Internal Assessment

Mathematics Higher Level

The Pascal's Pyramid

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Introduction

During one of my Mathematics classes, when talking about the Binomial Expansion I got fascinated in the Pascal's triangle. All the properties within this representation, for me, seemed to be extremely interesting. I started wondering how does something of this kind works for $(a+b+c)^n$, $n \in N$. After some research I found out that this topic is relatively unpopular, which made me even more curious about this issue. I wanted to delve into it earlier so that is why the Mathematics Internal Assessments seems to be my great opportunity to analyse this phenomenon.

The aim of this paper is to construct a three-dimensional model of the so-called "Pascal's Pyramid" and to conduct a mathematical proof of my findings. I will achieve it by analysing the most important mechanisms and properties within the pyramid, which seem to be relatively analogical to the ones in the Binomial Theorem.

Introduction To Trinomial Theorem

Knowing the mechanisms used to expand the binomial expression, it is possible to find the general representation of the trinomial expression. Firstly, let's expand the first five powers of the $(a+b+c)^n$:

For n=0

$$(a+b+c)^0 = 1 \tag{1}$$

For n=1

$$(a+b+c)^1 = a+b+c \tag{2}$$

For n=2

$$(a+b+c)^2 = (a+b+c)(a+b+c) = a^2+2ab+b^2+2bc+c^2+2ca \tag{3}$$

For n=3

$$\begin{aligned}(a+b+c)^3 = (a^2+2ab+b^2+2bc+c^2+2ca)(\mathrm{a}+\mathrm{b}+\mathrm{c}) = {} & a^3+2a^2b+ab^2 \\ & +2abc+ac^2+a^2b+2ab^2+b^3+2b^2c+bc^2+bc^2+2abc+a^2c \\ & +2abc+b^2c+2bc^2+c^3+2ac^2 \\ & = a^3+3a^2b+3ab^2+b^3+3b^2c+3bc^2+c^3+3ac^2+3a^2c+6abc\end{aligned} \tag{4}$$

For n=4

$$\begin{aligned}(a+b+c)^4 &= (a^3+3a^2b+3ab^2+b^3+3b^2c+3bc^2+c^3+3ac^2+3a^2c+6abc)(a+b+c) \\ &= a^4+4a^3b+6a^2b^2+4ab^3+b^4+4b^3c+6b^2c^2+4bc^3+c^4+4ac^3+6a^2c^2 \\ &\quad +4a^3c+12a^2bc+12ab^2c+12abc^2\end{aligned} \tag{5}$$

For n=5

$$(a+b+c)^5 = a^5 + 5a^4b + 10a^3b^2 + 10a^2b^3 + 5ab^4 + b^5 + 5b^4c + 10b^3c^2 + 10b^2c^3 + 5bc^4 + c^5 + 5ac^4 + 10a^2c^3 + 10a^3c^2 + 5a^4c + 20a^3bc + 20ab^3c + 20abc^3 + 30a^2b^2c + 30ab^2c^2 + 30a^2bc^2 \tag{6}$$

Analogically to the binomial expansion, patterns such as decreasing and increasing of powers of certain elements can be observed. These are followed by a change in the coefficient, which again, is relatively regular. Moreover, having expansion with three elements *a*, *b* and *c*, it seems as a logical conclusion that the graphic representation of certain expression should be presented in three dimensional form, which will be shown later on in this paper. The above expansions can be generalized by the Trinomial Theorem.

Theorem 1: Trinomial Theorem

Trinomial Theorem states that for $n \epsilon N$:

$$(x+y+z)^n = \sum_{\substack{0 \le a,b,c \le n \\ a+b+c=n}}^{n} \binom{a+b+c}{b+c}\binom{b+c}{c} x^a y^b z^c \tag{7}$$ [1]

Equation 7, in my notation will be stated as:

$$(a+b+c)^n = \sum_{p=0}^{n}\sum_{q=0}^{p} a^{n-p} b^{p-q} c^q \binom{n}{p}\binom{p}{q} \tag{8}$$

Proof of the Theorem 1:

On the example of the simple method of solving the equation $(a+b+c)^2$ when knowing only the solution of the squared sum of two elements $(a+b)^2$, I was able to conduct the proof of Trinomial Theorem. The method can be represented by the equation

$$(a+b+c)^2 = (a+(b+c))^2 = a^2 + 2a(b+c) + (b+c)^2 \tag{9}$$

Analogically, knowing the Binomial Theorem which is

$$(a+b)^n = a^n b^0 \binom{n}{0} + a^{n-1}b^1\binom{n}{1} + a^{n-2}b^2\binom{n}{2} + \cdots + a^0 b^n \binom{n}{n} = \sum_{k=0}^{n} a^{n-k} b^k \binom{n}{k} \tag{10}$$ [2]

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[1] R. L. Graham, D. E. Knuth, O.Patashnik, *Concrete Mathematic 2nd edn.*, Addison-Wesley Publishing Company, United States of America, 1988, p. 168.

[2] N. Buckle, I. Dunbar, *Mathematics Higher Level (Core) 3rd edn.*, IBID Press, Victoria, 2004, p.95.

I can represent the Trinomial Theorem in a similar way, initially treating *(b+c)* as a one element

$$(a+b+c)^n = (a+(b+c))^n =$$

$$= \binom{n}{0} a^n (b+c)^0 + a^{n-1}(b+c)^1 \binom{n}{1} + a^{n-2}(b+c)^2 \binom{n}{2} + \cdots$$

$$+ a^0 (b+c)^n \binom{n}{n} = \sum_{p=0}^{n} a^{n-p}(b+c)^p \binom{n}{p} = \sum_{p=0}^{n} a^{n-p} \binom{n}{p} (b+c)^p$$

$$= \sum_{p=0}^{n} a^{n-p} \binom{n}{p} \sum_{q=0}^{p} b^{p-q} c^q \binom{p}{q} = \sum_{p=0}^{n} \sum_{q=0}^{p} a^{n-p} b^{p-q} c^q \binom{n}{p} \binom{p}{q}$$

$$= \sum_{p=0}^{n} \sum_{q=0}^{p} a^{n-p} b^{p-q} c^q \frac{n!}{(n-p)!\,p!} \frac{p!}{(p-q)!\,q}$$

$$= \sum_{p=0}^{n} \sum_{q=0}^{p} a^{n-p} b^{p-q} c^q \frac{n!}{(n-p)!\,(p-q)!\,p!} \quad \blacksquare$$

(11)

Coefficients And Their Role

Now, let's focus on the mechanism within one plane, for *n* being constant. As I have already mentioned, some regularity in the values of trinomial coefficient can be observed. From the *Equation (11)* it can be examined that in the denominator, elements in factorial contributes to the power of *a*, *b* and *c*. Such phenomenon is, again, analogical to the binomial case. In the *Equation (10)*, it can be observed that the powers of *a* and *b*, contribute to the denominator of the expansion of the binomial coefficient. Therefore, *n-k* represents the power of *a* in certain element and *k* shows the power of *b* in the same element.

In trinomial theorem, however, the combinatorial number is presented as a product

$$\binom{n}{p}\binom{p}{q} = \frac{n!}{(n-p)!\,(p-q)!\,q!}$$

(12)

It is important to mention that the characteristic of *n* remains the same. It shows the number of all elements, meaning that the powers of *a*, *b* and *c* in a particular element of expansion should sum up to *n*. When taking into account the relationship between powers from the *Equation (8)* and the numbers in factorials in the denominator of *Equation (12)*, *(n-p)* represent the power of *a* in the certain element, *(p-q)* represent the power of *b* and *q*, the power of *c*. This information is crucial for determining the position of certain element on the graphical representation of the trinomial theorem.

Knowing that the coefficient for a^n, b^n and c^n is equal to 1, it can be deduced that those elements have to be at the end of the graphic representation of such expansion. This suggests the shape of such illustration – triangle. For the *Equation (5)*, after placing all elements of the expansion in a way that power of *a*, *b*, *c* increases while approaching the end of the triangle (for example, power of *a* increases when approaching the end a^4) it is possible to show the graphic representation:

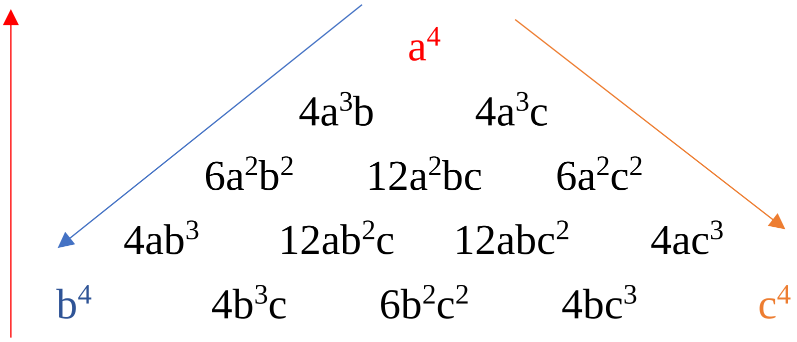

Figure 1

It can be presented in terms of *p*, *n*, *q*. Here, *n=4* and knowing that *q* is equivalent to the power of *c*, *p* can be calculated:

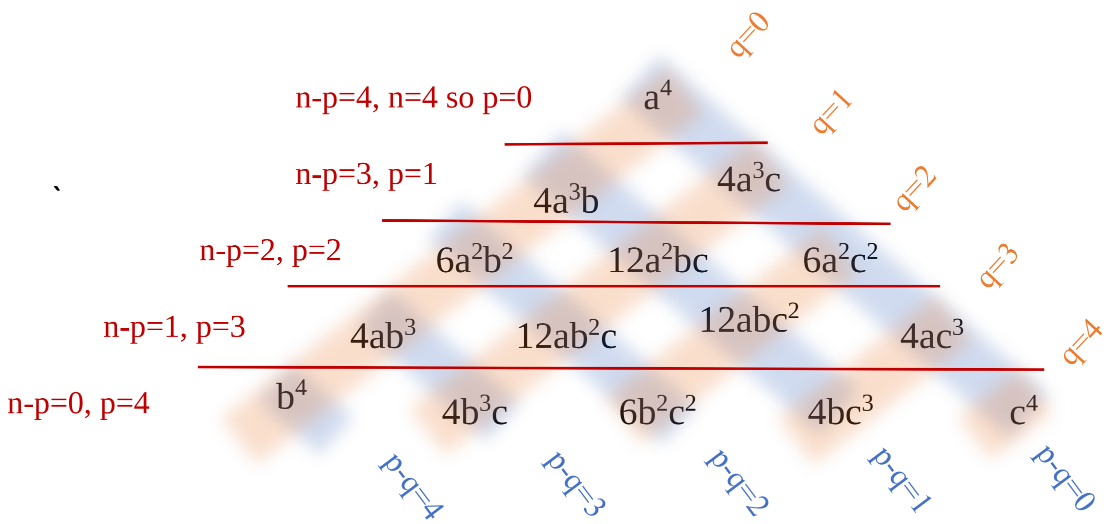

Figure 2

Figure 2 shows the graphical representation of $(a+b+c)^4$ with the relationship between *p*, *q* and the position within the same *n*-level. In the rotation presented above, *p* represents the vertical level where *p=0* is at the top tip of the triangle (a^4) and *q* represents an oblique level where *q=0* starts, again, from the tip of the triangle.

For example, if I want to find both the coefficient and the position of b^3c I can use the relation which I have already presented; here, full representation of the element is $a^0b^3c^1$, therefore:

$$n = 0 + 3 + 1 = 4$$

$$a^0 \;\; <=> n - p = 0 \;\;\; so, \qquad p = n - 0 = 4$$

The following means that vertically, the element is at the bottom of the triangle.

$$c^1 <=> q = 1$$

Therefore, the element is at the second oblique row (counting from left to right).

By using combinatorial number and knowing the powers of *a*, *b* and *c*, the coefficient of the element can be calculated.

$$\binom{n}{p}\binom{p}{q} = \frac{4!}{0!\,3!\,1!} = 4$$

To summarize the above, it can be stated that by knowing the powers of *a*, *b* and *c*, both coefficient and the position of a particular element can be calculated. Moreover, it allowed me to construct the generalized model for the *n*th layer of the pyramid shown on *Figure 3.*

$$a^n$$

$$\binom{n}{1} a^{n-1}b \qquad \binom{n}{1} a^{n-1}c$$

$$\binom{n}{2} a^{n-2}b^2 \qquad \binom{n}{2}\binom{2}{1} a^{n-2}bc \qquad \binom{n}{2} a^{n-2}c^2$$

$$\binom{n}{3} a^{n-3}b^3 \qquad \binom{n}{3}\binom{3}{1} a^{n-3}b^2c \qquad \binom{n}{3}\binom{3}{2} a^{n-3}bc^2 \qquad \binom{n}{3} a^{n-3}c^3$$

$$\vdots$$

$$\binom{n}{0} b^n \quad \binom{n}{1} b^{n-1}c \quad \binom{n}{2} b^{n-2}c^2 \quad \binom{n}{3} b^{n-3}c^3 \quad \cdots \quad \binom{n}{n-3} b^3c^{n-3} \quad \binom{n}{n-2} b^2c^{n-2} \quad \binom{n}{n-1} bc^{n-1} \quad \binom{n}{n} c^n$$

Figure 3

Relations between different layers of the Pascal's Pyramid

After understanding majority of the mechanisms working within one *n*-level, it can be added up to the actual three-dimensional model. However, there is another relationships between certain numbers which has to be mentioned to construct the pyramid correctly. In the Pascal's Triangle, one number is a sum of the two nearest number from the row above.

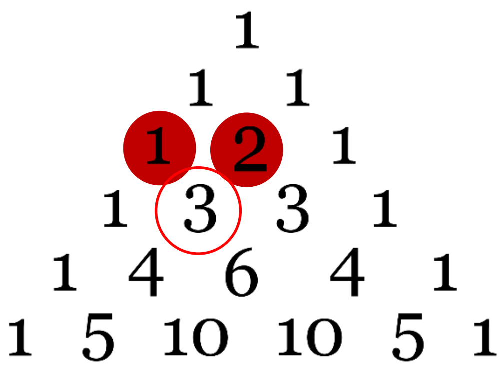

Figure 6

Similarly, in the pyramid, lower elements are the sum of the ones above. However, this relation will be observed between numbers from different *n*-levels, so, for example, the number from the n=4 will be the sum of numbers from n=3. Moreover, as it may be expected, these numbers will be a sum of three numbers from the higher *n*-level (not a sum of two, as it was in the Pascal's Triangle). Let's take pyramid's layers for *n=3* and *n=4*

1
3 3
3 6 3
1 3 3 1

Figure 5

1
4 4
6 12 6
4 12 12 4
1 4 6 4 1

Figure 5

After adding *Figure 6* (n=4) on top of the *Figure 5* (n=3) the relations between numbers from two layers can be shown. Let's state that the black level for n=3 is on the top and the orange one , for n=4, is on the bottom.

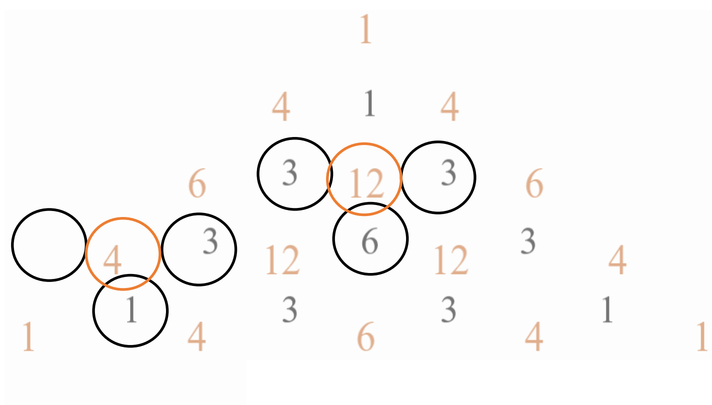

Figure 7

It can be observed that the three nearest numbers from the top (black) layer sum up to the number between them, which is beneath them (here, 3+3+6=12). When considering numbers on the edges of the layer, if there is no number above them, it counts as 0 (here, 0+3+1=4).

Using the previously explained relation between n, p, q and the coefficient's position in the triangle, relation from *Figure 7* can be generalized to the Theorem 2.

Theorem 2:

$$\binom{n}{p}\binom{p}{q} = \binom{n-1}{p-1}\binom{p-1}{q-1} + \binom{n-1}{p}\binom{p}{q} + \binom{n-1}{p-1}\binom{p-1}{q} \tag{13}$$

Proof of the Theorem2:

$$LHS = \binom{n}{p}\binom{p}{q} = \frac{n!}{(n-p)!\,(p-q)!\,q!}$$

$$\begin{aligned}
RHS &= \binom{n-1}{p-1}\binom{p-1}{q-1} + \binom{n-1}{p}\binom{p}{q} + \binom{n-1}{p-1}\binom{p-1}{q} \\
&= \frac{(n-1)!}{(p-1)!\,\big((n-1)-(p-1)\big)!}\cdot\frac{(p-1)!}{(q-1)!\,\big((p-1)-(q-1)\big)!} \\
&+ \frac{(n-1)!}{p!\,\big((n-1)-p\big)!}\cdot\frac{p!}{q!\,(p-q)!} + \frac{(n-1)!}{(p-1)!\,\big((n-1)-(p-1)\big)!} \\
&\cdot\frac{(p-1)!}{q!\,\big((p-1)-q\big)!} \\
&= \frac{(n-1)!}{(q-1)!\,(n-p)!\,(p-q)!} + \frac{(n-1)!}{q!\,(n-p-1)!\,(p-q)!} \\
&+ \frac{(n-1)!}{q!\,(n-p)!\,(p-q-1)!} \\
&= \frac{(n-1)!\,q}{q!\,(n-p)!\,(p-q)!} + \frac{(n-1)!\,(n-p)}{q!\,(n-p)!\,(p-q)!} + \frac{(n-1)!\,(p-q)}{q!\,(n-p)!\,(p-q)!} \\
&= \frac{(n-1)!\,(q+n-p+p-q)}{q!\,(n-p)!\,(p-q)!} = \frac{n(n-1)!}{q!\,(n-p)!\,(p-q)!} \\
&= \frac{n!}{q!\,(n-p)!\,(p-q)!} = LHS \quad \blacksquare
\end{aligned} \tag{14}$$

3D Model

Finally, knowing the relationship between different *n*-levels, the Pascal's Pyramid can be constructed. I decided to create such model by myself, using the SketchUp program which is a relatively popular modelling software. The Pascal's Pyramid has a shape of a tetrahedron which is determined by the fact that the layers within it (so also the base of the model) are triangular shape. However one may find it misleading, due to the fact that not all the faces are equal in terms of numbers which they include. Three faces on the sides include numbers which correspond to the ones from the Pascal's Triangle. One face which works as a base, however, includes numbers commensurate with the ones showed before, while explaining mechanism within one *n*-level.

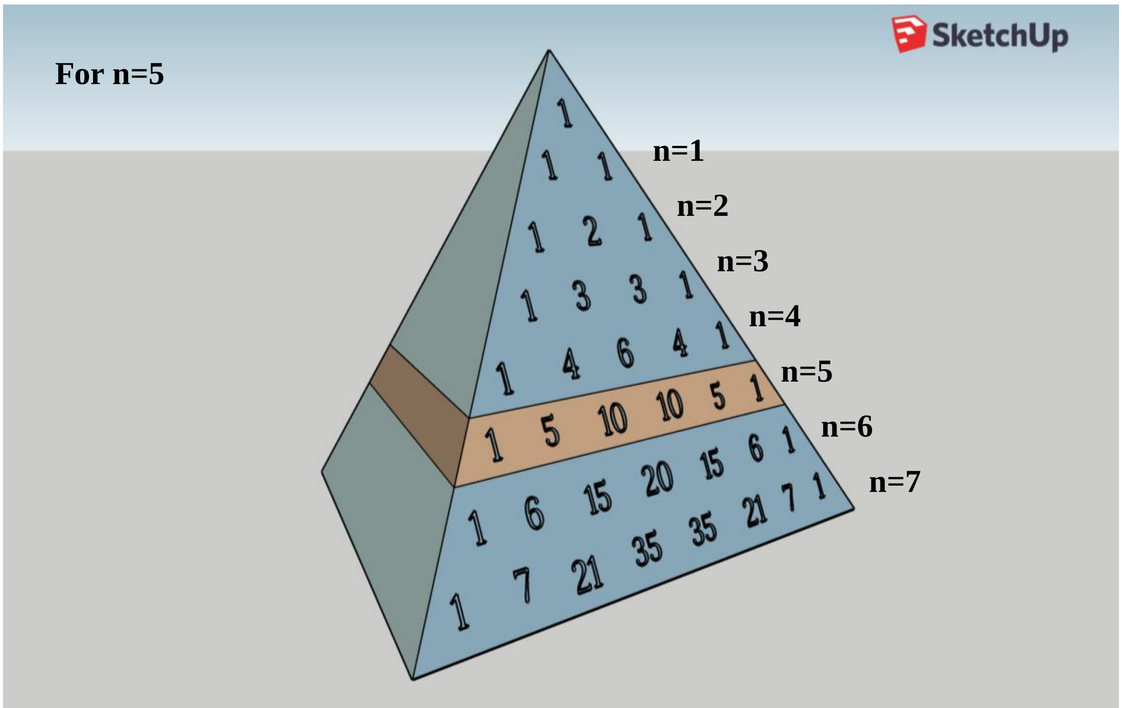

Figure 8

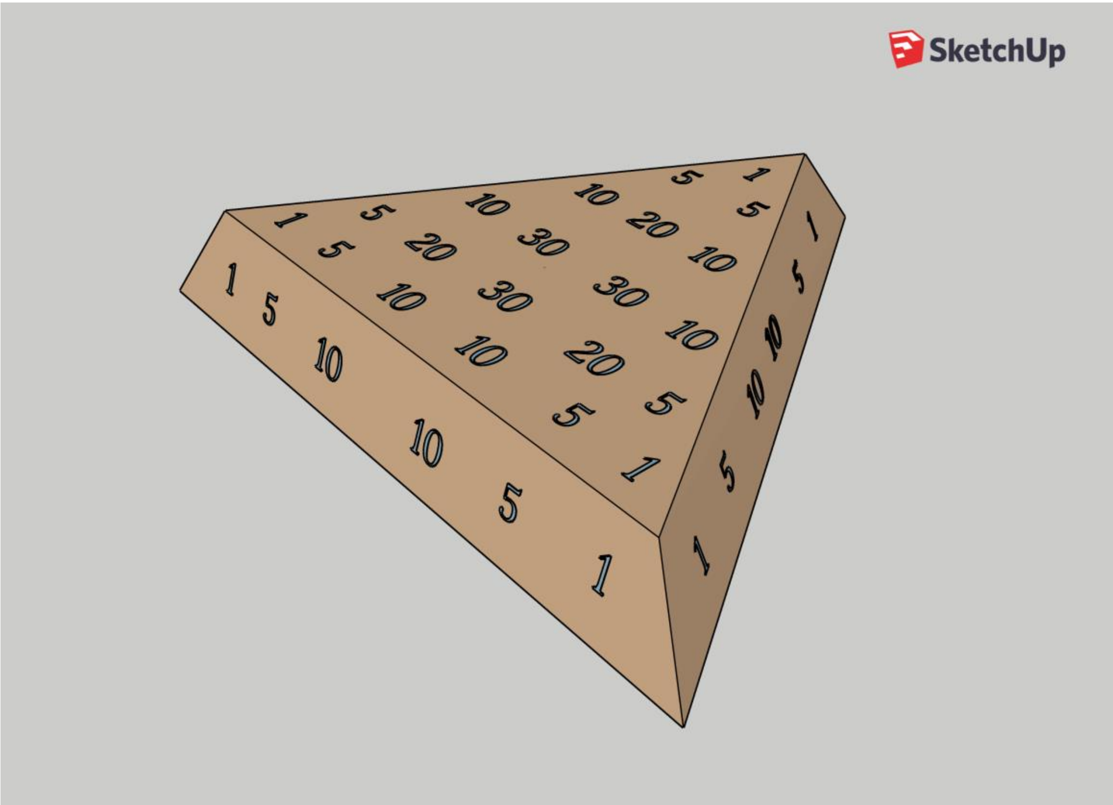

Figure 9

Conclusion

Due to the wide understanding of the Binomial Theorem, I was able to observe some universal patterns and mechanisms (for example the relationship between powers of elements and binomial coefficients) which helped me to deduce and prove the Trinomial Theorem. This, moreover, was useful to analyse the correlation between the binomial coefficients and the position of coefficients on one, horizontal plane, so for the *n* being constant (here usually called '*n*-level'). It also allowed me to create a generalization of a horizontal plane of the tetrahedron for the *n*th layer. Knowing the position of coefficients on single layers, the last aspect to deduce was the exact relation between two layers, meaning the dependence of one layer on the other one. Here, again, the analogy to the Pascal's Triangle was helpful. This was particularly difficult to present, due to lack of a noncomplicated tool to produce the two-dimensional model of the equilateral triangle (in order to show the relation between the top layer and the bottom one). However, I tried to make it as clear as possible with the usage of different colours and shapes. I was also able to conduct the Theorem 2 and prove it, despite me not being able to find such theorem in different Mathematical sources. All the above made the main aim of this paper, creating the three-dimensional model, undoubtful. The usage of SketchUp program helped me to illustrate the tetrahedron in a clear way, making this paper more perspicuous.

Bibliography

Buckle N., I. Dunbar, *Mathematics Higher Level (Core)*, 3rd edn., IBID Press, Victoria, 2004.

Graham R. L., Knuth D. E., Patashnik O., *Concrete Mathematic 2nd edn.*, Addison-Wesley Publishing Company, United States of America, 1988.