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Mathematical Exploration

Title:
Two-dimensional Transformations and Matrices — Drawing a Picture Mathematically.

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Rationale

Ten years ago my mother, motivated by my joyful sketching at home, signed me up for painting classes - I fell in love with the art - especially the shapes. Simultaneously at school, I discovered the concept of geometry - shapes again. Although geometry itself seemed not to be my favorite area of Mathematics, the way the graphs of functions change because of transforming them has inspired me: in particular, a situation when my teacher was introducing the concept of the arcus-functions - inverse trigonometric functions…

He drew sine function and line $y=x$ on the board, then he reflected some chosen points of sine function in the line and by connecting them, drew a new graph, whose part constituted new function: arcus-sine. Although I did not really focus on the way the function was drawn or what such transformation was really about, when a few months later my little sister asked me: "Could you draw a picture for me?", I recalled this lesson of Mathematics. As in the moment she asked I was using my computer to draw the graphs of functions for my physics laboratory report, three concepts: "functions", "computer" and "drawing" connected into one thought: let's use computer (a graphical program that would use some transformations of line-segments and points) to create a beautiful picture for my sister. I had heard the term "vector graphics" before, but I did not really know how such programs work, so I started researching. I stumbled upon matrices and linear transformations soon. I concluded that using a program and just clicking icons is one way of dealing with the task of drawing a picture, but the second way might be more interesting, and more developing, i.e. to draw the picture mathematically, by firstly thoroughly understanding the concept of transformations, linear transformations and matrices. Since I want both to deeply understand the mathematical concepts and use all of them to draw a particular picture (more details in Part 5.), I decided to focus on three basic transformations: rotation, reflection and scaling.

The exploration shows the way I dealt with drawing a picture for my sister (I certainly chose the second way). I see a great advantage in such a picture, because if I have the coordinates of a few points, I can easily redraw the same picture infinitely many times, but also, applying some transformations, I can change and ameliorate the picture; everything thanks to the creative use of Mathematics in Art.

The first 4 Parts of the Exploration focus on the purely Mathematical investigation of the transformations, then, in the fifth Part in which I describe what the process of drawing the picture looks like, I use the transformation matrices from the first parts and hence, at the end, I obtain a picture for my sister.

At the end, I just want to emphasize that the picture that I draw might be easily created in some graphical programs, as drawing basic transformations is one of these programs' function, but the understanding of the process of creating a picture, important for me as an artist and as a passionate student of Mathematics is possible only by creating and following the detailed step by step mathematical instruction described in Part 5.

1. Introduction

The exploration will base upon three popular transformations: rotation, reflection and scaling, and their matrices, which interested me because of the desire to draw a picture mathematically where these transformations are necessary. I will focus on the two-dimensional space and hence all used matrices will be of order *2x2.* Let's start from the concept of linear transformations (see [1]).

Definition 1.

The linear transformation L of the vector space U such that $L: R^2 \rightarrow R^2$ is linear, if and only if the below conditions are fulfilled

1. $L\left(\overrightarrow{u_1}+\overrightarrow{u_2}\right)=L\left(\overrightarrow{u_1}\right)+L\left(\overrightarrow{u_2}\right),\ \overrightarrow{u_1},\overrightarrow{u_2} \in R;$
2. $L\left(\alpha\overrightarrow{u_1}\right)=\alpha L\overrightarrow{u_1}$, for every $\overrightarrow{u_1} \in U$ and for any scalar α.

Since the transformations with which we will deal are in the two-dimensional space, the most convenient form of presenting them mathematically is the matrix form.

Theorem 1.

Consider a real matrix of order *2*x*2* as *A*. *T* is a linear transformation that maps R^2 to R^2 if and only if the equation

$$T\left(\vec{v}\right)=A\vec{v} \tag{1}$$

is true for some matrix *A*, which is called the transformation matrix. $\vec{v}$ is a position vector of a point with coordinates *(x,y)* and thus the equation (1) can be written in a form

$$\begin{bmatrix} x' \\ y' \end{bmatrix} = A \cdot \begin{bmatrix} x \\ y \end{bmatrix},$$

where:

$-x', y'-$ coordinates of a transformed point,

$-A-$ is a 2×2 matrix,

$-x, y-$ coordinates of an original point.

The proof of this theorem is standard and can be found in many papers regarding Linear Algebra, I found it in [2].

2. Matrices of popular linear transformations

The transformations such as rotations, scalings and reflections are mentioned in the first section, which also introduces the concept of linear transformations. However, not all of the mentioned transformations are the linear transformations. A linear transformation *L* must satisfy the following equation

$$L\left(\vec{0}\right)=\vec{0}. \tag{2}$$

Which might be proven using Definition 1. point 1. as

$$L\left(\vec{0}\right) = L\left(\vec{0}+\vec{0}\right) = L\left(\vec{0}\right) + L\left(\vec{0}\right),$$

which gives

$$L\left(\vec{0}\right)=\vec{0}.$$

Some of the mentioned transformations may satisfy (2) under certain conditions, which I will investigate in this section. I also predict that each transformation that satisfies (2) probably has its specific *2x2* transformation matrix, which I will also investigate in the Theorems 2. - 4. and their proofs.

Theorem 2.

If in the two dimensional plane we want to rotate the point *P* about the origin through the angle α and obtain point *P'*, then the transformation matrix *A* will be of a form

$A=\begin{pmatrix} \cos\beta & -\sin\beta \\ \sin\beta & \cos\beta \end{pmatrix}$, where β is an angle between the half-line l and k, which are defined as

k - the half-line orginating at the origin and passing through the point P,

l - the half-line originating at the origin and passing through the point P'.

Proof of Theorem *2*.

Firstly, consider Figure 1.

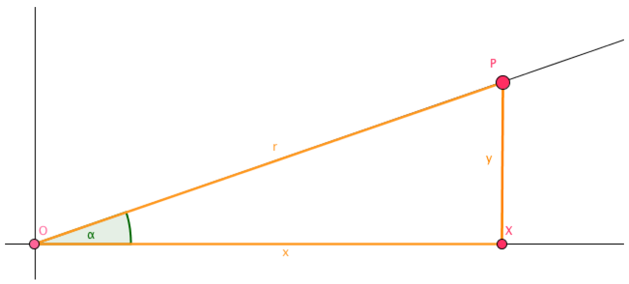

Figure 1.
Proof will base on the picture of the triangle OXP and will use the notation presented on the picture.

The triangle OXP is a right triangle, point *P* has coordinates *(x,y)*, and the hypothenuse of the triangle is *r.* We will have the following equations

$$x = r\cos\alpha, \tag{3}$$

$$y = r\sin\alpha. \tag{4}$$

Now, we apply the transformation of the graph: we rotate the point *P* through the angle β and we obtain the point *P'* (see Figure 2.).

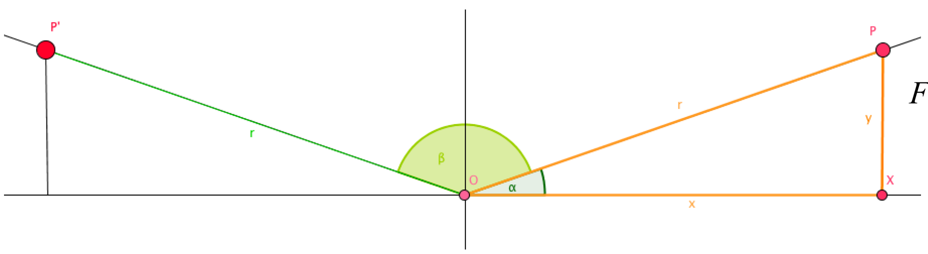

Figure 2. Point P rotated around point O.

The point *P'* has coordinates *(x', y')*. We will use trigonometry to represent these coordinates in terms of the coordinates of the original point *P* being *(x,y)*.

Hence, we will use the following trigonometric formulas for the sum of two angles

$$\begin{aligned}\cos(\alpha+\beta) &= \cos\alpha\cos\beta-\sin\alpha\sin\beta,\\ \sin(\alpha+\beta) &= \sin\alpha\cos\beta+\sin\beta\cos\alpha.\end{aligned} \tag{5}$$

We represent *x'* in terms of *x* and *y* as

$$x' = r\cos(\alpha+\beta) = r(\cos\alpha\cos\beta-\sin\alpha\sin\beta) = r\cos\alpha\cos\beta - r\sin\alpha\sin\beta,$$

and using (3), we finally have

$$x' = r\cos\alpha\cos\beta - r\sin\alpha\sin\beta = x\cos\beta - y\sin\beta. \tag{6}$$

Similarly, we represent *y'* in terms of *x* and *y* as

$$y' = r\sin(\alpha+\beta) = r(\sin\alpha\cos\beta+\sin\beta\cos\alpha) = r\sin\alpha\cos\beta + r\sin\beta\cos\alpha,$$

and using (4), we finally have

$$y' = r\sin\alpha\cos\beta + r\sin\beta\cos\alpha = y\cos\beta + x\sin\beta. \tag{7}$$

Therefore, from (6) and (7). we have the following equation

$$\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{pmatrix} \cos\beta & -\sin\beta \\ \sin\beta & \cos\beta \end{pmatrix}\begin{bmatrix} x \\ y \end{bmatrix}, \tag{8}$$

where the vector at the LHS represents the position vector of point *P'*, vector at the RHL the position vector of point *P* and the matrix at the RHL is the matrix of transformation - the matrix *A*. ∎

Theorem 3.

If in the two dimensional plane we want to reflect point *P* in line *k* which passes through the origin and thus is of a form either

$a)$ $k: y = x\tan\alpha$, where α is the angle between the *x*-axis and the line *k*,

or

$b)$ $k: x=0$ (here we take $\alpha = 90^o$),

then the transformation matrix - the matrix of reflection, matrix *A,* will be

$$A = \begin{pmatrix} \cos 2\alpha & \sin 2\alpha \\ \sin 2\alpha & -\cos 2\alpha \end{pmatrix}.$$

Proof of Theorem *3.*

Part a) of the Theorem 3.

We will base upon the graphical method. Consider Figure 3. first.

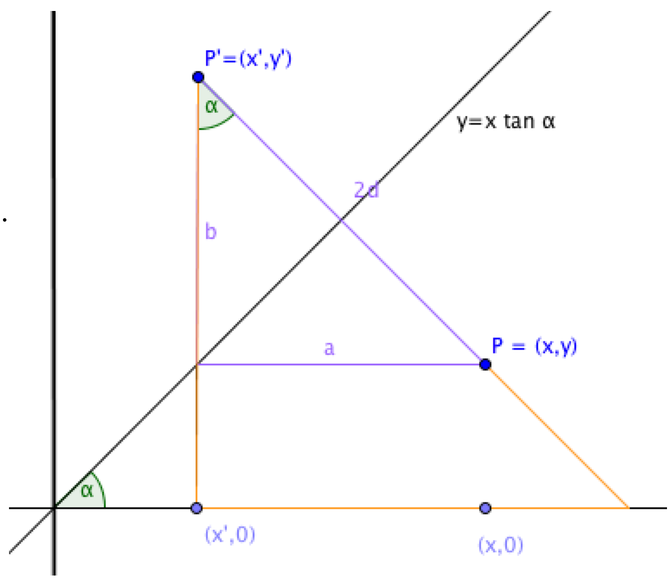

Figure 3.

(The picture is drawn not to scale).

The figure presents points P and P', P' is the reflection of P in the line y=x tan.

The distance between two points is *2d* (distance from a point to the line is *d*). We have

$a = x - x'$,

$b = y' - y$,

and

$\frac{a}{d} = \sin\alpha$.

Hence, from the basic analysis of the picture we have

$$a = 2d\sin\alpha,$$
$$b = 2d\cos\alpha, \tag{9}$$

thus,

$$x' = x - a = x - 2d\sin\alpha,$$
$$y' = y + b = y + 2d\cos\alpha. \tag{10}$$

We take the equation of line *k*

$y = x\tan\alpha = mx$,

its Cartesian form will be

$mx - y = 0$.

From a well known formula for a distance between a point and a line (see [3])

$$\text{distance} = \frac{Ax + B}{\sqrt{A^2 + B^2}}.$$

distance *d*, between point *P* and the line will be

$$d = \frac{mx - y}{\sqrt{1 + m^2}} = \frac{x\tan\alpha - y}{\sqrt{1 + \tan^2\alpha}},$$

and since

$$1 + \tan^2\alpha = \sec^2\alpha, \tag{11}$$

then

$$d = \frac{x\tan\alpha - y}{\sec\alpha} = x\frac{\tan\alpha}{\sec\alpha} - y\frac{1}{\sec\alpha} = x\sin\alpha - y\cos\alpha.$$

Hence, looking for the coordinates of point *P'*,

$$x' = x - 2d\sin\alpha = x - 2\left(x\sin\alpha - y\cos\alpha\right)\sin\alpha =$$
$$= x - 2x\sin^2\alpha + y\cdot 2\sin\alpha\cos\alpha = x(1 - 2\sin^2\alpha + y\cdot 2\sin\alpha\cos\alpha),$$

and we will use the double angle formulas (derived in (5)) being

$$\cos 2\alpha = 1 - 2\sin^2\alpha,$$
$$\sin 2\alpha = 2\sin\alpha\cos\alpha, \tag{12}$$

so,

$$x' = x\cos 2\alpha + y\sin 2\alpha, \tag{13}$$

and

$$y' = y + 2d\cos\alpha = y + 2(x\sin x - y\cos\alpha)\cos\alpha =$$
$$= y + x2\sin\alpha\cos\alpha - y2\cos^2\alpha = y(1 - 2\cos^2\alpha) + x\sin 2\alpha, \tag{14}$$

and since the formula for double angle is

$\cos 2\alpha = 2\cos^2\alpha - 1$,

then

$y' = x\sin 2\alpha - y\cos 2\alpha$.

So, from (13) and (14) we have

$$\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{pmatrix} \cos 2\alpha & \sin 2\alpha \\ \sin 2\alpha & -\cos 2\alpha \end{pmatrix} \begin{bmatrix} x \\ y \end{bmatrix}, \tag{15}$$

where the RHS matrix is the transformation matrix, hence part a) is proven.

Part b) of the Theorem 3.

The reflection in the y-axis cannot be proven according to the procedure in part a), because the angle between the x-axis and the y-axis is equal to 90° and the tangent function of 90° does not exist. However, as given on picture in Figure 4. we have the basic equation

$d = x$,

hence

$x' = x - 2d = -x$,

$y' = y$,

which leads to

$$\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix} \begin{bmatrix} x \\ y \end{bmatrix},$$

and since the angle between line k and the x-axis is 90°, we have

$$\begin{aligned} -1 &= \cos 180^\circ = \cos\left(2\cdot 90^\circ\right) \\ 1 &= -\cos 180^\circ = -\cos\left(2\cdot 90^\circ\right), \\ 0 &= \sin 180^\circ = \sin\left(2\cdot 90^\circ\right). \end{aligned} \tag{16}$$

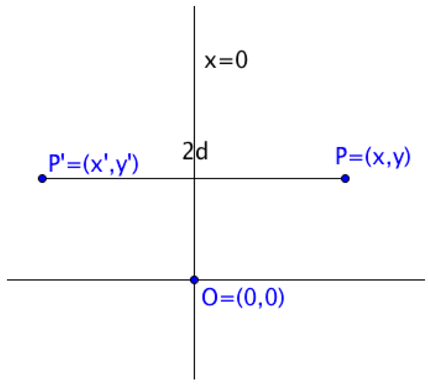

Figure 4.
Reflection in the y-axis.

Then we obtain

$$\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{pmatrix} \cos 2\alpha & \sin 2\alpha \\ \sin 2\alpha & -\cos 2\alpha \end{pmatrix} \begin{bmatrix} x \\ y \end{bmatrix}.$$

Hence, the transformation matrix from a) and b) is $A = \begin{pmatrix} \cos 2\alpha & \sin 2\alpha \\ \sin 2\alpha & -\cos 2\alpha \end{pmatrix}$,

which we wanted to prove. ∎

Theorem 4.

If in the two-dimensional plane we want to scale point P along the *x-axis* and along the *y-axis*, then the transformation matrix - the matrix of scaling, matrix *A,* will be

$$A = \begin{pmatrix} k & 0 \\ 0 & t \end{pmatrix},$$

$k, t > 0$, and k and t are the scale factors responsible for the scale along the x-axis and along the y-axis respectively. If $k = t$, then the point P will be scaled proportionally along two axes (it is also called a uniform scaling).

Proof of Theorem *4.*

Firstly, consider Figure 5., the notation used on the picture will be used in the proof.

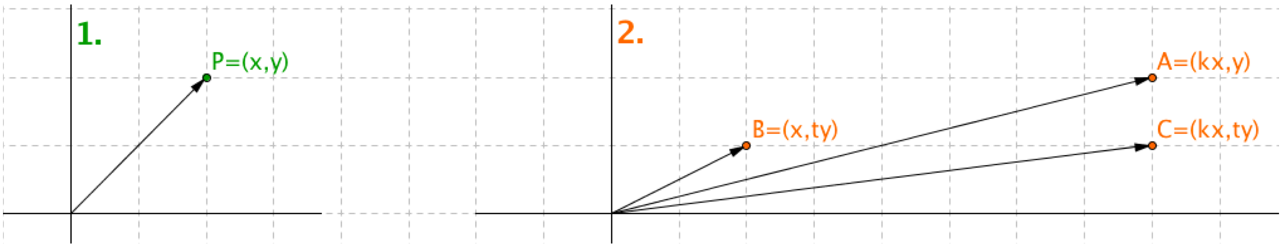

Figure 5.
1. *Point P, which will be scaled.*
2. *Three different versions of scaled point P*

A - scale along the x-axis by scale factor k
B - scale along the y-axis by scale factor t
C - scale along the x-axis by scale factor k and simultaneously along the y-axis by scale factor t.

If we want to scale the point $P=(x,y)$ by scale factor k along the x-axis, we will have
$x' = kx + 0y,$
$y' = 0x + y,$
and hence,

$$\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{pmatrix} k & 0 \\ 0 & 1 \end{pmatrix} \begin{bmatrix} x \\ y \end{bmatrix}. \tag{17}$$

Analogically, if we want to scale the point $P=(x,y)$ by scale factor t along the y-axis we will have
$x' = x + 0y,$
$y' = x + ty,$
and hence,

$$\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & t \end{pmatrix} \begin{bmatrix} x \\ y \end{bmatrix}. \tag{18}$$

If we apply both partial transformations, (17) and (18), simultaneously, we will have
$x' = kx + 0y,$
$y' = 0x + ty,$
and hence,

$$\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{pmatrix} k & 0 \\ 0 & t \end{pmatrix} \begin{bmatrix} x \\ y \end{bmatrix}. \tag{19}$$

The same applies as $k = t.$ ∎

3. More complicated transformations

In Part 2. we showed what are the matrices of linear transformations while only one transformation is applied at the time. However, I remember that at this very lesson about arcus-sine function that so much inspired me to delve into transformations, my teacher suggested that it may be easier to draw the reflection of the function in the line $y = x$ when we firstly rotate the graph of the function through the angle 90° and then reflect it in the y-axis. Basically, he said that a single transformation might be presented as two transformations applied one after another. It surprised me then, and now, when I am really interested in the matrix representations of transformations, I decided to investigate the topic further, plus, the method I will use, i.e. matrix multiplication, is inevitable to mathematically draw a picture - hence to achieve the aim of my exploration.

Lemma 1.

If the first transformation T_1 has a *2x2* matrix *A* and the second transformation T_2 has a *2x2* matrix *B*, then the composition of these transformations, $T = T_2 \circ T_1$, has a *2x2* matrix *C* which is the result of matrix multiplication in order $B \cdot A$.

Proof of Lemma *1*.

We have a position vector $\vec{u}$ of a point $P = (x,y)$, $\vec{u} \in R$,

so

$$T(\vec{u}) = T_2(T_1(\vec{u})) = T_2(A\vec{u}) = B(A\vec{u}) = (B \cdot A)\vec{u},$$

where $B \cdot A$ is the resultant transformation matrix. ∎

In Theorems 5.—7. I will focus on three different situations in which two transformations are applied at once, which are

a) Symmetry and rotation (and the second version: rotation and symmetry as matrix multiplication is non-commutative),
b) Symmetry and symmetry (both symmetries are through the different lines),
c) Rotation and rotation (both rotations are through different angles).

Theorem 5.

If, in a two dimensional plane, we want to[1]

a)

- firstly, rotate the point *P=(x,y)* through the angle β to obtain point *P'=(x',y'),*
- secondly, reflect the point *P'* in the line that creates an angle α with the positive part of the x-axis,

then the result of such transformation will be reflection of point *P* in the line that forms an angle $\alpha - \frac{\beta}{2}$ with the positive x-axis,

b)

- firstly, reflect the point *P=(x,y)* in the line that forms an angle α with the positive x-axis to obtain point *P'=(x',y'),*

[1] Refer to the variables used in Theorems 2. and 3.

• secondly, rotate the point *P'* through the angle β,
then the result of such transformation will be reflection of point *P* in the line the line that forms an angle $\alpha + \frac{\beta}{2}$ with the positive *x*-axis.

Proof of Theorem *5*.

Since we will composite two transformations, we will multiply their matrices of transformations
$B \cdot A = C$,
where
- B - matrix of symmetry, given in (15),
- A - matrix of rotation, given in (8).

and *T* will be resultant transformation.

$$a)\ B \cdot A = \begin{pmatrix} \cos 2\alpha & \sin 2\alpha \\ \sin 2\alpha & -\cos 2\alpha \end{pmatrix} \begin{pmatrix} \cos\beta & -\sin\beta \\ \sin\beta & \cos\beta \end{pmatrix} =$$

$$= \begin{pmatrix} \cos 2\alpha\cos\beta + \sin 2\alpha\cos\beta & \sin 2\alpha\cos\beta - \sin\beta\cos 2\alpha \\ \sin 2\alpha\cos\beta - \sin\beta\cos 2\alpha & -(\cos 2\alpha\cos\beta + \sin 2\alpha\sin\beta) \end{pmatrix} = \tag{20}$$

$$= \begin{pmatrix} \cos(2\alpha - \beta) & \sin(2\alpha - \beta) \\ \sin(2\alpha - \beta) & -\cos(2\alpha - \beta) \end{pmatrix},$$

by (15) we see that *T* is a reflection and it is in the line that forms an angle $\alpha - \frac{\beta}{2}$ with the positive *x*-axis.

$$\text{b)}\ A \cdot B = \begin{pmatrix} \cos\beta & -\sin\beta \\ \sin\beta & \cos\beta \end{pmatrix} \begin{pmatrix} \cos 2\alpha & \sin 2\alpha \\ \sin 2\alpha & -\cos 2\alpha \end{pmatrix} =$$

$$= \begin{pmatrix} \cos 2\alpha\cos\beta - \sin 2\alpha\sin\beta & \sin 2\alpha\cos\beta + \sin\beta\cos 2\alpha \\ \sin 2\alpha\cos\beta + \sin\beta\cos 2\alpha & -(\cos 2\alpha\cos\beta - \sin 2\alpha\sin\beta) \end{pmatrix} =$$

$$= \begin{pmatrix} \cos(2\alpha + \beta) & \sin(2\alpha + \beta) \\ \sin(2\alpha + \beta) & -\cos(2\alpha + \beta) \end{pmatrix}, \tag{21}$$

by (15) we see that *T* is a reflection and it is in the line that forms an angle $\alpha + \frac{\beta}{2}$ with the positive *x*-axis. ∎

Theorem 6.

If in a two dimensional plane, we want to

a) firstly, reflect the point *P=(x,y)* in the line that forms an angle α with the positive *x*-axis to obtain point *P'=(x',y')*,
b) secondly, reflect the point *P'* in the line that forms an angle β with the positive *x*-axis,

then we will rotate the point *P* through the angle $\gamma = 2\alpha - 2\beta$.

Proof of Theorem *6.*

According to the Lemma 1,
$B \cdot B' = C$,

$$B \cdot B' = \begin{pmatrix} \cos 2\alpha & \sin 2\alpha \\ \sin 2\alpha & -\cos 2\alpha \end{pmatrix} \begin{pmatrix} \cos 2\beta & \sin 2\beta \\ \sin 2\beta & -\cos 2\beta \end{pmatrix} =$$

$$= \begin{pmatrix} \cos 2\alpha \cos 2\beta + \sin 2\alpha \sin 2\beta & -(\sin 2\alpha \cos 2\beta - \sin 2\beta \cos 2\alpha) \\ \sin 2\alpha \cos 2\beta - \sin 2\beta \cos 2\alpha & \sin 2\alpha \sin 2\beta + \cos 2\alpha \cos 2\beta \end{pmatrix} = \tag{22}$$

$$= \begin{pmatrix} \cos(2\alpha - 2\beta) & -\sin(2\alpha - 2\beta) \\ \sin(2\alpha - 2\beta) & \cos(2\alpha - 2\beta) \end{pmatrix},$$

by (8) we see that *T* is a rotation and the angle of rotation is $\gamma = 2\alpha - 2\beta$. ∎

Theorem 7.

If in a two dimensional plane, we want to
a) firstly, rotate the point *P=(x,y)* through the angle α to obtain point *P'=(x',y'),*
b) secondly, rotate the point *P'* through the angle β,
then we will rotate the point *P* through the angle $\theta = \beta + \gamma$.

Proof of Theorem *7.*

From Lemma 1,
$A \cdot A' = C$,

$$A \cdot A' = \begin{pmatrix} \cos\beta & -\sin\beta \\ \sin\beta & \cos\beta \end{pmatrix} \begin{pmatrix} \cos\gamma & -\sin\gamma \\ \sin\gamma & \cos\gamma \end{pmatrix} =$$

$$= \begin{pmatrix} \cos\beta\cos\gamma - \sin\beta\sin\gamma & -(\sin\gamma\cos\beta + \sin\beta\cos\gamma) \\ \sin\beta\cos\gamma + \sin\gamma\cos\beta & \cos\beta\cos\gamma - \sin\beta\sin\gamma \end{pmatrix} = \tag{23}$$

$$= \begin{pmatrix} \cos(\beta + \gamma) & -\sin(\beta + \gamma) \\ \sin(\beta + \gamma) & \cos(\beta + \gamma) \end{pmatrix},$$

by (8) we see that *T* is the rotation and the angle of rotation is $\theta = \beta + \gamma$. ∎

4. Non-linear popular transformations

The Theorems 2.— 7. refer to the linear transformations and we will use the matrices from these Theorems in Part 5. However, what if we want to rotate or scale around the point that is not the origin or reflect in the line that does not pass through the origin? Well, such transformations have a similar form, slightly modified by addition of the translation vector. How this is applied to the particular transformations is shown in Theorems 8. —10.

Theorem 8.

In the two-dimensional space rotation around point *(a,b)*, different from point *(0,0)*, through angle β, will have a form

$$T\left(\vec{v}\right)=A\left(\vec{v}-\vec{u}\right)+\vec{u}, \tag{24}$$

where

$-A-$ rotation matrix, given in (8),

$-\vec{v}-$ position vector of the point $P=(x,y)$,

$-\vec{u}-$ translation vector of a form $\begin{bmatrix} a \\ b \end{bmatrix}$.

Proof of Theorem 8.

Let's say that we want to rotate point $P=(x,y)$ around point $Z=(a,b)$ with the angle of rotation α. We may equivalently, as rotation does not change the distances,

a) translate the point *P* by vector $\begin{bmatrix} -a \\ -b \end{bmatrix}$, so that point *(a,b)* is mapped on point *(0,0)*,
b) rotate the point *P'* linearly, around point *(0,0)*, using the rotation matrix given in Theorem 2.,
c) translate the point *P'* by vector $\begin{bmatrix} a \\ b \end{bmatrix}$,

and in fact

$$\begin{bmatrix} x' \\ y' \end{bmatrix}=A\cdot\begin{bmatrix} x-a \\ y-b \end{bmatrix}+\begin{bmatrix} a \\ b \end{bmatrix}, \tag{25}$$

where

- matrix of transformation (here rotation matrix, given in (8)),
- x', y' - coordinates of the rotated point,
- a, b - coordinates of the point around which the rotation takes place,
- x, y - coordinates of the point P,

which is equivalent to the equation in Theorem 8. ∎

Theorem 9.

In the two-dimensional space a symmetry through the line that passes point *(a,b)*, different from point *(0,0)*, will have a form

$$T\left(\vec{v}\right)=A\left(\vec{v}-\vec{u}\right)+\vec{u},$$

where

$-A-$ matrix of symmetry, given in (15),

$-\vec{v}-$ position vector of point $P=(x,y)$,

$-\vec{u}-$ translation vector of a form $\begin{bmatrix} a \\ b \end{bmatrix}$.

Proof of Theorem *9.* is analogical to the Proof of Theorem *8.,* uses the same procedure, points a)-c).

Theorem 10.

In two-dimensional space scaling around point *(a,b)*, different from point *(0,0)*, will have a form $T(\vec{v}) = A(\vec{v} - \vec{u}) + \vec{u}$,

where

$-A-$ scaling matrix, given in (19),

$-\vec{v}-$ position vector of point $P = (x, y)$,

$-\vec{u}-$ translation vector of a form $\begin{bmatrix} a \\ b \end{bmatrix}$.

Proof of Theorem *10.* is analogical to the Proof of Theorem *8.,* uses the same procedure, points a)-c).

5. Application of the theory drawing a picture

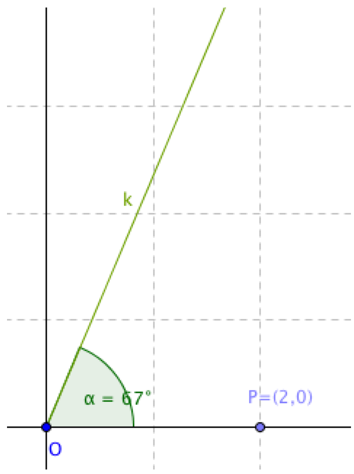

Figure 6.
There are two lines and one point in the picture. The coordinates of the point and the angle between the lines do not have any special properties, by starting drawing a picture by taking such data, I want to show that by using matrices of linear transformations, we can draw any desired picture.

Since the main aim of my Exploration is to show how the "mathematical drawing" works, this part allows me to achieve my aim. In this Part I will use the Theorems and Definitions given in the first four Parts of the Exploration. The Theorems and Proves gave me the opportunity to deeply understand the mathematical operations that I apply in the second part. The way I achieved my aim is a step by step process, which sometimes might be a bit monotonic and complicated, but after all, seems to be the best way to "draw" a desired picture.

I started the process of drawing the picture by establishing a task: take two lines, indicate a point on one of them. I found on my desk a picture similar to the one in Figure 6., the measured angle and distance are indicated on the picture. All pictures included in this Part of Exploration were drawn using the basic functions of the graphical program *Geogebra*. However, I did not use the functions such as linear transformations, I was only drawing points and line segments to illustrate my calculations.

Then I began the procedure aiming at drawing a cat most similar to the one presented on the picture (see Appendix). I predict the result of my work to be similar to the sketch in Figure 7. The steps of my working are given below.

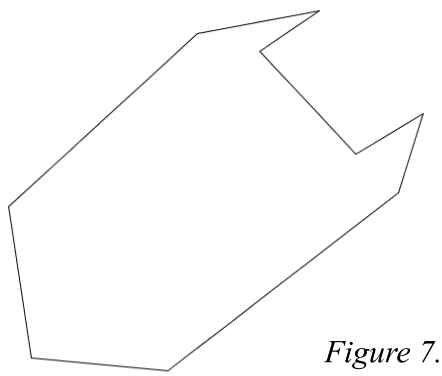

Figure 7.
Sketch of the picture which I predict to obtain, so that it is at least to some extent similar to the picture given in the Appendix.

Step 1. Anti-clockwise rotation of point P around the origin, point O, so that the line k is mapped onto the y-axis. We will use the matrix given in Theorem 2. The angle of rotation is $\beta = 90^\circ - 67^\circ = 23^\circ$. Point P has coordinates: $x = 2$, $y = 0$, and we want to find the coordinates of its image after applying rotation, which will be the coordinates of point A.

Hence, taking the matrix from (8), to find the coordinates of point A, x_A, y_A, we will have

$$\begin{bmatrix} x_A \\ y_A \end{bmatrix} = \begin{pmatrix} \cos 23^\circ & -\sin 23^\circ \\ \sin 23^\circ & \cos 23^\circ \end{pmatrix} \begin{bmatrix} 2 \\ 0 \end{bmatrix},$$

$$\begin{bmatrix} x_A \\ y_A \end{bmatrix} = \begin{pmatrix} 2\cos 23^\circ \\ 2\sin 23^\circ \end{pmatrix}, \tag{26}$$

$x_A = 2\cos 23^\circ$,

$y_A = 2\sin 23^\circ$.

After step 1. the picture from Figure 7. becomes vertical and thus now, it is possible to sketch the desired picture with a greater number of details, see Figure 8.

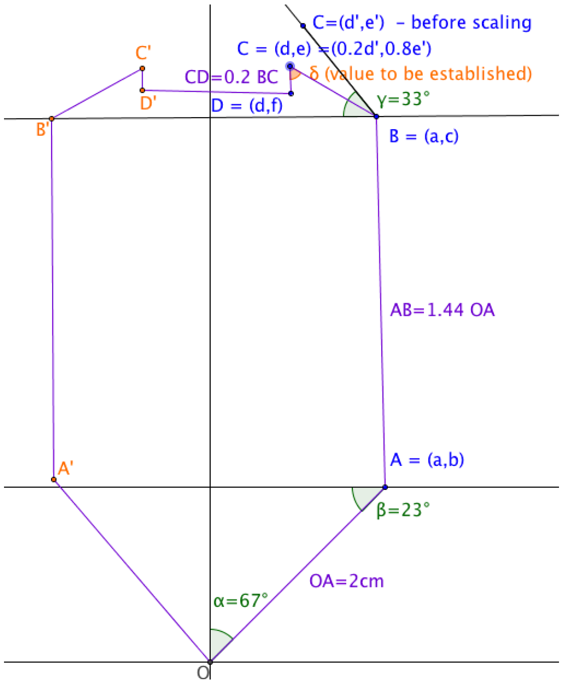

Figure 8.
More detailed sketch of the picture that I want to obtain. The values of the angles and line-segments are given and will be used while applying the transformations in the next steps.

Step 2. Clockwise rotation of point O around point A, scaling the rotated point O along the y-axis by scale factor 1.44, and obtaining point B. We will use similar matrix as in previous step, however, this rotation is clockwise so the transformation matrix is slightly changed. As indicated on the sketch in Figure 6, the angle of rotation is $\beta = 90^\circ + 23^\circ = 113^\circ$.

Hence, taking matrices from (19) and (8) but taking into account that the rotation is clockwise rotation, using Theorem 8. and Lemma 1., we have the following equation

$$\begin{bmatrix} x_B \\ y_B \end{bmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1.44 \end{pmatrix} \begin{pmatrix} \cos 113^\circ & \sin 113^\circ \\ -\sin 113^\circ & \cos 113^\circ \end{pmatrix} \begin{bmatrix} x - x_A \\ y - y_A \end{bmatrix} + \begin{bmatrix} x_A \\ y_A \end{bmatrix},$$

$$\begin{bmatrix} x_B \\ y_B \end{bmatrix} = \begin{pmatrix} \cos 113^\circ & \sin 113^\circ \\ -1.44\sin 113^\circ & 1.44\cos 113^\circ \end{pmatrix} \begin{bmatrix} x - 2\cos 23^\circ \\ y - 2\sin 23^\circ \end{bmatrix} + \begin{bmatrix} 2\cos 23^\circ \\ 2\sin 23^\circ \end{bmatrix} =$$

$$= \begin{bmatrix} \cos 113^\circ (x - 2\cos 23^\circ) + \sin 113^\circ (y - 2\sin 23^\circ) \\ -1.44\sin 113^\circ (x - 2\cos 23^\circ) + 1.44\cos 113^\circ (y - 2\sin 23^\circ) \end{bmatrix} + \begin{bmatrix} 2\cos 23^\circ \\ 2\sin 23^\circ \end{bmatrix}, \tag{27}$$

since $x = 0$ and $y = 0$, we will have

$x_B = 2\cos 23^\circ$,

$y_B = 2\sin 23^\circ + 2.88$.

Step 3. Clockwise rotation of point A around point B, scaling the rotated point A along the x-axis by scale factor 0.2 and along y-axis by scale factor 0.8 and obtaining point C. The angle of rotation is $\beta = 90^\circ + 33^\circ = 123^\circ$.

So that, similarly as in the previous point, we obtain the equation

$$\begin{bmatrix} x_C \\ y_C \end{bmatrix} = \begin{pmatrix} 0.2 & 0 \\ 0 & 0.8 \end{pmatrix} \begin{pmatrix} \cos 123^\circ & \sin 123^\circ \\ -\sin 123^\circ & \cos 123^\circ \end{pmatrix} \begin{bmatrix} x_A - x_B \\ y_A - y_B \end{bmatrix} + \begin{bmatrix} x_B \\ y_B \end{bmatrix},$$

$$\begin{bmatrix} x_C \\ y_C \end{bmatrix} = \begin{pmatrix} 0.4\cos 123^\circ & 0.3\sin 123^\circ \\ -0.8\sin 123^\circ & 0.8\cos 123^\circ \end{pmatrix} \begin{bmatrix} 2\cos 23^\circ - 2\cos 23^\circ \\ 2\sin 23^\circ - (2\sin 23^\circ + 2.88) \end{bmatrix} + \begin{bmatrix} 2\cos 23^\circ \\ 2\sin 23^\circ + 2.88 \end{bmatrix} =$$

$$= \begin{pmatrix} 0.4\cos 123^\circ & 0.4\sin 123^\circ \\ -0.8\sin 123^\circ & 0.8\cos 123^\circ \end{pmatrix} \begin{bmatrix} 0 \\ -2.88 \end{bmatrix} + \begin{bmatrix} 2\cos 23^\circ \\ 2\sin 23^\circ + 2.88 \end{bmatrix} = \tag{28}$$

$$= \begin{bmatrix} -1.15\sin 123^\circ \\ -2.30\cos 123^\circ \end{bmatrix} + \begin{bmatrix} 2\cos 23^\circ \\ 2\sin 23^\circ + 2.88 \end{bmatrix},$$

hence

$$x_C = -1.15\sin 123^\circ + 2\cos 23^\circ,$$
$$y_C = -2.30\cos 123^\circ + 2\sin 23^\circ + 2.88.$$

Step 4. Clockwise rotation of point *B* around point *C*, scaling the rotated point *B* along *y*-axis by scale factor 0.2 and obtaining point *D*. However, we do not know yet the measure of the angle through which this rotation will be done. According to the sketch in the Figure 8. we only know that the *x*-coordinate of point *D* has the same *x*-coordinate as the point *C*.

Firstly, we need to find an angle (see Figure 8.). We get

$$\gamma = \arctan\left(\frac{x_B - x_C}{y_C - y_B}\right) = \arctan\left(\frac{2\cos 23^\circ - \left(-1.15\sin 123^\circ + \left(2\cos 23^\circ\right)\right)}{-2.3\cos 123^\circ + 2\sin 23^\circ + 2.88 - \left(2\sin 23^\circ + 2.88\right)}\right) =$$

$$= \arctan\left(\frac{1.15\sin 123^\circ}{-2.3\cos 123^\circ}\right) \approx 37.6^\circ. \tag{29}$$

Having (29), we may apply both transformations as given in the description of step 4., thus

$$\begin{bmatrix} x_D \\ y_D \end{bmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 0.2 \end{pmatrix} \begin{pmatrix} \cos\gamma & \sin\gamma \\ -\sin\gamma & \cos\gamma \end{pmatrix} \begin{bmatrix} x_B - x_C \\ y_B - y_C \end{bmatrix} + \begin{bmatrix} x_C \\ y_C \end{bmatrix},$$

$$\begin{bmatrix} x_D \\ y_D \end{bmatrix} = \begin{pmatrix} \cos\gamma & \sin\gamma \\ -0.2\sin\gamma & 0.2\cos\gamma \end{pmatrix} \begin{bmatrix} 2\cos 23^\circ - (-1.15\sin 123^\circ + 2\cos 23^\circ) \\ 2\sin 23^\circ + 2.88 - \left(-2.3\cos 123^\circ + 2\sin 23^\circ + 2.88\right) \end{bmatrix} + \begin{bmatrix} x_C \\ y_C \end{bmatrix} =$$

$$= \begin{pmatrix} \cos\gamma & \mathrm{sins}\gamma \\ -0.2\sin\gamma & 0.2\cos\gamma \end{pmatrix} \begin{bmatrix} 1.15\sin 123^\circ \\ 2.3\cos 123^\circ \end{bmatrix} + \begin{bmatrix} -0.576\sin 123^\circ + 2\cos 23^\circ \\ -2.3\cos 123^\circ + 2\sin 23^\circ + 2.88 \end{bmatrix},$$

and as

$$\begin{bmatrix} x_D \\ y_D \end{bmatrix} = \begin{bmatrix} 1.15\cos\gamma\sin 123^\circ + 2.3\sin\gamma\cos 123^\circ \\ -0.23\sin\gamma\sin 123^\circ + 0.46\cos\gamma\cos 123^\circ \end{bmatrix} + \begin{bmatrix} -1.15\sin 123^\circ + 2\cos 23^\circ \\ -2.3\cos 123^\circ + 2\sin 23^\circ + 2.88 \end{bmatrix} =$$

$$= \begin{bmatrix} 0 \\ -0.23\sin\gamma\sin 123^\circ + 0.46\cos\gamma\cos 123^\circ \end{bmatrix} + \begin{bmatrix} -1.15\sin 123^\circ + 2\cos 23^\circ \\ -2.3\cos 123^\circ + 2\sin 23^\circ + 2.88 \end{bmatrix}, \tag{30}$$

$$x_D = x_C = -1.15\sin 123^\circ + 2\cos 23^\circ,$$
$$y_D = -0.23\sin\gamma(\sin 123^\circ) + 0.46\cos\gamma(\cos 123^\circ) - 2.3\cos 123^\circ + 2\sin 23^\circ + 2.88.$$

Therefore, according to the sketch in Figure 8. we have four basic points, their coordinates, rounded to three significant figures. are given in the Table 1. These points will be reflected in step 5. in the y-axis to create a desired picture.

Table 1. The points which will be reflected in the y-axis with given coordinates rounded to three significant figures.

Point	x - coordinate	y - coordinate
O	0	0
A	1,84	0,781
B	1,84	3,66
C	0,877	4,91
D	0,877	4,60

Step 5. Reflection of points O, A, B, C and D in the y-axis. See (16) from which we have

$$\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{pmatrix} \cos 180^\circ & \sin 180^\circ \\ \sin 180^\circ & -\cos 180^\circ \end{pmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix} \begin{bmatrix} x \\ y \end{bmatrix}, \qquad (31)$$

$x' = -x,$

$y' = y,$

where x and y are the coordinates of any point from points: A, B, C, D.

Table 2. Points and their coordinates needed to draw the picture.

	Point	x - coordinate	y - coordinate
1.	O	0	0
2.	A	1,84	0,781
3.	B	1,84	3,66
4.	C	0,877	4,91
5.	D	0,877	4,60
6.	D’	-0,877	4,60
7.	C’	-0,877	4,91
8.	B’	-1,84	3,66
9.	A’	-1,84	0,781
10.	O’	0	0

The coordinates of reflected points together with the initial points are given in Table 2.

Step 6. Now, using Table 2. and the given coordinates, we will connect each point with the subsequent one, i.e. point 1. with point 2. Point 1. and point 10. have the same coordinates, as reflection of point $O = (0,0)$ in the y-axis results in the point’s map onto itself.

Hence, when we input all the points and link them as given in the above instruction, we will obtain a desired picture of a cat (see Figure 9.). However, to achieve our aim, i.e. to make the picture more similar to the picture in Figure 7., we will rotate and then scale the picture.

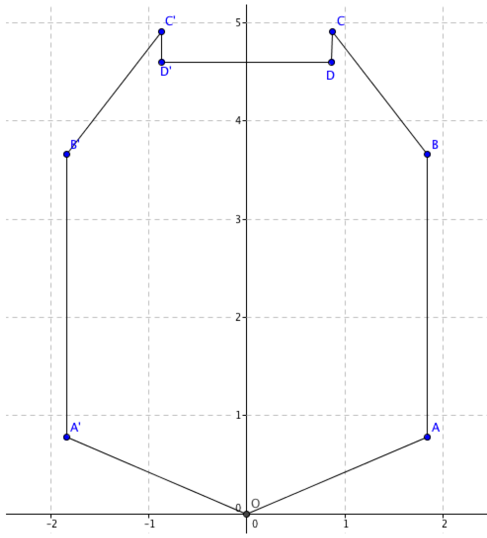

Figure 9. Picture obtained after applying transformations to respective points in steps 1 - 5 and after connecting the points according to the instruction given in step 6.

Step 7. Clockwise rotation of all points around the origin, angle of rotation is $\alpha = 56^\circ$, scaling all the points along the x-axis and the y-axis, both by scale factor 4.5.

We obtain the following equation

$$\begin{bmatrix} x'' \\ y'' \end{bmatrix} = \begin{pmatrix} 4.5 & 0 \\ 0 & 4.5 \end{pmatrix} \begin{pmatrix} \cos 56^\circ & \sin 56^\circ \\ -\sin 56^\circ & \cos 56^\circ \end{pmatrix} \begin{bmatrix} x \\ y \end{bmatrix} =$$

$$= \begin{pmatrix} 4.5\cos 56^\circ & 4.5\sin 56^\circ \\ -4.5\sin 56^\circ & 4.5\cos 56^\circ \end{pmatrix} \begin{bmatrix} x \\ y \end{bmatrix}, \qquad (32)$$

$x'' = 4.5\cos 56^\circ x + 4.5\sin 56^\circ y,$

$y'' = -4.5\sin 56 x + 4.5\cos 56^\circ y,$

where x and y are the coordinates of any point from points: A, B, C, D, A', B', C', D', and x'' and y'' are the coordinates of the final points of the picture (after being rotated and scaled).

Since we have the coordinates of all the points that constitute our picture in Table 2, we will use (32) to establish the coordinates of the final points of the picture. They are given in Table 3. and they are rounded to 3 significant figures.

Table 3. The coordinates of the final points of the desired picture calculated basing on (32).

Point	x - coordinate	y - coordinate
O	0	0
A	7.54	-4.90
B	18.3	2.35
C	20.5	9.08
D	19.4	8.30
D’	15.0	14.9
C’	16.1	15.6
B’	9.02	16.1
A’	-1.72	8.83
O’	0	0

The picture is ready now, i.e. if the step 6 was already done and if the graphics is being made in the computer program which adjusts the line-segments AB, BC, CD, etc. to the change of the coordinates of the points, if not the step 6 should be repeated. Step 6 must be also repeated if we are given only the points in Table 3. Picture given in Figure 10. should be the result.

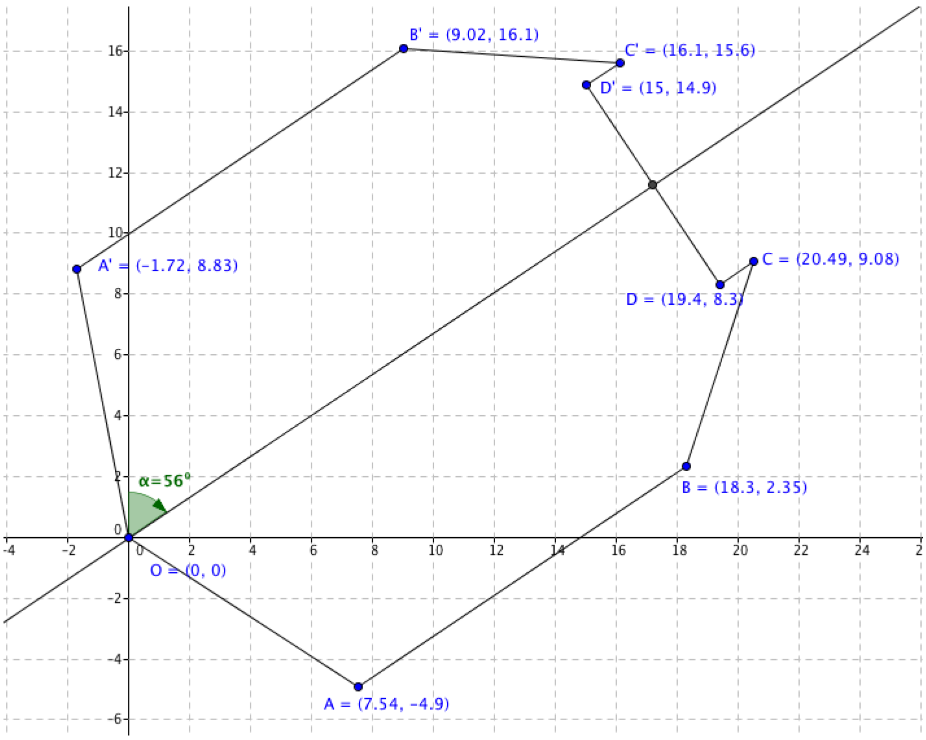

Figure 10. The final picture obtained by following steps 1-7 or by drawing and connecting the points given in Table 3 which makes the picture repeatable and showing that was one of the aim of the Exploration.

6. Reflections and conclusion

Applying linear and non-linear transformations, rotations, reflections and scalings in particular, and using the matrices of these transformations appeared to be an effective way of achieving the aim of my Exploration — drawing a picture, mathematically (see Figure 10). Simultaneously I gained the mathematical knowledge that allowed me to thoroughly understand what each applied transformation consists of and what is its specific matrix form. This basic knowledge followed by discovering the difference between the linear and non-linear transformations and investigating the concept of composition of transformations allowed me to create a step by step instruction of drawing a picture using the mathematical tools, when I was given only a point and a line at the beginning of the task.

Hence, I understood how the basic graphical programs work. Even though I could use a computer program instead of conducting calculations in part 5., I decided to create my own way of dealing with the problem. I drew a picture by applying the transformations and as a result I deepened my theoretical knowledge in the field of vector graphics, which I assess as an important area in the developing worlds of science, where the graphical models are often used, and entertainment, in computer gaming for example. I am also motivated to study in detail the concept of transformations and their practical applications in more complicated cases as three-dimensional spaces, which, however, would not be possible if I did not learn the foundations of the linear and non-linear transformations in the two-dimensional space first.

Moreover, the structure of my Exploration gave me a chance to pursue a new area of mathematics gradually and systematically which brought the desired effects. Hence, I improved a more general skill to logically and gradually learn new concepts.

Finally, the Exploration gave a lot of satisfaction, as I could use both my interest in Mathematics and Art to create a picture for my little sister, widened my knowledge and perspective on the modern world (computer graphics) and motivated to conduct further research in new for me field of Linear Algebra and transformations.

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Appendix

1. Picture on which the sketch from Figure 7. is based.

Source:
Available online:
http://www.koty.org.pl/zdjecia/dodane/49_53_87.jpg,
accessed 27th December 2013