R_4$, and events are mutually disjoint.

Using a value of probability of finding one particular Report in "musik" (10) probability of finding whole Report is:

$$P(R) = P(R_2) + P(R_3) + P(R_4) = \frac{3}{136} + \frac{3}{136} + \frac{3}{136} = \frac{9}{136}. \tag{11}$$

Other possibility for player A to get a Report in this case is to find a particular Queen matching the King he already has. Probability of finding one particular Queen in "musik" can be found with the formula (1) for hypergeometic distribution:

$$P(Q_1) = \frac{\binom{1}{1}\binom{17-1}{3-1}}{\binom{17}{3}} = \frac{3}{17}, \tag{12}$$

where:

$\binom{1}{1}$ – 1 particular Queen is needed and there is one such Queen,

$\binom{17-1}{3-1}$ – remaining 2 cards in "musik" are taken out of set of cards which are not present in set player A has and which are not this Queen,

$\binom{17}{3}$ - 3 cards taken from set of 17 cards player A does not have create whole "musik".

In this part as there is only 1 Queen available, $Q_1 = Q$.

Events of finding a Report and a Queen in “musik” are not disjoint and thus I will use a commonly known formula for union of two events:

$$P(R \cup Q) = P(R) + P(Q) - P(R \cap Q). \tag{13}$$

Probability of common part need to be found, so probability of finding a Report and a Queen in “musik” is:

$$P(R_1 \cap \mathrm{Q}) = P(R_2 \cap \mathrm{Q}) = P(R_3 \cap \mathrm{Q}) = \frac{\binom{2}{2}\binom{1}{1}}{\binom{17}{3}} = \frac{1}{680}, \tag{14}$$

where:

$\binom{2}{2}$ – Report chosen out of set of one such Reports,

$\binom{1}{1}$ – Queen chosen out of set of 1 such Queens,

$\binom{17}{3}$ – 3 cards taken from set of 17 cards player A does not have create whole “musik”.

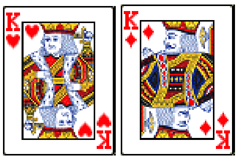
From the formula for a union of two events and values calculated in equations (11), (12) and (14):

$$P(R \cup Q) = P(R) + P(Q) - P(R_1 \cap \mathrm{Q}) - P(R_2 \cap \mathrm{Q}) - P(R_3 \cap \mathrm{Q}) = \frac{9}{136} + \frac{3}{17} - \frac{1}{680} \times 3 = \frac{81}{340} \approx 0.238.$$

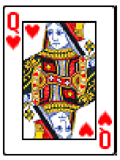
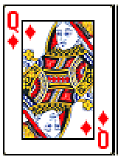
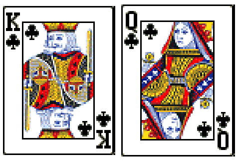
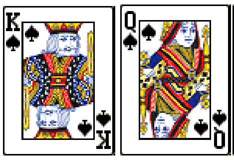
Having one half of a Report in set of cards, increases the probability of having whole Report with the aid of “musik” a lot. On the other hand, this probability is still rather low, so player should not depend on it if they want to win the game.

If player A has 2 parts of Reports.

According to made assumptions, player A has:



And is searching for:

 or  or  or  in “musik”.

In this case player A may get a Report by finding one of available 2 in “musik”. Thus let R denote any Report found in “musik”, so $R = R_3 \cup R_4$.

Using a value of probability of finding one particular Report in “musik” (10) and remembering about the mutual disjunction of Reports in “musik”, probability of finding whole Report is:

$$P(R) = \frac{3}{136} + \frac{3}{136} = \frac{6}{136}. \tag{15}$$

As here player A may find one of two matching Queens in “musik”, let Q denote any Queen found in “musik”, thus $Q = Q_1 \cup Q_2$. These events are not mutually disjoint and thus a I will use common formula for the union of two events:

$$P(Q) = P(Q_1) + P(Q_2) - P(Q_1 \cap Q_2). \tag{16}$$

Formula allowing to calculate union of events requires probability of common part of two events. Probability of finding 2 particular Queens in “musik” is:

$$P(Q_1 \cap Q_2) = \frac{\binom{2}{2}\binom{15}{1}}{\binom{17}{3}} = \frac{15}{680}, \tag{17}$$

where:

$\binom{2}{2}$ - 2 out of 2 Queens which may be present in “musik”,

$\binom{15}{1}$ - 1 card out of 15 which are not Queens and also can be present in “musik”,

$\binom{17}{3}$ - 3 cards in “musik” out of set of all 17 cards not known to player A, which may be present in “musik”.

Probability of finding a particular Queen in “musik” was calculated in equation (12). With formula (16):

$$P(Q) = P(Q_1) + P(Q_2) - P(Q_1 \cap Q_2) = \frac{3}{17} + \frac{3}{17} - \frac{15}{680} = \frac{45}{136}. \tag{18}$$

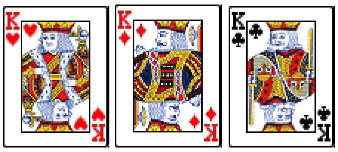
Probability of finding a whole Report or a Queen in “musik” can be calculated from the common formula for the union of not disjoint events (13), with the values calculated in equations (15), (18) and (14):

$$P(R \cup Q) = P(R) + P(Q) - P(R_3 \cap Q_1) - P(R_3 \cap Q_2) - P(R_4 \cap Q_1) - P(R_4 \cap Q_2) = \frac{6}{136} + \frac{45}{136} - \frac{1}{680} \times 4 = \frac{251}{680} \approx 0.369.$$

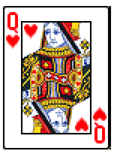
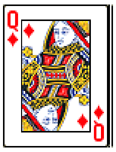
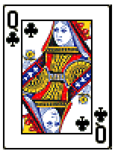
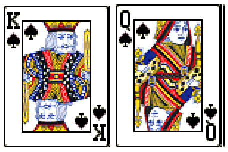
The probability of having a whole Report thanks to cards in “musik” continues to increase as more parts of the Reports player A already has in the set. With two parts of Reports obtained value is still not the highest and should not be relied on.

<u>If player A has 3 parts of Reports.</u>

According to made assumptions, player A has:



And is searching for:

 or  or  or  in “musik”.

In this set, player A may only find one Report and thus, $R = R_4$.

Here, player A to create a Report may find in “musik” one of 3 Queens available. Let then Q denote any Queen found in “musik”, thus $Q = Q_1 \cup Q_2 \cup Q_3$. These events are not mutually disjoint and thus a I will use common formula for the union of three events (6) with changed variables:

$$P(Q) = P(Q_1) + P(Q_2) + P(Q_3) - P(Q_1 \cap Q_2) - P(Q_1 \cap Q_3) - P(Q_2 \cap Q_3) + P(Q_1 \cap Q_2 \cap Q_3)\ . \tag{19}$$

It is important to note that
$P(Q_1 \cap Q_2) = P(Q_1 \cap Q_3) = P(Q_2 \cap Q_3)$.

Probability of finding 3 Queens in “musik” is:

$$P(Q_1 \cap Q_2 \cap Q_3) = \frac{\binom{3}{3}}{\binom{17}{3}} = \frac{1}{680}, \tag{20}$$

where:
$\binom{3}{3}$ - 3 out of 3 Queens which may be present in “musik”,
$\binom{17}{3}$ - 3 cards in “musik” out of set of all 17 cards not known to player A, which may be present in musik.

Probability of finding a Queen in “musik” can be calculated from the formula (19) with the usage of values calculated in equations (12), (17) and (20):

$$P(Q) = \frac{3}{17} \times 3 - \frac{15}{680} \times 3 + \frac{1}{680} = \frac{79}{170}. \tag{21}$$

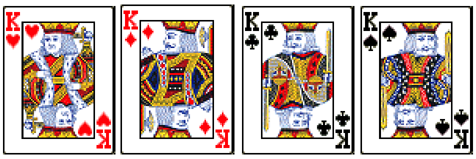
Probability of finding a whole Report or a Queen in “musik” can be calculated from the formula (13) with the values calculated in equations (10), (21) and (14):

$$P(R \cup Q) = P(R) + P(Q) - P(R_4 \cap \mathrm{Q}_1) - P(R_4 \cap \mathrm{Q}_2) - P(R_4 \cap \mathrm{Q}_3) = \frac{3}{136} + \frac{79}{170} - \frac{1}{680} \times 3 = \frac{41}{85} \approx 0.482.$$

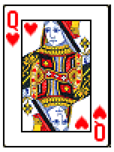
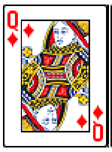
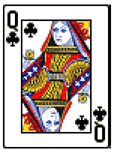
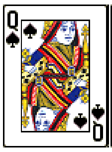
With 3 parts of the Reports, probability of obtaining a whole Report using “musik” increases as previously. Value reaches almost 0.5 what shows that it is more and more probable to have Report thanks to “musik”.

<u>If player A has 4 parts of the Reports.</u>

According to made assumptions, player A has:



And is searching for:

 or  or  or  in “musik”.

With this set of cards, player A may only find a Queen in “musik”. Q denote any Queen found in “musik”, thus $Q = Q_1 \cup Q_2 \cup Q_3 \cup Q_4$ which are not disjoint events as Queen is a single card and 3 such cards may be present in “musik”. Thus I will use a formula for union of four events, which is

$$P(Q) = P(Q_1) + P(Q_2) + P(Q_3) + P(Q_4) - P(Q_1 \cap Q_2) - P(Q_1 \cap Q_3) - P(Q_1 \cap Q_4) - P(Q_2 \cap Q_3) - P(Q_2 \cap Q_4) - P(Q_3 \cap Q_4) + P(Q_1 \cap Q_2 \cap Q_3) + P(Q_1 \cap Q_2 \cap Q_4) + P(Q_1 \cap Q_3 \cap Q_4) + P(Q_2 \cap Q_3 \cap Q_4) - P(Q_1 \cap Q_2 \cap Q_3 \cap Q_4). \quad (22)$$

In this case it is however important to note that:
$Q_1 \cap Q_2 \cap Q_3 \cap Q_4 = \emptyset$,
because there are only 3 cards present in “musik”.
Thus,
$P(Q_1 \cap Q_2 \cap Q_3 \cap Q_4) = 0$.

Probability of finding a Queen in “musik” may be calculated from the formula (22) with the usage of values calculated in equations (12), (17) and (20):
$P(Q) = \frac{3}{17} \times 4 - \frac{15}{680} \times 6 + \frac{1}{680} \times 4 = \frac{197}{340} \approx 0.579.$

Case in which player A holds all Kings in his or her set appears to be the most successful if player wants to obtain a Report using “musik”. I was always tempted to try the luck and declare high score just to win the Auction and obtain “musik” in this case and it appears that there is a reasonale explanation for my feeling. In this case it is more probable to find in “musik” matching part of the Report than not to find it. On the other hand, in almost half of the situations part of the Report will not be found in “musik” so players should not base their whole strategy on Report which they wish to find in “musik”.

Analysis of possibility of obtaining a Report using “musik” clearly shows that in any case the likelihood of such event is not as high as one would wish. Players should then resist the temptation declaring high scores during the Auction to pick up “musik” because it usually will not pay-off. Much better strategy is to relay on cards one already has and declare the score according to them.

Conclusions.

Game of 1000 is a quite advanced game. There are many possible arrangements of cards - in the Auction stage, for every of 346104 arrangements of known to the player 7 cards, there many possible arrangements of 3 unknown cards which are in “musik” and are thus interesting to the player. Because of this high numbers of possible card arrangements it is pracitically to analyze all of them. On the other hand, such analysis would be also useless, because nobody would be able to use such analysis (neither remembering of all possible arrangement is possible nor arranging a guide clear enough to use it during the game). Instead, a set of more general rules was investigated to determine rules according to which the Game of 1000 works. This analysis gives insight into what is possible during a game and shows an estimation of probability of occurrence of many events.

Analysis of strategies using one suit only, shows that there are many situations in which keeping all cards in this chosen suit is either sure or impossible. From all considered arrangements of cards in one suit, I found only 4 arrangement where success is possible but not sure. There are 2 arrangements in which probability of success exceeds 0.5, and in one of them it even reaches 0.8. In case of other 2 sets of cards it is probably unwise to risk it, as probability of keeping all cards is about 0.267 and 0.123. Of course considered arrangements are connected with classification of sets of cards into groups, but this classification allows for using derived examples quickly and easily. Change of perspective from which sets of cards are considered changes the probability of success by just a small fraction, yet it even better shows that the situation with the smallest possibility of occurrence should not be taken into account while playing game of 1000. Having 2 cards in one suit, it is almost impossible not to lose them (unless these are an Ace and a Ten).

Analysis of 2 full sets of cards shows that some tempting sets of cards in fact do not show a high possibility of collecting the highest possible number of points. On the other hand, there are some with which it is possible to collect many points. Sometimes to increase the probability of winning it is worth to decrease the level of expectations a bit, just to be more sure of success.

Then, the Auction stage was analyzed. On this stage the most important question I am always asking myself is if there is a whole Report in “musik” or if there is a part of Report which suits me. When it is necessary to count on full Report in “musik” because player has no parts of Reports in his or her set, the probability of finding such is rather low – about 0.0882. The more parts of the Reports player finds in his or her set, the greater chance there is to have a report thanks to “musik”. Unfortunately, in the best case, so when player holds 4 parts of the Reports, the probability of finding a matching cards is less than 0.6. This probability is too low to base whole strategy on it. in general, calculated probabilities give an idea of to what extent can a player count on the content of “musik” and it shows that it is often not wise to do so.

In general, this analysis taught me a lot about the Game of 1000 and showed me in which parts it is wise to risk something and in which rather not. I think that this analysis, together with a lot of experience in playing can ensure many successes. Thanks to this analysis I also know how to calculate probabilities of winning in other interesting cases such as if it is possible to find an Ace in “musik” or how to act while having 2 Reports which do not give a certainty of winning. Thanks to the Exploration I will be able to solve all my doubts concerning the strategy I should use while playing Game of 1000, what will for sure allow me to win it more often.