MATHEMATICS
HIGHER LEVEL
OPTION 10

SERIES & DIFFERENTIAL EQUATIONS

PREFACE

This book has been specifically written to meet the demands of the Mathematics Higher Level Option 10: Series and Differential Equations. It presents an extensive and comprehensive coverage of the syllabus and in some sections goes beyond what is required of the student. However, this is for the teacher to decide how best to use these sections with their students.

The text has been written in a conversational style so that students will find that they are not simply making reference to an encyclopedia filled with mathematical facts, but rather find that they are in some way participating in or listening in on a discussion of the subject matter.

There is an extensive set of examples for students to refer to when learning the subject matter for the first time or during their revision period. There is an abundance of well-graded exercises for students to hone their skills. Of course, it is not expected that any one student work through every question in this text - such a task would be quite a feat. Again it is hoped then that teachers will guide the students as to which questions to attempt. The questions serve to develop routine skills, reinforce concepts introduced in the topic and develop skills in making appropriate use of the graphics calculator.

Throughout the text the subject matter is presented using graphical, numerical, algebraic and verbal means whenever appropriate. Classical approaches have been judiciously combined with modern approaches reflecting new technology - in particular the use of the graphics calculator.

A detailed solution manual has also been included as part of this text book. The solution manual has been made available on a disc, which can be found on the inside of the back cover. To open the files you will need to have Adobe® Acrobat® loaded on your system as well as Microsoft Excel® & Microsoft Word®.

As always, we welcome and encourage teachers and students to contact IBID Press with feedback, not only on their likes and dislikes but suggestions on how the book can be improved as well as where errors and misprints occur. There will be updates on the IBID Press website in relation to errors that have been located in the book as well as other relevant material for this option (for example, alternative solutions to those presented in the solution manual) – so we suggest that you visit the IBID website at **www.ibid.com.au.** If you believe you have located an error or misprint please email Fabio Cirrito at **fabio@ibid.com.au.**

Fabio Cirrito, September 2006

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Dedication

I dedicate this book to my dearest wife Manola and to my beloved children Aram and Igor. The three of them are the sunshine of my life and my source of inspiration. They have also been a source of support and dedication during the difficult times of my illness and my present recovery. There are no words to fully convey what this has meant to me.

Eduardo Balanovski

1.1 SEQUENCES

1.1.1 INTRODUCTION

Consider the following sets (a) $\left\{\frac{1}{2}, \frac{2}{3}, \frac{3}{4}, \frac{4}{5}, \ldots\right\}$

(b) $\{2, 5, 9, 25, 42, \ldots\}$.

While both sets in (a) and (b) represent a **sequence** of numbers, set (a) gives the impression that an 'obvious' pattern exists. That is, it seems to represent a well defined sequence of numbers. We say this because each number in this set can be defined by a rule that not only generates that number but also tells us the position of that number in the sequence.

The first number in set (a), which we call the first term, is $\frac{1}{2}$. To indicate this we use the following notation; $u_1 = \frac{1}{2}$, with the subscript '1' indicating that it is the first term. The second term, $\frac{2}{3}$ is written as $u_2 = \frac{2}{3}$. Similarly, the third term is $u_3 = \frac{3}{4}$ and so on. In fact, close observation leads us to deduce that the **general term**, called the ***n*th term** – denoted by u_n, is given by

$$u_n = \frac{n}{n+1}, \text{ where } n = 1, 2, 3, \ldots$$

or

$$u_n = \frac{n}{n+1}, \text{ where } n \in \mathbb{Z}^+$$

The domain of this sequence is given by the values of n – i.e., the set of positive integers, while the range (i.e., the numbers in the sequence) is given by the set of rational numbers. Of course, the range will depend on the rule, u_n. The three dots at the end of the sequence mean 'and so on'. That is, we are referring to an **infinite sequence** – which is what we will be dealing with in this book. Because of this, the sequence $\{u_1, u_2, u_3, \ldots\}$ is sometimes written as $\{u_n\}_{n=1}^{\infty}$.

We are now in the position to have the following definition:

A real infinite sequence is a function whose domain is the set of positive integers and whose range is the set of real numbers.

i.e., $u_n : \mathbb{Z}^+ \mapsto \mathbb{R}$, where u_n is a function of n

In this course we will only be interested in sequences where $u_n = f(n)$, i.e., the sequence is given by a well defined function.

EXAMPLE 1.1

List the first four terms of the following sequences:

(a) $u_n = \frac{n}{2^n}$ (b) $u_n = \frac{n-1}{n}$ (c) $u_n = 2$

Solution

(a) The nth term of the sequence is given by $u_n = \frac{n}{2^n}$.

The first term, when $n = 1$, is given by $u_1 = \frac{1}{2^1} = \frac{1}{2}$.

The second term, when $n = 2$, is given by $u_2 = \frac{2}{2^2} = \frac{2}{4} = \frac{1}{2}$.

The third term, when $n = 3$, is given by $u_3 = \frac{3}{2^3} = \frac{3}{8}$.

The fourth term, when $n = 4$, is given by $u_4 = \frac{4}{2^4} = \frac{4}{16} = \frac{1}{4}$.

Therefore the first four terms of the sequence are $\frac{1}{2}, \frac{1}{2}, \frac{3}{8}, \frac{1}{4}$.

(b) With $u_n = \frac{n-1}{n}$ we have, $u_1 = \frac{1-1}{1} = 0$;

$$u_2 = \frac{2-1}{2} = \frac{1}{2};$$

$$u_3 = \frac{3-1}{3} = \frac{2}{3};$$

$$u_4 = \frac{4-1}{4} = \frac{3}{4}.$$

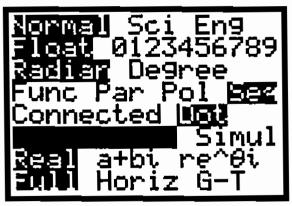
Therefore the first four terms of the sequence are $0, \frac{1}{2}, \frac{2}{3}, \frac{3}{4}$.

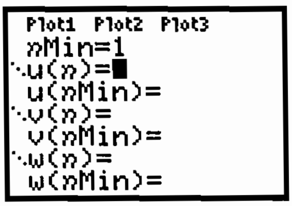
(c) This time $u_n = 2$ so that $u_1 = 2$, $u_2 = 2$, $u_3 = 2$, $u_4 = 2$.

The first four terms are 2, 2, 2, 2.

That is, each term has a value of 2. In fact, every term in this sequence will be 2 – this is known as a **constant sequence**

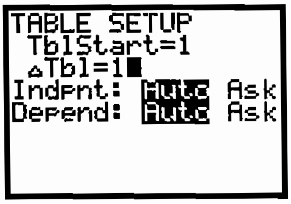
It is important to make appropriate use of a graphics calculator when working with sequences. To do this we need to first set the calculator mode to **seq** (i.e., sequence mode) and **Dot** (this is for when we want to plot the sequence). Once that is completed, by pressing the equation editor we can enter the sequence rule and proceed as required.

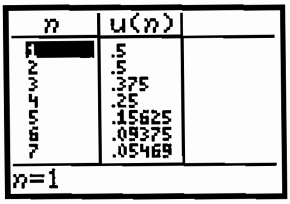




Using the examples in Example 1.1 we have:

Step 1: **Step 2:** **Step 3:**





Step 1: Enter the expression for the sequence.
If you wish, you may also enter the value of u_1 (i.e., u(*n*Min) = 0.5)

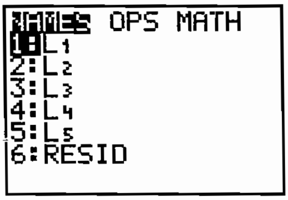
Step 2: To view the sequence using the Table format you need to first set up the Table parameters. In this case, set TblStart to 1 (as this represents $n = 1$) and then increase the values of n by 1 (i.e., set ΔTbl = 1).

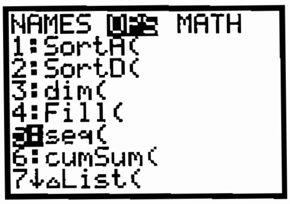
Step 3: View the entries by calling up the TABLE.

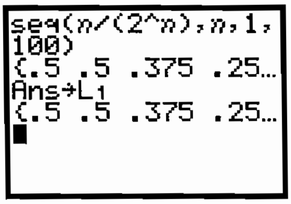
It is also possible to generate the sequence by using the following process:

Step 1: Press **2nd STAT** (this will bring up the **LIST** screen).
Step 2: Move cursor to **OPS** and then select **seq(**.
Step 3: Enter the expression for the sequence using the following syntax:
seq(expression, variable, term to start from, last term).
Then store the sequence as a list.

Step 1: **Step 2:** **Step 3:**







EXAMPLE 1.2

List the first four terms of the following sequences

(a) $u_n = (-1)^{n+1}$ (b) $u_n = (-1)^{n+1}n$ (c) $u_n = 1+\frac{(-1)^n}{n}$

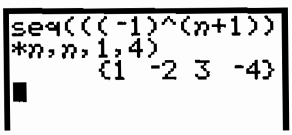
Solution

(a) Using $u_n = (-1)^{n+1}$ we have:

$u_1 = (-1)^2 = 1$;
$u_2 = (-1)^3 = -1$;
$u_3 = (-1)^4 = 1$;
$u_4 = (-1)^5 = -1$.

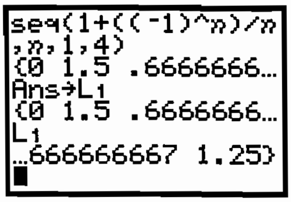
Therefore the first four terms of the sequence are $1, -1, 1, -1$

(b) Using the graphics calculator we have:
So the first four terms of the sequence are; $1, -2, 3, -4$.



(c) Again, making use of the graphics calculator we have:

Giving the first four terms as $0, \frac{3}{2}, \frac{2}{3}, \frac{5}{4}$.



EXAMPLE 1.3

List the first four terms of the following sequences.

(a) $\left\{\frac{2n}{n+1}\right\}_{n=1}^{\infty}$ (b) $\left\{\sqrt{n+4}\right\}_{n=1}^{\infty}$ (c) $\left\{\sin\frac{n\pi}{6}\right\}_{n=1}^{\infty}$

Solution

(a) With $u_n = \frac{2n}{n+1}$ we have: $u_1 = \frac{2 \times 1}{1+1} = \frac{2}{2} = 1$; $u_2 = \frac{2 \times 2}{2+1} = \frac{4}{3}$;

$u_3 = \frac{2 \times 3}{3+1} = \frac{3}{2}$ and $u_4 = \frac{2 \times 4}{4+1} = \frac{8}{5}$.

That is, the first four terms are $1, \frac{4}{3}, \frac{3}{2}, \frac{8}{5}$.

(b) With $u_n = \sqrt{n+4}$ we have: $u_1 = \sqrt{1+4} = \sqrt{5}$; $u_2 = \sqrt{2+4} = \sqrt{6}$;

$u_3 = \sqrt{3+4} = \sqrt{7}$ and $u_4 = \sqrt{4+4} = \sqrt{8}$.

That is, the first four terms are $\sqrt{5}, \sqrt{6}, \sqrt{7}, \sqrt{8}$.

(c) With $u_n = \sin\frac{n\pi}{6}$ we have: $u_1 = \sin\frac{\pi}{6} = \frac{1}{2}$; $u_2 = \sin\frac{2\pi}{6} = \frac{\sqrt{3}}{2}$;

$u_3 = \sin\frac{3\pi}{6} = 1$ and $u_4 = \sin\frac{4\pi}{6} = \frac{\sqrt{3}}{2}$.

That is, the first four terms are $\frac{1}{2}, \frac{\sqrt{3}}{2}, 1, \frac{\sqrt{3}}{2}$.

So far we have only considered sequences whose rules were written explicitly in terms of n. However, it is possible to define sequences implicitly. The process of evaluating successive terms in these sequences does not require too much more effort. The following examples illustrate this.

EXAMPLE 1.4

List the first four terms of the following sequences.

(a) $u_{n+1} = u_n + \frac{1}{2}, u_1 = \frac{1}{2}, n = 1, 2, 3$.

(b) $u_{n+1} = \frac{1}{2}u_n + 1, u_1 = 2, n = 1, 2, 3$.

Solution

(a) Using the expression $u_{n+1} = u_n + \frac{1}{2^n}, u_1 = \frac{1}{2}, n = 1, 2, 3, \ldots$ we have:

$n = 1, u_1 = \frac{1}{2}$ and $u_2 = u_1 + \frac{1}{2^1} = \frac{1}{2} + \frac{1}{2} = 1$;

$n = 2, u_2 = 1$ and $u_3 = u_2 + \frac{1}{2^2} = 1 + \frac{1}{4} = \frac{5}{4}$;

$n = 3, u_3 = \frac{5}{4}$ and $u_4 = u_3 + \frac{1}{2^3} = \frac{5}{4} + \frac{1}{8} = \frac{11}{8}$.

Therefore, the first four terms are $\frac{1}{2}, 1, \frac{5}{4}, \frac{11}{8}$.

(b) Using the expression $u_{n+1} = \frac{1}{2}u_n - 1, u_1 = 2, n = 1, 2, 3, \ldots$ we have:

$n = 1, u_1 = 2$ and $u_2 = \frac{1}{2}u_1 - 1 = \frac{1}{2} \times 2 - 1 = 0$;

$n = 2, u_2 = 0$ and $u_3 = \frac{1}{2}u_2 - 1 = \frac{1}{2} \times 0 - 1 = -1$;

$n = 3, u_3 = -1$ and $u_4 = \frac{1}{2}u_3 - 1 = \frac{1}{2} \times -1 - 1 = -\frac{3}{2}$.

Threfore, the first four terms are $2, 0, -\frac{1}{2}, -\frac{3}{2}$.

EXERCISES 1.1.1

1. Write down the first four terms of the sequences

(a) $u_n = \frac{1}{n(n+1)}, n \in \mathbb{Z}^+$ (b) $u_n = 3 + \frac{n}{3}, n \in \mathbb{Z}^+$

(c) $u_n = 5 + \frac{1}{n}, n \in \mathbb{Z}^+$ (d) $u_n = n + 2^n, n \in \mathbb{Z}^+$

2. Write down the first four terms of the following sequences, where $n \in \mathbb{Z}^+$.

(a) $u_n = \frac{4n^2}{2n^2 - n}$ (b) $u_n = \frac{\sqrt{n}}{n}$ (c) $u_n = \frac{n}{(n+1)^2}$

3. Determine an explicit expression for the general term of the sequence

(a) $1, \frac{1}{3}, \frac{1}{9}, \frac{1}{27}, \ldots$ (b) $1, -\frac{1}{3}, \frac{1}{9}, -\frac{1}{27}, \ldots$ (c) $\frac{1}{2}, \frac{1}{5}, \frac{1}{10}, \frac{1}{17}, \ldots$

4. Calculate u_1, u_5, u_{10} and u_{50} (to four decimal places) for each of the following

(a) $u_n = \sqrt{2n} + 2$ (b) $u_n = \frac{n}{\sqrt{n}}$ (c) $u_n = \arctan(n)$

5. Write down the first four terms of the following sequences, where $n \in \mathbb{Z}^+$.

(a) $u_n = n^2 \sin\left(\frac{\pi}{2^n}\right)$ (b) $u_n = \sqrt{n+2} - \sqrt{n}$ (c) $u_n = \frac{n}{(n+1)!}$

6. Write down the first five terms of the following sequences.

(a) $\left\{\frac{n}{2}+1\right\}_{n=1}^{\infty}$ (b) $\left\{\frac{5}{n^2}\right\}_{n=1}^{\infty}$ (c) $\left\{1-\frac{1}{2^n}\right\}_{n=1}^{\infty}$ (d) $\left\{\sin\left(\frac{\pi}{n}\right)\right\}_{n=1}^{\infty}$

7. Write down the first five terms of the following sequences.

(a) $\left\{\left(1+\frac{1}{2n}\right)^{2n}\right\}_{n=1}^{\infty}$ (b) $\left\{\frac{\ln(n)}{n}\right\}_{n=1}^{\infty}$ (c) $\left\{n\sin\left(\frac{\pi}{n}\right)\right\}_{n=1}^{\infty}$

8. Consider the sequence $\{u_n\}_{n=1}^{\infty}$ where $u_{n+1} = 2(\sqrt{u_n}+1)$ and $u_1 = 1$. Find the first four terms of $\{u_n\}_{n=1}^{\infty}$.

9. Find the first five terms of the sequence defined by $u_{n+2} = \sqrt{u_n \cdot u_{n+1}}$ where $u_1 = 2$ and $u_2 = 4$.

10. Find the first six terms of the sequence defined by the relation

$$u_n = u_{n-1} + u_{n-2}, n \geq 3 \text{, where } u_1 = 1 \text{ and } u_2 = 1.$$

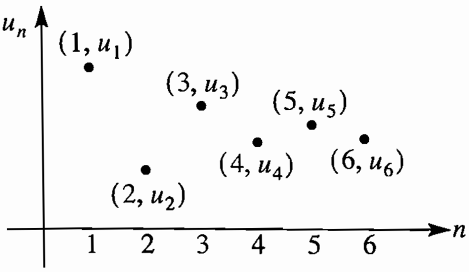
11. Consider the sequence $\{u_n\}_{n=0}^{\infty}$ defined by $u_{n+1} = \frac{1}{2}(u_n + u_{n-1}), n \geq 1$ where $u_0 = b$ and $u_1 = a$.

Show that $u_n - u_{n-1} = \left(-\frac{1}{2}\right)^{n-1}(a-b), n > 1$.

1.1.2 VISUALISING SEQUENCES – GRAPHS OF SEQUENCES

While it is relatively straight forward to calculate the terms of a well defined sequence, the ability to produce a visual representation of the sequence will prove to have additional benefits – not least, the ability to deduce the behaviour of the sequence as the value of n increases.

The graph of a sequence $\{u_n\}_{n=1}^{\infty}$ is plotted as the set of points (n, u_n) for $n \in \mathbb{Z}^+$.



Obviously, the graphics calculator comes into its own when plotting sequences. Once the rule for the sequence has been entered an appropriate window needs to be defined so that we can 'capture' the sequence's behaviour.

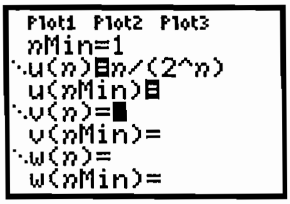
EXAMPLE 1.5

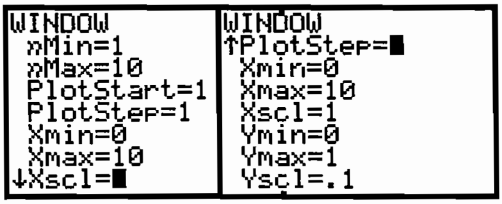
Plot each of the following sequences and describe the behaviour of u_n as n increases indefinitely. (a) $u_n = \frac{n}{2^n}$ (b) $u_n = \frac{n-1}{n}$

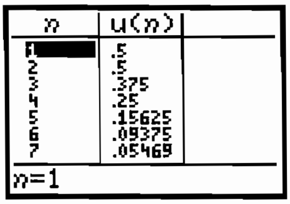
Solution

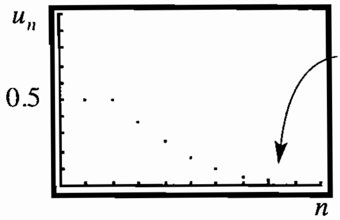
Step 1: Define the sequence.
Step 2: Set up an appropriate window (i.e., the domain and range)

(a) **Step 1:** **Step 2:**



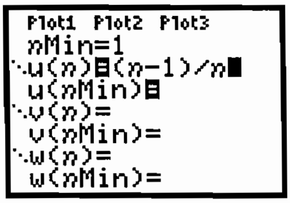


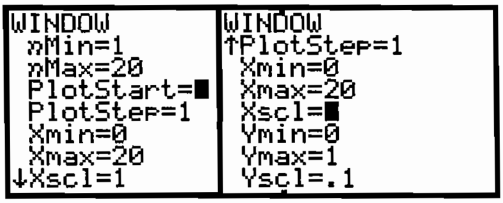


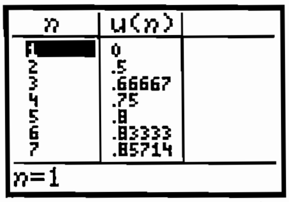


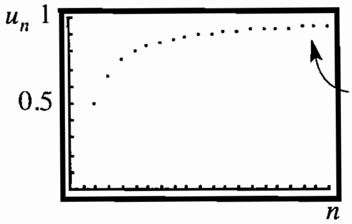
It appears that as n increases indefinitely, the values of u_n tend to 0.

(b) **Step 1:** **Step 2:**









It appears that as n increases indefinitely, the values of u_n tend to 1.

EXERCISES 1.1.2

1. Plot the following sequences and where possible, describe the behaviour of the sequence as n increases indefinitely.

(a) $u_n = \frac{n^2}{n^2+10}$ (b) $u_n = \frac{10}{n+2}$ (c) $u_n = 3 + \frac{1}{2^n}$

(d) $u_n = \frac{\sin(n)}{n}$ (e) $u_n = \sin(n)$ (f) $u_n = 2 + (1.2)^n$

1.2 LIMITS OF SEQUENCES

1.2.1 INTRODUCTION

One aim in studying sequences is to investigate the behaviour of their terms as n increases indefinitely.

EXAMPLE 1.6

Investigate the behaviour of the sequence $u_n = \frac{n}{2n+1}$ where $n \in \mathbb{Z}^+$.

Solution

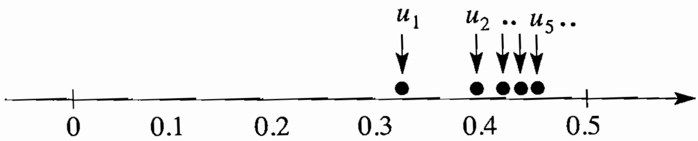
(a) We start by evaluating the terms in the sequence:

$$n = 1, u_1 = \frac{1}{2+1} = \frac{1}{3}$$

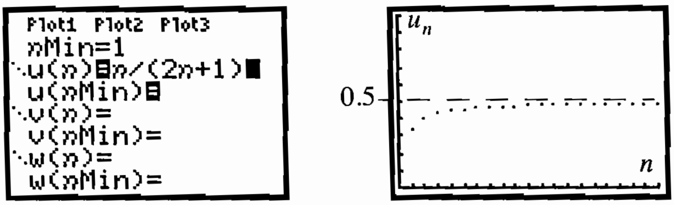
$$n = 2, u_2 = \frac{2}{4+1} = \frac{2}{5}$$

Similarly, we have $u_3 = \frac{3}{7}$, $u_4 = \frac{4}{9}$, $u_5 = \frac{5}{11}$ and so on.

If we consider these numbers on the real number line we have:



While from this diagram it appears that the values of the terms in this sequence approach 0.5 we should plot a graph of this sequence that shows more than just the first 5 terms.



From the plot we see that the sequence is **asymptotic** to the value 0.5. In this instance we say that the limit of the sequence is 0.5. We can express this as follows: $\lim_{n \to \infty} u_n = \lim_{n \to \infty} \frac{n}{2n+1} = 0.5$.

We make the following observation: $$u_n = \frac{n}{2n+1} = \frac{\frac{n}{n}}{\frac{2n+1}{n}} = \frac{1}{2+\frac{1}{n}}$$

Then, as n increases indefinitely, i.e., as $n \to \infty$ the term $\frac{1}{n} \to 0$ and so, $u_n \to \frac{1}{2+0} = \frac{1}{2}$.

So, while plotting a graph of the sequence will provide a visual display of the behaviour of the sequence as $n \to \infty$, we see that it is possible to use algebraic manipulation to help in the investigation of the behaviour of the sequence. We will explore such algebraic manipulations in more detail as we encounter sequences for which such an approach is appropriate.

It is also worth noting the similarity between the sequence $u_n = \frac{n}{2n+1}$ and the function $f(x) = \frac{x}{2x+1}$. In the same way that $u_n \to \frac{1}{2}$ as $n \to \infty$ we have that $f(x) \to \frac{1}{2}$ as $x \to \infty$.

EXAMPLE 1.7

Investigate the behaviour of the sequence $u_n = \frac{4n^2}{2n^2+5}$ where $n \in \mathbb{Z}^+$.

Solution

While we could make use of a graph of the sequence, this time we proceed as in the previous example and use an algebraic approach.

$$u_n = \frac{4n^2}{2n^2+5} = \frac{4}{2+\frac{5}{n^2}} \text{ (dividing numerator and denominator by } n^2\text{)}.$$

Next, as $n \to \infty$, $\frac{5}{n^2} \to 0$ so that $\frac{4}{2+\frac{5}{n^2}} \to \frac{4}{2+0} = 2$.

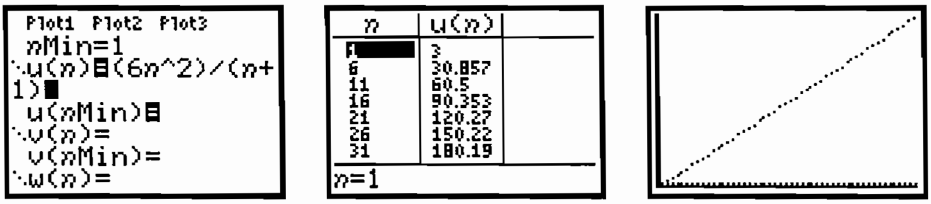
That is, as $n \to \infty$, $u_n = \frac{4n^2}{2n^2+5} \to 2$. Or we could write $\lim\limits_{n\to\infty} \frac{4n^2}{2n^2+5} = 2$.

EXAMPLE 1.8

Investigate the behaviour of the sequence $u_n = \frac{5n^2}{n+1}$ where $n \in \mathbb{Z}^+$.

Solution

We first make use of the graph of the sequence:



From the graph it appears that as n increases indefinitely so too does the sequence. Therefore, we write that as $n \to \infty$, $u_n \to \infty$.

Using an algebraic approach we have:

$$u_n = \frac{5n^2}{n+1} = \frac{5n}{1+\frac{1}{n}} \text{ (after dividing the numerator and denominator by } n\text{)}$$

Then, as $n \to \infty$, $\frac{1}{n} \to 0$ so that $\frac{5n}{1+\frac{1}{n}} \to \frac{5n}{1+0} = 5n$. Therefore, as $n \to \infty$, $u_n \to \infty$.

Note: As ∞ does not represent a number, we say that the sequence does not have a limit.

EXERCISES 1.2.1

1. Determine the limit (where it exists) of u_n as $n \to \infty$.

(a) $u_n = \dfrac{n}{2n+1}$ (b) $u_n = \dfrac{n^2}{2n^2+1}$ (c) $u_n = \dfrac{4-n}{2+3n}$

(d) $u_n = 4\left(\dfrac{4}{5}\right)^n$ (e) $u_n = \dfrac{n^3}{n^2+10}$ (f) $u_n = \dfrac{n^2}{2^n}$

2. Determine the limit of u_n as $n \to \infty$ for each of the following.

(a) $u_n = \dfrac{2\sin(n)}{n}$ (b) $u_n = 2n\sin\left(\dfrac{1}{n}\right)$

3. Determine the limit (where it exists) of u_n as $n \to \infty$ for each of the following.

(a) $u_n = \left(\dfrac{n}{2}\right)^{1/n}$ (b) $u_n = \left(\dfrac{n}{2^n}\right)^{1/n}$ (c) $u_n = \dfrac{n!}{2^{n^2}}$

(d) $u_n = \dfrac{n!}{2^n}$ (e) $u_n = \dfrac{10!}{n!}$ (f) $u_n = \log_{10}\left(\dfrac{1}{n}\right)$

4. Determine the limit of u_n as $n \to \infty$ for each of the following.

(a) $u_n = \dfrac{a+bn}{c+dn}$ (b) $u_n = \dfrac{n^2}{(2n+1)^2}$ (c) $u_n = \dfrac{(n+1)^n}{n^{n+1}}$

5. Determine the limit of $\{u_n\}_{n=1}^{\infty}$ as $n \to \infty$ for each of the following sequences:

(a) $\left\{\dfrac{2n-4}{3n+5}\right\}_{n=1}^{\infty}$ (b) $\left\{\dfrac{2+3\cdot 5^n}{7+4\cdot 5^n}\right\}_{n=1}^{\infty}$

6. Consider the sequence $u_{n+1} = \left(\dfrac{n}{n+1}\right)u_n,\ u_1 = 1,\ n \in \mathbb{Z}^+$.

(a) Calculate u_2, u_3, u_4. (b) Determine the limit of u_n as $n \to \infty$.

7. Determine the limit of u_n as $n \to \infty$ where $u_n = \sqrt{n+1} - \sqrt{n}$.

8. Consider the sequence defined by $u_{n+1} = \dfrac{4u_n - 9}{u_n - 2},\ u_1 = 1,\ n \in \mathbb{Z}^+$.

(a) Calculate the values of u_2, u_3, u_4 and u_5.

(b) Given that $u_n = \dfrac{f(n)}{g(n)}$, where both f and g are linear functions,

i. find an explicit expression for u_n in terms of n;

ii. determine the limit of u_n as $n \to \infty$.

1.2.2 DEFINING THE LIMIT OF A SEQUENCE

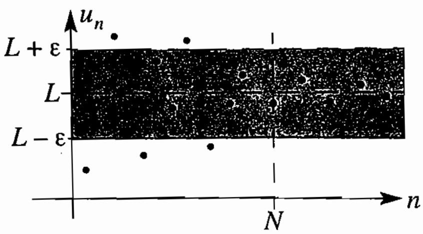
Let us return to Example 1.6, $u_n = \frac{n}{2n+1}$ where $n \in \mathbb{Z}^+$. We observed that as $n \to \infty$, $u_n \to \frac{1}{2}$. The question that remains is how close to 0.5 does u_n eventually get? We set up a table to show what happens to the terms in the sequence as n increases.

1	$\frac{1}{3}$	$\left\|\frac{1}{3} - \frac{1}{2}\right\| = 0.1666667$
10	$\frac{10}{21}$	$\left\|\frac{10}{21} - \frac{1}{2}\right\| = 0.0238095$
50	$\frac{50}{101}$	$\left\|\frac{50}{101} - \frac{1}{2}\right\| = 0.0049505$
100	$\frac{100}{201}$	$\left\|\frac{100}{201} - \frac{1}{2}\right\| = 0.0024876$
1000	$\frac{1000}{2001}$	$\left\|\frac{1000}{2001} - \frac{1}{2}\right\| = 0.0002499$
10000	$\frac{10000}{20001}$	$\left\|\frac{10000}{20001} - \frac{1}{2}\right\| = 0.0000250$

We see that as n increases the difference between the limit value and u_n decreases. We make use of the absolute value of the difference to include the scenario where the values of u_n approach the limit from *above* the limit or *below* the limit. In this instance we see that the values of u_n approach 0.5 from below. We then have that as $n \to \infty$, $|u_n - 0.5| \to 0$, telling us that the limit is 0.5. In other words, to determine if a limit exists we need to find values of n such that, as n increases, the absolute value of the differences between u_n and its limit, L (say), tend to zero. Formally we have:

> Let $\{u_n\}$ be a sequence and L be a fixed number.
>
> The values of the sequence $\{u_n\}$ converge to the limit L as $n \to \infty$, i.e., $\lim_{n \to \infty} u_n = L$,
>
> if for any $\varepsilon > 0$ there is a positive integer N, such that $|u_n - L| < \varepsilon$ for all $n > N$

Note: Our definition states that for every positive number ε the values of u_n must lie in the band shown in the diagram, i.e., in the range $]L - \varepsilon, L + \varepsilon[$ for all values of n beyond some threshold N.



Note that there is no necessity for N to be the least positive number beyond which $|u_n - L| < \varepsilon$. In fact, N will depend on the value of ε where the smaller the value of ε is the larger the N will be.

EXAMPLE 1.9

Prove that if $u_n = 4 + \frac{5}{n}$ where $n \in \mathbb{Z}^+$ then $\lim_{n \to \infty} u_n = 4$.

Solution

We must show that we can make u_n as close to 4 as we want by making n sufficiently large. Using our definition we have:
For **any** $\varepsilon > 0$ we must find a positive integer, N, sufficiently large so that $|u_n - L| < \varepsilon$ **for all** $n > N$.

That is, $$\left|\left(4 + \frac{5}{n}\right) - 4\right| < \varepsilon$$

Giving $$\left|\frac{5}{n}\right| < \varepsilon$$

Then, as $n > 0$ we have that $\frac{5}{n} < \varepsilon \Leftrightarrow n > \frac{5}{\varepsilon}$.

For example, if we choose $\varepsilon = 0.1$ then $n > 50$ (i.e., $N = 50$). Similarly, with $\varepsilon = 0.001$ we have $n > 5000$ (i.e., $N = 5000$).

A number of important observations are now made. To show that a limit, L, exists we must be able to show the following:

1. Determine the value of n such that $|u_n - L| < \varepsilon$ exists.
2. Show that the value of n depends on ε.
3. Show that as ε decreases, n increases.

EXAMPLE 1.10

Prove that if $u_n = \frac{n}{2n+1}$ where $n \in \mathbb{Z}^+$ then $\lim_{n \to \infty} u_n = \frac{1}{2}$.

Solution

Using the definition we must show that for any $\varepsilon > 0$ there exists a number N sufficiently large such that $\left|\frac{n}{2n+1} - \frac{1}{2}\right| < \varepsilon$ for every integer $n > N$.

Now, $$\left|\frac{n}{2n+1} - \frac{1}{2}\right| = \left|\frac{2n}{2(2n+1)} - \frac{(2n+1)}{2(2n+1)}\right|$$
$$= \left|\frac{-1}{2(2n+1)}\right|$$
$$= \frac{1}{4n+2}$$

We now need to find a number $N > 0$ such that $\frac{1}{4n+2} < \varepsilon$ for every integer $n > N$.

From $\frac{1}{4n+2} < \varepsilon$ we have that $4n + 2 > \frac{1}{\varepsilon} \Leftrightarrow n > \frac{1}{4}\left(\frac{1}{\varepsilon} - 2\right)$.

We can then see that as ε decreases, n increases.

As we were able to find a value of n satisfying the limit definition we have that $\lim_{n \to \infty} u_n = \frac{1}{2}$.

With $\varepsilon = 0.01$, $n > \dfrac{1 - 2 \times 0.01}{4 \times 0.01} = 24.5$. i.e., $N = 24$.

Similarly, with $\varepsilon = 0.0001$, $n > 2499.5$. i.e., $N = 2499$.

Depending on the value of ε we choose we obtain a different value of n. Again, the smaller the value of ε the larger the value of n.

Having shown how to go about proving that a limit exists, let us now consider the situation when a wrong limit is assumed. Using the sequence in Example 1.9, $u_n = 4 + \frac{5}{n}$, we now make a guess that $\lim_{n \to \infty} u_n = 2$. If this is true we must be able to show that for any $\varepsilon > 0$ there exists a number N sufficiently large such that $|u_n - 2| = \left|\left(4 + \frac{5}{n}\right) - 2\right| < \varepsilon$ for every positive integer $n > N$.

Let us proceed by choosing a value of $\varepsilon > 0$, say $\varepsilon = 1$:

$$|u_n - 2| = \left|\left(4 + \frac{5}{n}\right) - 2\right| < 1 \Leftrightarrow \left|\frac{5}{n} + 2\right| < 1$$

As $n > 0$, $\frac{5}{n} + 2 > 0$ and so we have that

$$\left|\frac{5}{n} + 2\right| < 1 \Rightarrow \frac{5}{n} + 2 < 1$$

$$\Leftrightarrow \frac{5}{n} < -1$$

$$\Leftrightarrow n < -5$$

However, we must have that $n > 0$ and so the original assumption is false. That is, $\lim_{n \to \infty} u_n \neq 2$.

Note that this is not saying that a limit does not exist, for we know from Example 1.9 that a limit does exist, it is only telling us that the limit (if it exists) is not 2. It is important then to appreciate the importance of the phrase '*For all* $\varepsilon > 0$.' in the definition of the limit.

We now provide a definition that will help in our use of mathematical language.

If $\lim_{n \to \infty} u_n = L$ and L is finite we say that the sequence converges (or is convergent).
Otherwise, we say that the sequence diverges (or is divergent).

In fact, not only do we require that L is finite, but to be able to state that the sequence converges the value of L must be **unique**. We consider a number of sequences to illustrate this point.

EXAMPLE 1.11 Which of the following sequences converge and which diverge?

(a) $u_n = \frac{n^3+2n}{n^2+4}$ (b) $u_n = \frac{3n}{n+2}$ (c) $u_n = \frac{(-1)^n}{2^n}$ (d) $u_n = \sin(n)$.

Solution

(a) Dividing numerator and denominator through by n^2 we have:

$$u_n = \frac{n^3+2n}{n^2+4} = \frac{\frac{n^3+2n}{n^2}}{\frac{n^2+4}{n^2}} = \frac{n+\frac{2}{n}}{1+\frac{4}{n^2}}$$

As both $\frac{2}{n} \to 0$ and $\frac{4}{n^2} \to 0$ as $n \to \infty$ we have that $u_n \sim \frac{n}{1}$ as $n \to \infty$.

So, we have that $u_n \to \infty$ as $n \to \infty$. That is, $\lim\limits_{n \to \infty} u_n = \infty$.

And so, the sequence **diverges**.

Note: We use the expression '$= \infty$' rather loosely, because the fact remains that '∞' is not a number and the limit L in the expression $\lim\limits_{n \to \infty} u_n = L$ needs to be a unique finite number.

When we write an expression such as $\lim\limits_{n \to \infty} u_n = \infty$ it will be understood to mean that $u_n \to \infty$ as $n \to \infty$.

(b) With the sequence $u_n = \frac{3n}{n+2}$ we first apply an algebraic 'trick':

$$u_n = \frac{3n}{n+2} = \frac{3(n+2)-6}{n+2}$$
$$= \frac{3(n+2)}{n+2} - \frac{6}{n+2}$$
$$= 3 - \frac{6}{n+2}$$

Of course the same thing can be accomplished by making use of the long division process.

Next, as $n \to \infty$ the term $\frac{6}{n+2} \to 0$ and so we have that as $n \to \infty$, $u_n \to 3 - 0 = 3$.

That is, the sequence **converges**.

(c) With the sequence $u_n = \frac{(-1)^n}{2^n}$ we first observe that if n is even then $(-1)^n = 1$, however, if n is odd then $(-1)^n = -1$.

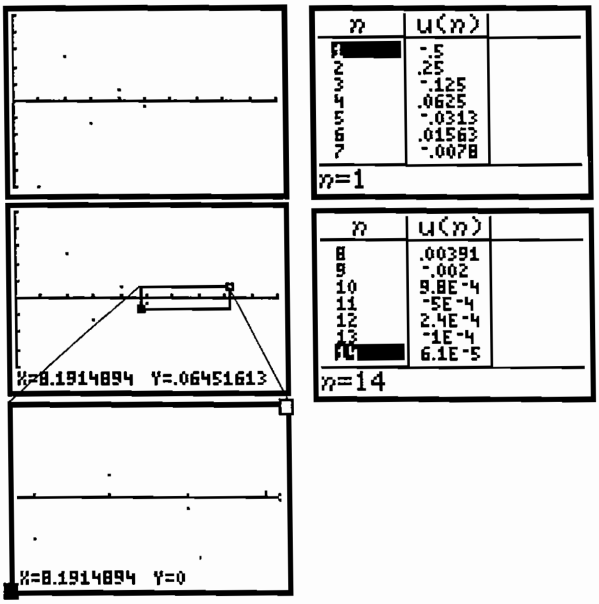
That is, we can rewrite the sequence as $u_n = \begin{cases} \frac{1}{2^n} & \text{if } n \text{ is even} \\ -\frac{1}{2^n} & \text{if } n \text{ is odd} \end{cases}$

Next, as $n \to \infty$, $2^n \to \infty$ and so, if n is even, as $n \to \infty$, $u_n \to 0$ (from above)

and if n is odd, as $n \to \infty$, $u_n \to 0$ (from below).

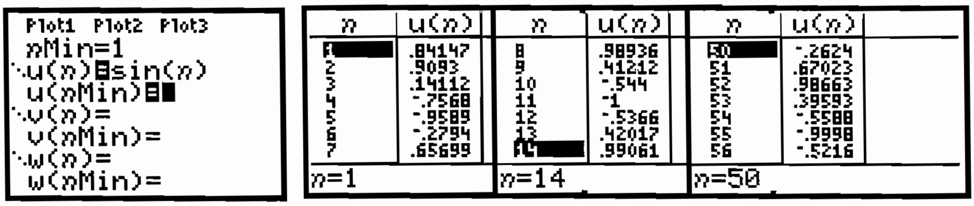
In either case, as $n \to \infty$, $u_n \to 0$ and so the sequence **converges**.

We display the results using the graphics calculator:



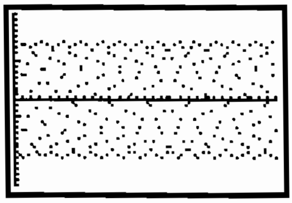
Using both a visual display and a table of values it is clearly the case that the sequence converges to 0.

(d) For the sequence $u_n = \sin n$ we make immediate use of the graphics calculator:



From the table of values we observe that while the values of the sequence are **bounded** between two values, being –1 and +1, there does not appear to be a value to which the sequence converges.

In fact, we observe that the graph of the sequence never settles on a single unique limit. The graph alongside shows the first three hundred values.



In this case we say that the sequence **diverges**.

The important point here is that for a sequence to diverge it does not necessarily mean that it tends to '±∞'. If there is no **unique** value to which the sequence tends, the operative word being 'unique', then the sequence diverges.

EXAMPLE 1.12

Consider the sequence $\left\{\dfrac{8n}{2n+3}\right\}_{n=1}^{\infty}$

(a) i. Show that the sequence converges.
ii. State the limit, L, of this sequence.

(b) Use the definition of a limit to prove that the sequence has the limit L from (a) ii.

Solution

(a) i. Let $u_n = \frac{8n}{2n+3} = \frac{4(2n+3)-12}{2n+3} = 4 - \frac{12}{2n+3}$.

As $n \to \infty$, $\frac{12}{2n+3} \to \frac{12}{\infty} = 0 \therefore u_n \to 4 - 0 = 4$.

Because u_n approaches a fixed, unique value, the sequence converges.

ii. From i., the limit is 4. That is, $\lim\limits_{n \to \infty} u_n = 4$.

(b) We must show that for any $\varepsilon > 0$ there exists a number N sufficiently large such that $\left|\frac{8n}{2n+3} - 4\right| < \varepsilon$ for every integer $n > N$.

Now,

$$\left|\frac{8n}{2n+3} - 4\right| < \varepsilon \Leftrightarrow \left|\frac{8n - 4(2n+3)}{2n+3}\right| < \varepsilon$$

$$\Leftrightarrow \left|\frac{-12}{2n+3}\right| < \varepsilon$$

As $n > 0$, $\left|\frac{-12}{2n+3}\right| = \frac{12}{2n+3}$ so:

$$\Leftrightarrow \frac{12}{2n+3} < \varepsilon$$

$$\Leftrightarrow \frac{12}{\varepsilon} < 2n+3$$

$$\Leftrightarrow n > \frac{1}{2}\left(\frac{12}{\varepsilon} - 3\right)$$

For example, if $\varepsilon = 0.001$, $n > \frac{1}{2}\left(\frac{12}{0.001} - 3\right) = 5998.5$. That is, $N = 5999$.

Therefore, as we have been able to determine n (as a function of $\varepsilon > 0$) such that $\left|\frac{8n}{2n+3} - 4\right| < \varepsilon$ we have shown that the limit of $\left\{\frac{8n}{2n+3}\right\}_{n=1}^{\infty}$ is indeed 4.

EXERCISES 1.2.2

1. Use the definition of a limit of a sequence to prove that the following sequences have the limit indicated.

(a) $u_n = \frac{1}{n^2}; L = 0$ (b) $u_n = \frac{4}{n+1}; L = 0$ (c) $u_n = \frac{1}{\sqrt{n}}; L = 0$

(d) $u_n = \frac{2n}{8n+1}; L = \frac{1}{4}$ (e) $u_n = \frac{2-n}{3+n}; L = -1$ (f) $u_n = \frac{2^n}{1+2^n}; L = 1$

(g) $u_n = 2 + \frac{(-1)^n}{n}; L = 2$ (h) $u_n = \frac{3n}{n+1}; L = 3$ (i) $u_n = \frac{1}{2^n}; L = 0$

(j) $u_n = \frac{n!}{(n+1)!}; L = 0$

2. For the following sequences, find th elimit if it exists and hence decide on the convergence or divergence of the sequence.

(a) $\left\{\dfrac{4}{\sqrt{n+2}}\right\}_{n=1}^{\infty}$ (b) $\left\{n\sin\left(\dfrac{\pi}{n}\right)\right\}_{n=1}^{\infty}$ (c) $\left\{\dfrac{3n^2+1}{2n^2-n}\right\}_{n=1}^{\infty}$

(d) $\{\cos(n)\}_{n=1}^{\infty}$ (e) $\left\{\dfrac{n!}{n^n}\right\}_{n=1}^{\infty}$ (f) $\left\{\dfrac{3n^2+2n+5}{n^2+4}\right\}_{n=1}^{\infty}$

(g) $\{(-1)^n\}_{n=1}^{\infty}$ (h) $\left\{e\cdot\left(1-\dfrac{1}{n}\right)^n\right\}_{n=1}^{\infty}$ (i) $\left\{\dfrac{n!}{e^n}\right\}_{n=1}^{\infty}$

(j) $\left\{\dfrac{1}{n}\sin(n)\right\}_{n=1}^{\infty}$ (k) $\{\cos(n\pi)\}_{n=1}^{\infty}$ (l) $\{e^{1/n}\}_{n=1}^{\infty}$

(m) $\left\{\dfrac{n+1}{2\sqrt{n}}\right\}_{n=1}^{\infty}$ (n) $\left\{\dfrac{3^n}{e^n+2}\right\}_{n=1}^{\infty}$ (o) $\{\sqrt{n+1}-\sqrt{n}\}_{n=1}^{\infty}$

3. Using the fact that $\lim\limits_{n\to\infty}\left(1+\dfrac{k}{n}\right)^n = e^k$, find the limits of the following sequences.

(a) $u_n = \left(1+\dfrac{3}{n}\right)^n$ (b) $u_n = \left(1-\dfrac{2}{n}\right)^n$ (c) $u_n = \left(1+\dfrac{1}{2n}\right)^n$

(d) $u_n = \left(1+\dfrac{2}{n}\right)^{-n}$ (e) $u_n = \left(1+\dfrac{3}{4n}\right)^{-n}$ (f) $u_n = \left(1+\dfrac{2}{n+1}\right)^{-n}$

4. Which of the following sequences converge? For those that do, state their limit.

(a) $\{a^n\}_{n=1}^{\infty}, 0<a<1$

(b) $\{a^n\}_{n=1}^{\infty}, a>1$

(c) $\{\sqrt[n]{a}+1\}_{n=1}^{\infty}, a>0$

5. Consider the sequences

(a) $\left\{\dfrac{n+1}{3n+1}\right\}_{n=1}^{\infty}$ (b) $\left\{2+\left(-\dfrac{1}{2}\right)^n\right\}_{n=1}^{\infty}$ (c) $\left\{\left(\dfrac{2}{3}\right)^n-\dfrac{1}{2}\right\}_{n=1}^{\infty}$

i. Show that the sequence converges.
ii. State the limit, L, of this sequence.
iii. Use the definition of a limit to prove that the sequence has the limit L.

6. Consider the sequence $u_n = \dfrac{1+2\cdot 5^n}{4+3\cdot 5^n}$.

(a) i. Show that the sequence converges.
ii. State the limit, L, of this sequence.

(b) Use the definition of a limit to prove that the sequence has the limit L.

1.2.3 MORE LIMIT DEFINITIONS

We start by considering increasing and decreasing sequences. For example, if we consider the sequence $\left\{\frac{4}{n}\right\}_{n=1}^{\infty}$ its terms are given by $u_1 = 4$, $u_2 = 2$, $u_3 = \frac{4}{3}$, and so on. Our first observation is that $u_1 > u_2 > u_3 > \ldots$. That is, the terms of the sequence are *decreasing*. Similarly, if we consider the sequence $\left\{2-\left(\frac{1}{2}\right)^{n-1}\right\}_{n=1}^{\infty}$, its terms are $u_1 = 1$, $u_2 = \frac{3}{2}$, $u_3 = \frac{7}{4}$, and so on. However, this time we have that $u_1 < u_2 < u_3 < \ldots$. That is, the terms of the sequence are *increasing*.

(a) The sequence $\{u_n\}_{n=1}^{\infty}$ is **decreasing** if $u_1 \geq u_2 \geq u_3 \geq \ldots \geq u_n \geq u_{n+1} \geq \ldots$

(b) The sequence $\{u_n\}_{n=1}^{\infty}$ is **increasing** if $u_1 \leq u_2 \leq u_3 \leq \ldots \leq u_n \leq u_{n+1} \leq \ldots$

A sequence that is either **increasing** or **decreasing**, it is called **monotonic**.

Note: If the sequences are such that $u_1 > u_2 > u_3 > \ldots$ or $u_1 < u_2 < u_3 < \ldots$ we say that the sequences are **strictly decreasing** or **strictly increasing** respectively.

In Example 1.11 (d) we introduced the term **bounded** sequence – we now formalise this notion by way of definition.

We say that the sequence $\{u_n\}_{n=1}^{\infty}$ is **bounded** if there is a number $M > 0$ for which $|u_n| \leq M$, for all n. (M is called a **bound**)

It should be pointed out that a bound is not the same as a limit – although the two may coincide. For example, in the sequence $\left\{\frac{4}{n}\right\}_{n=1}^{\infty}$, the bound is 4 whereas the limit is 0.

The question remains, how do these definitions help us? The reason we consider whether a sequence is monotonic or bounded or . . . , is that very often we cannot compute the limit of a given sequence directly and so we must rely on indirect methods to help us determine whether or not the sequence is convergent.

A powerful tool that is used in the investigation of sequences is provided by the following theorem:

Every bounded, monotonic sequence converges.

EXAMPLE 1.13

Consider the sequence $\left\{\frac{2^n}{n!}\right\}_{n=1}^{\infty}$. Determine whether the sequence is

(a) increasing or decreasing,
(b) bounded and if so, determine the minimum bound,
(c) convergent.

Solution

(a) We start by evaluating the first few terms of the sequence:

$$u_1 = \frac{2^1}{1!} = 2\,,\ u_2 = \frac{2^2}{2!} = 2\,,\ u_3 = \frac{2^3}{3!} = \frac{4}{3}\,,\ u_4 = \frac{2^4}{4!} = \frac{2}{3}$$

So, at first glance it appears that the sequence is decreasing.
However, to prove that the sequence is decreasing we need to show that $u_{n+1} \le u_n$ for all values of $n \ge 1$ (or that $u_{n+1} - u_n \le 0$).

With $u_n = \frac{2^n}{n!}$ we have that $u_{n+1} = \frac{2^{n+1}}{(n+1)!}$.

$$\text{Therefore, } u_{n+1} - u_n = \frac{2^{n+1}}{(n+1)!} - \frac{2^n}{n!} = \frac{2 \cdot 2^n}{(n+1)n!} - \frac{2^n}{n!}$$

$$= \frac{2^n}{n!}\left[\frac{2}{n+1} - 1\right]$$

$$= \frac{2^n}{n!}\left[\frac{1-n}{n+1}\right]$$

For $n \ge 1$, $1 - n \le 0$ and $n + 1 \ge 2$ $\therefore \frac{1-n}{n+1} \le 0$.

Then, as $\frac{2^n}{n!} > 0$ for $n \ge 1$ $\Rightarrow \frac{2^n}{n!}\left[\frac{1-n}{n+1}\right] \le 0$ for all $n \ge 1$.

Therefore, as we have shown that $u_{n+1} - u_n \le 0$ for all $n \ge 1$, the sequence decreases.

(b) As u_1 is the largest value of the sequence (together with u_2), the minimum upper bound is 2.

(c) The first thing we note is that we are not able to evaluate $\lim\limits_{n \to \infty} \frac{2^n}{n!}$ directly (as it would take on the form $\frac{\infty}{\infty}$).

However we can write, $\frac{2^n}{n!} = \frac{2 \times 2 \times 2 \ldots \times 2 \times 2 \times 2}{1 \times 2 \times 3 \ldots \times (n-1) \times n} = \frac{2}{1} \times \frac{2}{2} \times \frac{2}{3} \times \ldots \times \frac{2}{n-1} \times \frac{2}{n}$.

Then, as $\frac{2}{n} \to 0$ as $n \to \infty$ and every other term in the product is positive and less than 2 the overall product must also tend to zero.
That is, the sequence converges (and has a limit of 0).

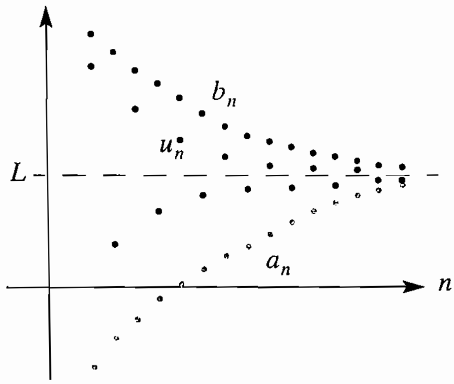
Note: All that was required was to state if the sequence converged. This could have been done by simply noting that we had a bounded, monotonic sequence and so it had to converge.

While we have now introduced some tools (or rather definitions) to help us deal with limit problems, the more interesting sequences are the ones that resist our attempts to determine their limits.

A second indirect tool that we can rely on is given by the following theorem – known as **The Squeeze Theorem**:

If the sequences $\{a_n\}_{n=1}^{\infty}$ and $\{b_n\}_{n=1}^{\infty}$ are convergent sequences, both converging to the limit L and if there exists an integer $n \geq n_1$ such that for all $n \geq n_1$, $a_n \leq u_n \leq b_n$, then $\{u_n\}_{n=1}^{\infty}$ also converges to L.

The above theorem can be visualised as follows:



As the value of n increases we see that the sequences $\{a_n\}_{n=1}^{\infty}$ and $\{b_n\}_{n=1}^{\infty}$ converge to a limit L. Then, as the sequence $\{u_n\}_{n=1}^{\infty}$ is squeezed between $\{a_n\}_{n=1}^{\infty}$ and $\{b_n\}_{n=1}^{\infty}$ as $n \to \infty$ it too converges to the limit L.

EXAMPLE 1.14

Use the squeeze theorem to show that $\lim\limits_{n \to \infty} \dfrac{\ln(n)}{n^2} = 0$

Solution

We first note that $0 < \ln(n) < n$ for all $n > 1$.

Then, dividing through by n^2 we have: $\dfrac{0}{n^2} < \dfrac{\ln(n)}{n^2} < \dfrac{n}{n^2}$

$$\Leftrightarrow 0 < \frac{\ln(n)}{n^2} < \frac{1}{n}$$

If we consider the three sequences, $\{a_n\}_{n=1}^{\infty} \equiv \{0\}_{n=1}^{\infty}$, $\{u_n\}_{n=1}^{\infty} \equiv \left\{\dfrac{\ln(n)}{n^2}\right\}_{n=1}^{\infty}$ and

$\{b_n\}_{n=1}^{\infty} \equiv \left\{\dfrac{1}{n}\right\}_{n=1}^{\infty}$ we have that $a_n \leq u_n \leq b_n$ for all $n \geq 1$.

Then, as $\lim\limits_{n \to \infty} a_n = 0$ and $\lim\limits_{n \to \infty} b_n = 0$, by the squeeze theorem we must have that $\lim\limits_{n \to \infty} u_n = 0$.

Therefore, we have that $\lim\limits_{n \to \infty} \dfrac{\ln(n)}{n^2} = 0$.

EXERCISES 1.2.3

1. Determine which of the following infinite sequences (i.e., $n = 1, 2, 3, \ldots$) are
(a) increasing;
(b) decreasing;
(c) not monotonic;
(d) bounded;
(e) convergent.

i. $\left\{\frac{2n-1}{3n+2}\right\}$ ii. $\{\cos(n)\}$ iii. $\left\{\frac{3^n}{1+3^n}\right\}$ iv. $\left\{\frac{n^2-1}{n}\right\}$

v. $\left\{\frac{2^n}{n^2}\right\}$ vi. $\left\{4-\left(\frac{1}{2}\right)^n\right\}$ vii. $\left\{2+\left(\frac{1}{2}\right)^n\right\}$ viii. $\left\{\frac{n}{\sin(n)}\right\}$

ix. $\left\{\frac{\sin(n^2)}{n+1}\right\}$ x. $\left\{\frac{n}{e^n}\right\}$ xi. $\{(ne)^{1/n}\}$ xii. $\left\{\frac{3^n}{2^{n+1}}\right\}$

2. (a) Prove that $\left\{\frac{4}{n+5}\right\}_{n=1}^{\infty}$ is a decreasing sequence.
(b) Determine its minimum upper bound.
(c) i. State if the sequence converges ii. Determine its limit.

3. (a) Prove that $\left\{\frac{n}{n^2+1}\right\}_{n=1}^{\infty}$ is a decreasing sequence.
(b) Determine its minimum upper bound.
(c) i. State if the sequence converges ii. Determine its limit.

4. Make use of the squeeze theorem to investigate the convergence or divergence of the sequence $\left\{\frac{\sin(n)}{n^2}\right\}_{n=1}^{\infty}$.

5. Make use of the squeeze theorem to prove that if $0 < k < 1$, the sequence $\{u_n\}_{n=1}^{\infty}$, where $u_n = (n+1)^k - n^k$, has a limit 0.

6. Consider the sequence $\{u_n\}_{n=1}^{\infty}$ where $u_n = \frac{n}{\sqrt{n^2+an+b}}$, $a > 0$ and $b \geq 0$.

(a) Use the squeeze theorem to show that $\lim_{n\to\infty} u_n = 1$.

(b) Hence, evaluate $\lim_{n\to\infty}\left(\frac{1}{\sqrt{n^2+1}} + \frac{1}{\sqrt{n^2+2}} + \ldots + \frac{1}{\sqrt{n^2+n}}\right)$.

1.2.4 ALGEBRA OF LIMITS

So far we have relied on an intuitive approach to determine some of the limits. For example, when dealing with a sequence such as $u_n = \dfrac{2n^2+3n}{6n^2-n}$, we have proceeded along the lines:

Step 1: Divide through by n^2 to give $u_n = \dfrac{2+\frac{3}{n}}{6-\frac{1}{n}}$

Step 2: Observing that as $n \to \infty$, $\frac{3}{n} \to 0$ and $\frac{1}{n} \to 0$.

Step 3: Deducing that as $n \to \infty$, $u_n \to \frac{2+0}{6-0} = \frac{2}{6} = \frac{1}{3}$.

In fact, these steps, although somewhat intuitive, are based on a number of rules that apply to limits. We provide a summary of these rules, so that they may then be confidently applied to more demanding problems.

If the sequences $\{a_n\}_{n=1}^{\infty}$ and $\{b_n\}_{n=1}^{\infty}$ are convergent sequences then:

1. $\lim\limits_{n\to\infty} ka_n = k \lim\limits_{n\to\infty} a_n$, where k is a real number.

2. $\lim\limits_{n\to\infty}(a_n + b_n) = \lim\limits_{n\to\infty} a_n + \lim\limits_{n\to\infty} b_n$

3. $\lim\limits_{n\to\infty}(a_n - b_n) = \lim\limits_{n\to\infty} a_n - \lim\limits_{n\to\infty} b_n$

4. $\lim\limits_{n\to\infty}(a_n \cdot b_n) = (\lim\limits_{n\to\infty} a_n)(\lim\limits_{n\to\infty} b_n)$.

5. $\lim\limits_{n\to\infty}\left(\dfrac{a_n}{b_n}\right) = \dfrac{\lim\limits_{n\to\infty} a_n}{\lim\limits_{n\to\infty} b_n}$, assuming that $\lim\limits_{n\to\infty} b_n \neq 0$

Note: If $\lim\limits_{n\to\infty} a_n \neq 0$ and $\lim\limits_{n\to\infty} b_n = 0$, then $\lim\limits_{n\to\infty}\left(\dfrac{a_n}{b_n}\right)$ does not exist.

If $\lim\limits_{n\to\infty} a_n = 0$ and $\lim\limits_{n\to\infty} b_n = 0$, then $\lim\limits_{n\to\infty}\left(\dfrac{a_n}{b_n}\right)$ may or may not exist.

6. $\lim\limits_{n\to\infty}(a_n)^k = (\lim\limits_{n\to\infty} a_n)^k$, for any real number k (and if $(\lim\limits_{n\to\infty} a_n)^k$ exists)

7. $\lim\limits_{n\to\infty}(k^{a_n}) = k^{(\lim\limits_{n\to\infty} a_n)}$, for any real number k (and if $k^{(\lim\limits_{n\to\infty} a_n)}$ exists).

To allow for continuity we leave the proofs to some of these statements (or theorems) to be dealt with elsewhere.

Again, we quote, leaving out the proof, a number of other useful limit results which you should call on as part of your mathematical arsenal.

1. $\lim_{n \to \infty} n^{1/n} = 1$
2. $\lim_{n \to \infty} k^{1/n} = 1$ for all $k > 0$
3. $\lim_{n \to \infty} \left(\frac{1}{n^k}\right) = 0$ for all $k > 0$
4. $\lim_{n \to \infty} k^n = 0$ for all $-1 < k < 1$
5. $\lim_{n \to \infty} \left(1 + \frac{k}{n}\right)^n = e^k$
6. If $\lim_{n \to \infty} |u_n| = 0$, then $\lim_{n \to \infty} u_n = 0$ also.

EXAMPLE 1.15

Find the limits of the following sequences:

(a) $u_n = 5 + \left(\frac{1}{2}\right)^n + \frac{2}{n^2},\ n \in \mathbb{Z}^+$ (b) $u_n = \frac{n + \sin(n)}{n^2 + 2^n},\ n \in \mathbb{Z}^+$

Solution

(a)
$$\lim_{n \to \infty}\left(5 + \left(\frac{1}{2}\right)^n + \frac{2}{n^2}\right) = \lim_{n \to \infty}(5) + \lim_{n \to \infty}\left(\left(\frac{1}{2}\right)^n\right) + \lim_{n \to \infty}\left(\frac{2}{n^2}\right)$$
$$= 5 + 0 + 0$$
$$= 5$$

(b) Care needs to be taken here, for if we blindly use the rule $\lim_{n \to \infty}\left(\frac{a_n}{b_n}\right) = \frac{\lim_{n \to \infty} a_n}{\lim_{n \to \infty} b_n}$ we end up with the expression $\lim_{n \to \infty}\left(\frac{n + \sin(n)}{n^2 + 2^n}\right) = \frac{\lim_{n \to \infty}(n + \sin(n))}{\lim_{n \to \infty}(n^2 + 2^n)}$ which is of the form $\frac{\infty}{\infty}$ and this is undefined!

To determine this limit we first re-arrange the expression:

$$u_n = \frac{n + \sin(n)}{n^2 + 2^n} = \frac{\frac{n + \sin(n)}{n}}{\frac{n^2 + 2^n}{n}} \text{ (after dividing through by } n\text{)}$$
$$= \frac{1 + \frac{\sin(n)}{n}}{n + \frac{2^n}{n}}$$

Next, $\lim_{n \to \infty}\left(1 + \frac{\sin(n)}{n}\right) = \lim_{n \to \infty}(1) + \lim_{n \to \infty}\left(\frac{\sin(n)}{n}\right) = 1 + 0 = 1$

and $\lim_{n \to \infty}\left(n + \frac{2^n}{n}\right) = \lim_{n \to \infty}(n) + \lim_{n \to \infty}\left(\frac{2^n}{n}\right) = \infty$ (or rather, as $n \to \infty$, $n + \frac{2^n}{n} \to \infty$).

So that the limit now takes on the form $\frac{1}{\infty}$ and so, $\lim_{n \to \infty}\left(\frac{n + \sin(n)}{n^2 + 2^n}\right) = 0$.

EXERCISES 1.2.4

1. Determine the limit of the following sequences.

(a) $u_n = 4 + \left(\frac{2}{3}\right)^n + \frac{5}{n}, n \in \mathbb{Z}^+$

(a) $u_n = 8n^{1/n} + \frac{\cos(n)}{n} - \sqrt{\frac{1}{n}}, n \in \mathbb{Z}^+$

(c) $u_n = \sqrt{n^2 + 4} - n, n \in \mathbb{Z}^+$

(d) $u_n = \left(1 - \frac{1}{n}\right)^n \left(\frac{\ln(n)}{n} - 1\right)$

(e) $\left\{\frac{n^2 + n}{3n^2 - 2n}\right\}_{n=1}^{\infty}, n \in \mathbb{Z}^+$

(f) $\left\{\frac{4n^2}{2n^3 - 1}\right\}_{n=1}^{\infty}, n \in \mathbb{Z}^+$

(g) $\left\{\frac{1}{2^n}\left(2 + \frac{1}{n}\right)^n\right\}_{n=1}^{\infty}, n \in \mathbb{Z}^+$

(h) $\left\{\frac{\sqrt{n} - 1}{\sqrt{n} + 1}\right\}_{n=1}^{\infty}, n \in \mathbb{Z}^+$

(i) $u_n = \frac{(3n + 2)(2n - 5)}{n^2}, n \in \mathbb{Z}^+$

(j) $u_n = \frac{3n^2}{(n - 1)(n - 2)}, n \in \mathbb{Z}^+, n \geq 3.$

(k) $u_n = \frac{1}{n}\left(\frac{n + 1}{2} + \frac{n^2}{3^n}\right), n \in \mathbb{Z}^+$

(l) $u_n = \left(\frac{n}{2^n}\right)^{1/n}, n \in \mathbb{Z}^+$

EXERCISES 1.2.5 – MISCELLANEOUS QUESTIONS

1. (a) Determine the nature of the sequence $u_n = \cos\left(\frac{n\pi}{2}\right) + \frac{1}{n}, n \in \mathbb{Z}^+$.

(b) Find the minimum upper bound and maximum lower bound of this sequence.

2. Consider the sequence $u_n = \frac{3^n - n^2 \cdot 2^n}{(n+1) \cdot 3^n}$.

(a) Plot this sequence.
(b) Find an upper and lower bound for this sequence.
(c) Use (a) and (b) to determine if the sequence is convergent or divergent and find its limit if it exists.

3. Prove, using the definition of a limit, that $\lim_{n \to \infty} \frac{3 + 2\sqrt{n}}{\sqrt{n}} = 2$.

4. Use the definition of a limit to show that $\lim_{n \to \infty} \frac{3n^2 + 1}{n^2 - 5n} = 3$.

5. Consider the sequence $\{n(0.9)^n\}_{n=1}^{\infty}$.

(a) Plot a graph of this sequence.
(b) Determine the values of n for which the sequence is
  i. increasing
  ii. decreasing.
(c) Determine the minimum upper bound for this sequence.
(d) Deduce that the sequence converges.
(e) State the limit of this sequence.

6. Determine

(a) $\lim_{n \to \infty} \left(\frac{3n - 1}{4n + 1}\right)^4$ (b) $\lim_{n \to \infty} \left(\frac{1 + 2 + 3 + \ldots + n}{n^2}\right)$

(c) $\lim_{n \to \infty} \left(\frac{1^2 + 2^2 + 3^2 + \ldots + n^2}{n^3}\right)$ (d) $\lim_{n \to \infty} (n\sqrt{n^2 + 1} - n)$

7. Prove formally that $\lim_{n \to \infty} \frac{3n}{2n + 1} = \frac{3}{2}$.

8. Using the definition of a limit, prove that $\lim_{n \to \infty} \frac{n^2}{n^2 + n - 1} = 1$.

9. (a) Use the definition of a limit to prove that $\lim_{n \to \infty} \frac{1 + 2 \cdot 5^n}{5 + 3 \cdot 5^n} = \frac{2}{3}$.

(b) Prove, using the definition of a limit, that $\lim_{n \to \infty} (\sqrt{n^2 + n} - n) = \frac{1}{2}$.

10. Determine $\lim_{n \to \infty} u_n$ where $u_n = \dfrac{(3n+1)(n-2)}{n(n+3)}$, $n \in \mathbb{Z}^+$.

11. Consider the sequence $\left\{\dfrac{a^n}{1+a^{n+1}}\right\}_{n=1}^{\infty}$. Determine the limit of this sequence if

(a) $|a| < 1$ (b) $|a| = 1$ (c) $|a| > 1$.

12. Consider the sequence $\{u_n\}_{n=1}^{\infty}$, where $2u_{n+2} - u_{n+1} - u_n = 0$, $u_1 = 1$ and $u_2 = -2$.

(a) Using mathematical induction, show that $u_n = -1 - \left(-\frac{1}{2}\right)^{n-2}$, $n \in \mathbb{Z}^+$.

(b) Hence, determine $\lim_{n \to \infty} u_n$.

13. Given the real sequence $\{u_n\}_{n=1}^{\infty} = \left\{(3^n + 4^n)^{\frac{1}{n}}\right\}_{n=1}^{\infty}$.

(a) Prove that $\{u_n\}_{n=1}^{\infty}$ is bounded.

(b) i. Prove that $(3^n + 4^n)^{\frac{n+1}{n}} > 3^{n+1} + 4^{n+1}$, $n \geq 1$.

ii. Hence deduce that $u_{n+1} < u_n$, $n \geq 1$.

(c) Determine whether $\{u_n\}_{n=1}^{\infty}$ is convergent, giving the reason for your decision.

14. Consider the sequence $\{u_n\}_{n=1}^{\infty}$ where $u_n = \sqrt{2u_{n-1}}$, $u_1 = \sqrt{2}$.

(a) Make use of mathematical induction show that $u_n > 0$ and $u_{n+1} > u_n$ for $n \in \mathbb{Z}^+$.

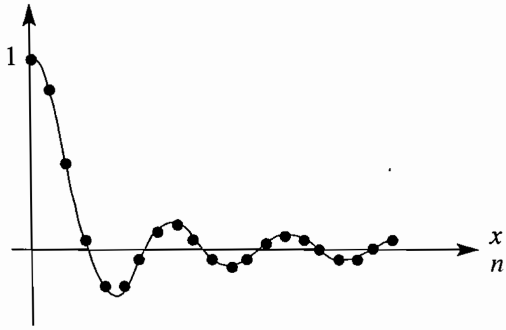
(b) From part (a), show that $u_n < 2$.

(c) Hence, show that the sequence converges and find its limit.

1.3 LIMIT OF A FUNCTION

1.3.1 SEQUENCES AS REAL-VALUED FUNCTIONS

Consider the sequence $\{u_n\}_{n=1}^{\infty}$ where $u_n = \frac{\sin(n)}{n}$ and the function f where $f(x) = \frac{\sin(x)}{x}$, $x > 0$ and x is real. The graph of the function f consists of the set of points $(x, f(x))$ which is depicted by a continuous curve, whereas the graph of the sequence $\{u_n\}_{n=1}^{\infty}$ consists of the set of points (n, u_n) which is depicted by discrete points:



From the graphs we see that the sequence $\{u_n\}_{n=1}^{\infty}$ represents a discrete sample of its corresponding continuous function. The key observation is that in both instances, the same behaviour is displayed. This reflects the sense in which sequences are special subsets of their corresponding functions. The benefit of this observation is that for such sequences we can often make use of their functional counterpart. This leads us to the following statement:

If the function, f, is well defined and is such that $\lim_{x \to \infty} f(x) = L$, then $\lim_{n \to \infty} u_n = L$.

It must be noted that the above statement only considers the case where $n \to \infty$ (i.e., $x \to \infty$).

So, how else can this observation help when working with limits of sequences? Let us return to Exercise 1.2.3 Q.3. The corresponding real valued function of the sequence $\left\{\frac{n}{n^2+1}\right\}_{n=1}^{\infty}$ is given by $f(x) = \frac{x}{x^2+1}$. If we can show that the function f decreases for some $x \geq x_0$ we can then deduce that the sequence $\left\{\frac{n}{n^2+1}\right\}_{n=1}^{\infty}$ will also decrease for some integer value $n \geq n_0$. To show that f decreases for $x \geq x_0$ we need to show that $f'(x) < 0$ for $x > x_0$.

So, with $f(x) = \frac{x}{x^2+1} \Rightarrow f'(x) = \frac{1 \times (x^2+1) - x \times (2x)}{(x^2+1)^2} = \frac{1-x^2}{(x^2+1)^2}$

Then, for $x > 1$, $1 - x^2 < 0 \Rightarrow f'(x) < 0$. That is, f decreases for values of $x > 1$. Which in turn corresponds to saying that $\left\{\frac{n}{n^2+1}\right\}_{n=1}^{\infty}$ decreases for $n \geq 2$. In fact, the sequence decreases for $n \geq 1$ – but all that matters is that we can identify some value n_0 for which $u_{n+1} < u_n$. Using calculus to show that $f'(x) < 0$ for $x > x_0$ is easier than showing that $u_{n+1} < u_n$.

EXAMPLE 1.16

Show that $u_n = \frac{\ln(n)}{n^2}, n \in \mathbb{Z}^+$ is a decreasing sequence.

Solution

We start by considering its corresponding real-valued function, $f(x) = \frac{\ln x}{x^2}, x > 0$.

$$\text{Now, } f'(x) = \frac{\frac{1}{x} \times x^2 - 2x \times \ln x}{(x^2)^2} = \frac{x - 2x\ln x}{x^4} = \frac{1 - 2\ln x}{x^3}$$

As $x > 0 \Rightarrow x^3 > 0$ and so all we need to consider is the sign of $1 - 2\ln x$.

Solving for $1 - 2\ln x < 0$ we have:

$$1 - 2\ln x < 0 \Leftrightarrow 1 < 2\ln x$$
$$\Leftrightarrow 0.5 < \ln x$$
$$\Leftrightarrow e^{0.5} < x$$

Now, $e^{0.5} \approx 1.648\ldots$ and so we can say that $f(x) = \frac{\ln x}{x^2}$ is a decreasing for $x \geq 2$ (after electing to choose the first integer value of x).

So, we can also conclude that the sequence $u_n = \frac{\ln(n)}{n^2}$ is a decreasing sequence for $n \geq 2$.

In the above example we also note that just because we have used the corresponding real-valued function for the given sequence, it has not rid us of the problem we faced when trying to determine aspects of its behaviour. That is, in trying to directly evaluate the limit, $\lim_{n \to \infty} \frac{\ln(n)}{n^2}$, we end up with the expression $\frac{\infty}{\infty}$ which we are unable to determine. The same happens if we use the function $f(x) = \frac{\ln x}{x^2}$. That is, evaluating $\lim_{x \to \infty} \frac{\ln x}{x^2}$ also leads us to the expression $\frac{\infty}{\infty}$. And yet, both $\lim_{n \to \infty} \frac{\ln(n)}{n^2} = 0$ and $\lim_{x \to \infty} \frac{\ln x}{x^2} = 0$.

In fact, as we have seen, we strike difficulties when trying to directly determine expressions that produce any one of the following results:

$$\frac{0}{0};\ \frac{\infty}{\infty};\ \frac{0}{\infty};\ \frac{\infty}{0};\ 0 \times \infty;\ 0^{\infty};\ \infty^0;\ 0^0;\ 1^{\infty}$$

Such expressions are referred to as an **indeterminate form**. When dealing with real-valued functions, these 'difficulties' need not occur only when trying to determine $\lim_{x \to \infty} f(x)$. They occur just as readily when considering limit expressions of the form $\lim_{x \to c} f(x)$ where c is a real number. For example, consider the function $f(x) = \frac{x + \sin x}{x}$. In trying to evaluate $\lim_{x \to 0} f(x)$ directly we produce the result $\frac{0}{0}$, which is of an indeterminate form – and yet we actually end up with $\lim_{x \to 0} f(x) = 2$!

The following tables show some of the forms for which limits cannot be evaluated directly.

$\lim_{x \to c} f(x)$	$\lim_{x \to c} g(x)$	$h(x)$	$\lim_{x \to c} h(x)$
$a \in \mathbb{R}$	$b \in \mathbb{R}, b \neq 0$	$\frac{f(x)}{g(x)}$	$\frac{a}{b}$
$a \in \mathbb{R}, a \neq 0$	0	$\frac{f(x)}{g(x)}$	$\pm\infty$
0	0	$\frac{f(x)}{g(x)}$	$\frac{0}{0}$ **indeterminate**
$a \in \mathbb{R}, a \neq 0$	$\pm\infty$	$\frac{f(x)}{g(x)}$	0
$\pm\infty$	$b \in \mathbb{R}, b \neq 0$	$\frac{f(x)}{g(x)}$	$\pm\infty$
$\pm\infty$	$\pm\infty$	$\frac{f(x)}{g(x)}$	$\pm\frac{\infty}{\infty}$ **indeterminate**

$\lim_{x \to c} f(x)$	$\lim_{x \to c} g(x)$	$h(x)$	$\lim_{x \to c} h(x)$
$a \in \mathbb{R}$	$b \in \mathbb{R}$	$f(x) \pm g(x)$	$a \pm b$
$a \in \mathbb{R}$	$\pm\infty$	$f(x) \pm g(x)$	$\pm\infty$
∞	∞	$f(x) + g(x)$	∞
∞	$-\infty$	$f(x) - g(x)$	∞
∞	∞	$f(x) - g(x)$	**indeterminate**
∞	$-\infty$	$f(x) + g(x)$	**indeterminate**

$\lim_{x\to c} f(x)$	$\lim_{x\to c} g(x)$	$h(x)$	$\lim_{x\to c} h(x)$
$a \in \mathbb{R}$	$b \in \mathbb{R}$	$f(x) \times g(x)$	$a \times b$
$a \in \mathbb{R}, a \neq 0$	$\pm\infty$	$f(x) \times g(x)$	$\pm\infty$
$a = 0$	$\pm\infty$	$f(x) \times g(x)$	$0 \times \pm\infty$, **indeterminate**

Again, we see from these tables that there are cases when the limit cannot be evaluated – these are known as the indeterminate forms and take on the forms, $\infty - \infty$, $0 \times \pm\infty$, $\frac{\infty}{\infty}$ and $\frac{0}{0}$. Of course, there are other forms for which the limit cannot be determined, e.g., 0^0, however these will be dealt with separately and via the use of examples.

1.3.2 L'HOSPITAL'S RULE – LIMITS OF INDETERMINATE FORMS

In this section, crucial for solving the 'seemingly unsolvable', we want to introduce a rule which is a major tool for calculation of limits that seem to be undefined, such as $\frac{0}{0}, \frac{\infty}{\infty}, 0^{\infty}, \infty^0$ etc...

In particular, the rule that we will refer to holds for limits that take on the form $\frac{0}{0}$ and $\frac{\pm\infty}{\pm\infty}$. The rule is attributed to the XVIIIth century French mathematician Guillaume de l'Hospital and as such carries his name.

L'Hospital's rule

If $\lim_{x\to c} \frac{f(x)}{g(x)}$ takes on an indeterminate form $\frac{0}{0}$ or $\frac{\pm\infty}{\pm\infty}$ then $\lim_{x\to c} \frac{f(x)}{g(x)} = \lim_{x\to c} \frac{f'(x)}{g'(x)}$

Note: Should the result $\lim_{x\to c} \frac{f'(x)}{g'(x)}$ still be of an indeterminate form $\frac{0}{0}$ or $\frac{\pm\infty}{\pm\infty}$, the process can be repeated a second time, so that we then evaluate the limit $\lim_{x\to c} \frac{f''(x)}{g''(x)}$. In fact we can reapply the rule again (and again – if need be) until it is not of the form $\frac{0}{0}$ or $\frac{\pm\infty}{\pm\infty}$.

While we will not provide a rigourous proof of L'Hospital's rule, we will show that the result holds true for the indeterminate form when $f(c) = 0$ and $g(c) = 0$

If $f(c) = 0$ and $g(c) = 0$ we have $\lim_{x\to c} \frac{f(x)}{g(x)} = \lim_{x\to c} \frac{f(x)-0}{g(x)-0}$

$$= \lim_{x\to c} \frac{f(x)-f(c)}{g(x)-g(c)}$$

$$\therefore \lim_{x \to c} \frac{f(x)}{g(x)} = \lim_{x \to c} \frac{\frac{f(x)-f(c)}{x-c}}{\frac{g(x)-g(c)}{x-c}}$$

$$= \frac{\lim_{x \to c} \frac{f(x)-f(c)}{x-c}}{\lim_{x \to c} \frac{g(x)-g(c)}{x-c}}$$

$$= \frac{\lim_{x \to c} f'(x)}{\lim_{x \to c} g'(x)}$$

$$= \lim_{x \to c} \frac{f'(x)}{g'(x)}$$

So, as long as $g'(c) \neq 0$, the result is complete. If the quotient of the derivatives is still of the form $\frac{0}{0}$ or $\frac{\pm\infty}{\pm\infty}$ we have to apply L'Hospital's rule again and calculate the quotient of the second, third,.... derivatives at $x = c$ until the quotient yields a properly defined value.

EXAMPLE 1.17

One of the classic examples of the use of L'Hospital's rule is the calculation of $\lim_{x \to 0} \frac{\sin x}{x}$. Later we shall give an alternative way of calculating this limit by using Maclaurin's expansion.

Solution

As $\frac{\sin 0}{0}$ is of the form $\frac{0}{0}$ we will need to apply L'Hospital's rule to solve this limit.

Letting $f(x) = \sin x$ we have $f'(x) = \cos x$ and letting $g(x) = x$ we have $g'(x) = 1$.

Then, $\lim_{x \to 0} \frac{\sin x}{x} = \lim_{x \to 0} \frac{\cos x}{1}$

$$= \frac{1}{1}$$

$$= 1$$

EXAMPLE 1.18

Evaluate $\lim_{x \to 0} \frac{4x - 2\sin(2x)}{5x^3}$.

Solution

Let $f(x) = 4x - 2\sin(2x)$ and $g(x) = 5x^3$. Then, as $\frac{f(0)}{g(0)} = \frac{0}{0}$ we need to apply L'Hospital's rule. We obtain $f'(x) = 4 - 4\cos 2x$ and $g'(x) = 15x^2$ to give

$$\lim_{x \to 0} \frac{f(x)}{g(x)} = \lim_{x \to 0} \frac{f'(x)}{g'(x)} = \lim_{x \to 0} \frac{4 - 4\cos(2x)}{15x^2} = \frac{4-4}{0} = \frac{0}{0}$$

This quotient is still of the form $\frac{0}{0}$, so we apply our rule again.

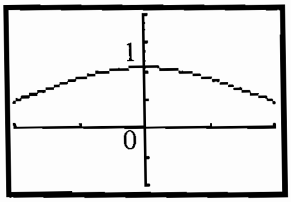
That is, $\lim_{x\to 0}\frac{f''(x)}{g''(x)} = \lim_{x\to 0}\frac{8\sin(2x)}{30x}$ which is still of the form $\frac{0}{0}$.

Upon applying the rule a third time we see that it is possible to get a quotient that is properly defined, viz. $\lim_{x\to 0}\frac{8\sin(2x)}{30x} = \lim_{x\to 0}\frac{16\cos(2x)}{30} = \frac{16}{30}$.

Therefore, $\lim_{x\to 0}\frac{4x-2\sin(2x)}{5x^3} = \frac{8}{15}$.

This is not an obvious result, especially if we look at the form of the original limit that we wanted to calculate.

In order to convince ourselves, we can plot the function $f(x) = \frac{4x-2\sin(2x)}{5x^3}$ and check the value that the graphics calculator produces as $x \to 0$. We see that this value coincides with that found by using L'Hospital's rule.



EXAMPLE 1.19

Evaluate $\lim_{x\to 1}\frac{\ln x}{x-1}$.

Solution

Setting $f(x) = \ln x$ and $g(x) = x-1$ we have $f(1) = \ln 1 = 0$ and $g(1) = 1-1 = 0$.

Again we end up with $\lim_{x\to 1}\frac{\ln x}{x-1} = \frac{0}{0}$ and so L'Hospital's rule is applicable.

We have $f'(x) = \frac{1}{x}$ and $g'(x) = 1$. Using the L'Hospital's rule, $\lim_{x\to 1}\frac{\left(\frac{1}{x}\right)}{x-1} = \frac{1}{1} = 1$.

Therefore, $\lim_{x\to 1}\frac{\ln x}{x-1} = 1$.

EXAMPLE 1.20

Evaluate $\lim\limits_{x \to 0} \frac{\tan x - x}{x - \sin x}$

Solution

When $x \to 0$, this quotient is again of the form $\frac{0}{0}$ and so L'Hospital's rule is applicable.

Setting $f(x) = \tan x - x$ and $g(x) = x - \sin x$ we have $f'(x) = \sec^2 x - 1$ and $g'(x) = 1 - \cos x$.

Applying L'Hospital's rule we get $\lim\limits_{x \to 0} \frac{\sec^2 x - 1}{1 - \cos x} = \lim\limits_{x \to 0} \frac{\tan^2 x}{1 - \cos x} = \frac{0}{0}$.

As we still have an indeterminate form we apply L'Hospital's rule again.
Now, $f''(x) = 2\tan x \sec^2 x$ and $g''(x) = \sin x$.

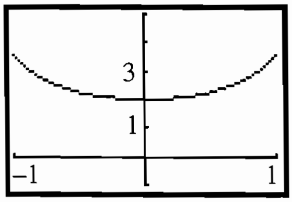
However, $\lim\limits_{x \to 0} \frac{2\tan x \sec^2 x}{\sin x} = \frac{0}{0}$ and so we need to make use of L'Hospital's rule again.

$f'''(x) = 2\sec^2 x \sec^2 x + 2\tan x \times 2\tan x \sec^2 x = 2\sec^4 x + 4\tan^2 x \sec^2 x$ and $g'''(x) = \cos x$.

Finally, $\lim\limits_{x \to 0} \frac{2\sec^4 x + 4\tan^2 x \sec^2 x}{\cos x} = \frac{2}{1} = 2$.

Therefore we have that $\lim\limits_{x \to 0} \frac{\tan x - x}{x - \sin x} = 2$.

Again, by no means an obvious result. We can check what the graphics calculator gives us if we graph the function $f(x) = \frac{\tan x - x}{x - \sin x}$ by looking at the value of the function when in the vicinity of $x = 0$.



We can see that the value found using the graphics calculator coincides with that obtained by L'Hospital's rule.

We shall not give the proof here, but it can be shown that L'Hospital's rule also applies to quotients of the form $\frac{\pm\infty}{\pm\infty}$.

EXAMPLE 1.21

Evaluate $\lim\limits_{x \to 0} \frac{\ln x}{\cot x}$

Solution

As $x \to 0$, the limit, $\lim\limits_{x \to 0} \frac{\ln x}{\cot x}$, is of the form $\frac{-\infty}{\infty}$.

With $f(x) = \ln x$ and $g(x) = \cot x$ we have $f'(x) = \frac{1}{x}$ and $g'(x) = -\frac{1}{\sin^2 x}$

So, applying L'Hospital's rule to this quotient, we can see that it becomes

$$\lim_{x \to 0} \frac{\left(\frac{1}{x}\right)}{\left(-\frac{1}{\sin^2 x}\right)} = \lim_{x \to 0} -\frac{\sin^2 x}{x}$$

Which is of the form 0/0 and so we again apply L'Hospital's rule to this quotient.

However, this time we note that $\frac{\sin^2 x}{x} = \frac{\sin x}{x} \times \sin x$.

So, $\lim_{x \to 0} -\frac{\sin^2 x}{x} = -\lim_{x \to 0} \frac{\sin^2 x}{x} = -\lim_{x \to 0}\left(\frac{\sin x}{x} \times \sin x\right) = -1 \times 0 = 0$.

Therefore, $\lim_{x \to 0} \frac{\ln x}{\cot x} = 0$.

EXAMPLE 1.22

Evaluate $\lim_{x \to 0} \frac{\cot x}{\ln x}$.

Solution

We shall now see the difference with $\lim_{x \to 0} \frac{\cot x}{\ln x}$, which is a ratio of the form $\frac{\infty}{-\infty}$.

Applying L'Hospital's rule we obtain $\lim_{x \to 0} \frac{\cot x}{\ln x} = \lim_{x \to 0} \frac{\left(-\frac{1}{\sin^2 x}\right)}{\left(\frac{1}{x}\right)} = -\lim_{x \to 0}\left(\frac{x}{\sin^2 x}\right)$,

which is of the form $\frac{0}{0}$. Repeating the procedure, we get

$$-\lim_{x \to 0}\left(\frac{x}{\sin^2 x}\right) = -\lim_{x \to 0}\left(\frac{1}{2\sin x \cos x}\right) = -\lim_{x \to 0}\left(\frac{1}{\sin 2x}\right) = -\infty.$$

EXAMPLE 1.23

Evaluate $\lim_{x \to 0} \frac{\ln(\tan x)}{\ln(\sin x)}$.

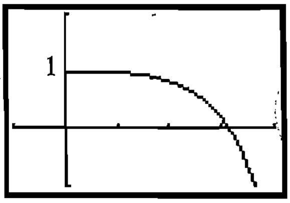
Solution

This limit is of the form $\frac{-\infty}{-\infty}$. Applying L'Hospital's rule we have:

$f(x) = \ln(\tan x) \therefore f'(x) = \frac{1}{\tan x} \cdot \frac{1}{\cos^2 x}$ and $g(x) = \ln(\sin x) \therefore g'(x) = \frac{\cos x}{\sin x}$

$$\therefore \lim_{x \to 0} \frac{\ln(\tan x)}{\ln(\sin x)} = \lim_{x \to 0} \frac{\frac{1}{\tan x} \cdot \frac{1}{\cos^2 x}}{\frac{\cos x}{\sin x}} = \lim_{x \to 0}\left(\frac{1}{\cos^2 x}\right) = \frac{1}{1} = 1.$$

We can check with the graphics calculator by plotting $f(x) = \dfrac{\ln(\tan x)}{\ln(\sin x)}$ and observing that this function, in the vicinity of $x = 0$, tends to 1. Which is again not an obvious result given our starting point.



EXAMPLE 1.24

Investigate $\lim\limits_{x \to \infty} \dfrac{\ln x}{x^n}$.

Solution

Again we have a limit of the form $\dfrac{\infty}{\infty}$.

Applying L'Hospital's rule, we have $\lim\limits_{x \to \infty} \dfrac{\ln x}{x^n} = \lim\limits_{x \to \infty} \dfrac{\left(\frac{1}{x}\right)}{(nx^{n-1})} = \lim\limits_{x \to \infty} \dfrac{1}{(nx^n)} = 0$.

This is quite a logical result, as $x^n \to \infty$ more rapidly than $\ln x$ would. Therefore we can say that for some n, x^n would reach infinity long before $\ln x$ (which would still be finite) and thus the quotient would tend to 0.

The same type of reasoning would apply to $\lim\limits_{x \to \infty} \dfrac{e^x}{x^n}$, $n > 1$, as $x^n \to \infty$ more rapidly than e^x, the quotient would tend to 0.

It is clear that L'Hospital's rule can also be applied to a product of the form $0 \times \infty$ as this product can always be converted to the form $\dfrac{0}{0}$ or $\dfrac{\infty}{\infty}$.

EXAMPLE 1.25

Evaluate $\lim\limits_{x \to 0} x \ln x$.

Solution

This product is of the form $0 \times -\infty$. To be able to use L'Hospital's rule we need to rearrange the expression so that we may obtain an indeterminate form.

We can rewrite $\lim\limits_{x \to 0} x \ln x$ as $\lim\limits_{x \to 0} \dfrac{\ln x}{\left(\frac{1}{x}\right)}$ which is of the form $\dfrac{-\infty}{\infty}$.

Applying L'Hospital's rule we have $\lim\limits_{x \to 0} \dfrac{\left(\frac{1}{x}\right)}{\left(-\frac{1}{x^2}\right)} = \lim\limits_{x \to 0} (-x) = 0$

Therefore, $\lim\limits_{x \to 0} x \ln x = 0$.

EXAMPLE 1.26

Evaluate $\lim_{x \to \infty}\left(x\sin\frac{\pi}{x}\right)$.

Solution

This limit is of the form $0 \times \infty$. But to use L'Hospital's rule we need to rewrite the expression in the appropriate indeterminate form.

In this instance we rewrite $\lim_{x \to \infty}\left(x\sin\frac{\pi}{x}\right)$ as $\lim_{x \to \infty}\frac{\sin\frac{\pi}{x}}{\frac{1}{x}}$ (which is of the form $\frac{0}{0}$).

Applying L'Hospital's rule we obtain $\lim_{x \to \infty}\left(\frac{-\frac{\pi}{x^2}\cos\frac{\pi}{x}}{-\frac{1}{x^2}}\right) = \lim_{x \to \infty}\left(\pi\cos\frac{\pi}{x}\right) = \pi$.

Again, not an obvious result.

We now solve Example 1.26 by making use of a known limit, namely, $\lim_{x \to 0}\frac{\sin x}{x} = 1$. We do this by observing that the statements $\lim_{x \to 0}\frac{\sin x}{x} = 1$ and $\lim_{x \to \infty}\frac{\sin\left(\frac{1}{x}\right)}{\left(\frac{1}{x}\right)} = 1$ are equivalent.

That is, letting $u = \frac{1}{x}$ we have that $\lim_{x \to \infty}\frac{\sin\left(\frac{1}{x}\right)}{\left(\frac{1}{x}\right)} = \lim_{u \to 0}\frac{\sin u}{u} = 1$.

So, $\lim_{x \to \infty}\left(x\sin\frac{\pi}{x}\right) = \lim_{x \to \infty}\frac{\sin\left(\frac{\pi}{x}\right)}{\left(\frac{1}{x}\right)} = \lim_{x \to \infty}\pi\cdot\frac{\sin\left(\frac{\pi}{x}\right)}{\left(\frac{\pi}{x}\right)}$

$$= \pi\lim_{x \to \infty}\frac{\sin\left(\frac{\pi}{x}\right)}{\left(\frac{\pi}{x}\right)}$$

$$= \pi\lim_{u \to 0}\frac{\sin u}{u}, \text{ where } u = \frac{\pi}{x}$$

$$= \pi$$

EXAMPLE 1.27

Evaluate $\lim_{x \to 0} (5x \cdot 3\cot x)$

Solution

This is of the form $0 \times \infty$, but it can easily be transformed into the form $\frac{\infty}{\infty}$. This is done by rewriting $5x \cdot 3\cot x$ as $15 \cdot \frac{\cot x}{\left(\frac{1}{x}\right)}$. Therefore, using L'Hospital's rule we have:

$$\lim_{x \to 0} 15 \cdot \frac{\cot x}{\left(\frac{1}{x}\right)} = 15 \times \lim_{x \to 0} \frac{\cot x}{\left(\frac{1}{x}\right)} = 15 \lim_{x \to 0} \frac{\left(\frac{-1}{\sin^2 x}\right)}{\left(\frac{-1}{x^2}\right)}$$

$$= 15 \times \lim_{x \to 0} \left(\frac{x^2}{\sin^2 x}\right)$$

$$= 15 \times \lim_{x \to 0} \left(\frac{x}{\sin x}\right)\left(\frac{x}{\sin x}\right)$$

$$= 15 \times \left[\lim_{x \to 0} \left(\frac{x}{\sin x}\right)\right]\left[\lim_{x \to 0} \left(\frac{x}{\sin x}\right)\right]$$

$$= 15 \times 1 \times 1$$

$$= 15$$

Note that in this problem we applied L'Hospital's rule once.

EXAMPLE 1.28

Evaluate $\lim_{x \to 0} (1 - \cos x)\ln x$

Solution

This is of the form $0 \times \infty$, but can be easily transformed into the form $\frac{-\infty}{\infty}$. We do this by rewriting $(1 - \cos x)\ln x$ as $\frac{\ln x}{\frac{1}{(1 - \cos x)}}$.

Therefore, $\lim_{x \to 0} (1 - \cos x)\ln x = \lim_{x \to 0} \left(\frac{\ln x}{\frac{1}{(1 - \cos x)}}\right)$

Applying L'Hospital's rule, we obtain $\lim_{x \to 0} \left(\frac{\frac{1}{x}}{\frac{-\sin x}{(1 - \cos x)^2}}\right) = \lim_{x \to 0} \left(-\frac{(1 - \cos x)^2}{x \sin x}\right)$

As this is now of the form $\frac{0}{0}$ we can apply L'Hospital's rule:

That is, $\lim_{x \to 0}\left(-\frac{(1-\cos x)^2}{x\sin x}\right) = \lim_{x \to 0}\frac{2(1-\cos x)\sin x}{\sin x + x\cos x} = \lim_{x \to 0}\frac{2\sin x - \sin 2x}{\sin x + x\cos x}$

As this is still of the form $\frac{0}{0}$ we once again apply L'Hospital's rule, which will give us:

$$\lim_{x \to 0}\frac{2\sin x - \sin 2x}{\sin x + x\cos x} = \lim_{x \to 0}\frac{2\cos x - 2\cos 2x}{2\cos x - x\sin x} = \frac{0}{1} = 0$$

EXAMPLE 1.29

Evaluate $\lim_{x \to \frac{\pi}{4}}\frac{1-\tan x}{\cos 2x}$.

Solution

This is a limit of the form $\frac{0}{0}$, therefore we are in a position to apply L'Hospital's rule.

With $f(x) = 1 - \tan x$ and $g(x) = \cos 2x$ then $f'(x) = -\sec^2 x$ and $g'(x) = -2\sin 2x$.

So, $\lim_{x \to \frac{\pi}{4}}\frac{1-\tan x}{\cos 2x} = \lim_{x \to \frac{\pi}{4}}\frac{-\sec^2 x}{-2\sin 2x} = \frac{2}{2} = 1$.

EXAMPLE 1.30

Evaluate $\lim_{x \to 0} x^x$.

Solution

This is an interesting case as it appears to be 0^0, but we can transform it into the form $\frac{\infty}{\infty}$.

If we take the function $y = x^x$, then $\ln y = x\ln x = \frac{\ln x}{\frac{1}{x}}$, where $y > 0$.

In this expression we have that as x tends to 0, it takes on the form $\frac{-\infty}{\infty}$ and so we can apply L'Hospital's rule, giving: $\lim_{x \to 0}\left(\frac{\frac{1}{x}}{\frac{-1}{x^2}}\right) = \lim_{x \to 0}(-x) = 0$.

So it follows that $\lim_{x \to 0}\ln y = 0$, which means that $\lim_{x \to 0} y = 1$ that is, $\lim_{x \to 0} x^x = 1$.

To help us determine the limit we made use of a special 'transformation', i.e., we used $\ln y$ instead of using $y = x^x$. This worked because the log function is a strictly increasing function and so the behaviour of $y = x^x$ is reflected by the behaviour of $\ln y$.

Also, it should be noted that the limit $\lim_{x \to 0} x^x$ should more correctly be written as $\lim_{x \to 0^+} x^x$.

It is also important to once again realise that before applying L'Hospital's rule we need to make sure that the expression is indeterminate and of the required form, i.e., of the form $\frac{0}{0}$ or $\frac{\pm\infty}{\pm\infty}$. However, as we have seen in Examples 1.27, 1.28 and 1.30 it is possible to manipulate an expression that is not of the form $\frac{0}{0}$ or $\frac{\infty}{\infty}$ into one that is of the required form for L'Hospital's rule to apply.

EXERCISES 1.3

1. Determine the following limits.

(a) $\lim_{x \to 0} \left(\frac{x + \sin 2x}{x - \sin 2x}\right)$ (b) $\lim_{x \to \pi} \left(\frac{x - \pi}{\sin x}\right)$ (c) $\lim_{x \to \frac{\pi}{2}} \left(\frac{\sin 2x}{\cos x}\right)$

2. Determine the following limits.

(a) $\lim_{x \to \infty} \left(\frac{x}{e^{2x}}\right)$ (b) $\lim_{x \to \infty} \frac{\ln x}{x}$ (c) $\lim_{x \to \infty} \frac{2x}{x + \ln x}$

3. Determine the following limits.

(a) $\lim_{x \to 0} \left(\frac{2x}{x + \sin x}\right)$ (b) $\lim_{x \to 0} \left(\frac{\cos x - 1}{x^2}\right)$ (c) $\lim_{x \to 0} \left(\frac{x - \sin x}{x^3}\right)$

4. Determine the following limits.

(a) $\lim_{x \to \frac{\pi}{2}} \left(\frac{\sin x - 1}{\cos x}\right)$ (b) $\lim_{x \to 0^+} x \ln\left(1 + \frac{1}{x}\right)$ (c) $\lim_{x \to 1} \frac{\ln x - (x - 1)}{x - 1}$

5. Determine the following limits, if they exist.

(a) $\lim_{x \to \frac{\pi}{2}} (\tan x + \sec x)$ (b) $\lim_{x \to 1} \left(\frac{1}{\ln x} - \frac{1}{x - 1}\right)$ (c) $\lim_{x \to 1} \left(\frac{\ln x}{x^2 - x}\right)$

6. What is wrong in the calculation $\lim_{x \to 0} \frac{\cos x}{x^2} = \lim_{x \to 0} \frac{-\sin x}{2x} = \lim_{x \to 0} \frac{-\cos x}{2} = -\frac{1}{2}$?

7. Determine the following limits, if they exist.

(a) $\lim_{x \to \infty} \frac{1}{x} e^x$ (b) $\lim_{x \to \infty} \frac{x^2}{e^x}$ (c) $\lim_{x \to 0^+} (\sin x)(\ln x)$

8. Evaluate the following limits, if they exist.

(a) $\lim_{x \to 0} \frac{x - \sin x}{x^2 e^x}$ (b) $\lim_{x \to 0} \frac{1 - \cos x}{\sin^2 x}$ (c) $\lim_{x \to 1} \left(\frac{x^4 - 7x^3 + 8x^2 - 2}{x^3 + 5x - 6}\right)$

(d) $\lim_{x \to 0} \left(\operatorname{cosec} x - \frac{1}{x}\right)$ (e) $\lim_{x \to 0} x^2 \ln x$ (f) $\lim_{x \to 0} \left(\frac{e^x - e^{-x} - x^2 - 2}{\sin^2 x - x^2}\right)$

(g) $\lim_{x \to 0} \left(\frac{\cot x}{\cot 2x}\right)$ (h) $\lim_{x \to \infty} \left(\frac{5x + 2\ln x}{x + 3\ln x}\right)$ (i) $\lim_{x \to 0} \left(\frac{\cos 2x - \cos x}{\sin^2 x}\right)$

9. (a) Determine i. $\lim_{x \to 8} \left(\frac{x - 8}{\sqrt[3]{x} - 2}\right)$ ii. $\lim_{x \to 1} \left(\frac{e^x - e}{x - 1}\right)$

(b) Consider the continuous function f with a continuous first derivative such that $f(\pi) = 0$.

Given that $\lim_{x \to \pi} \frac{f(x)}{\sin x} = 2$, calculate the value $f'(\pi)$.

10. Determine $\lim_{x \to 0} x^{\sin x}$. [Hint: Let $z = x^{\sin x}$ and take $\ln z$ to transform it to the form $\frac{\infty}{\infty}$].

11. Show that $(1 + x)^{1/x} = e^{\left(\frac{1}{x}\right)\ln(1 + x)}$. Hence, prove that $\lim_{x \to 0^+} (1 + x)^{1/x} = e$.

12. Determine $\lim_{x \to \infty} x(a^{1/x} - 1)$.

13. Determine the following limits.

(a) $\lim_{x \to 1^+} x^{\frac{1}{x-1}}$ (b) $\lim_{x \to 0} (\sin x)^x$ (c) $\lim_{x \to \infty} (x + 1)^{2/x}$

14. Determine (a) $\lim_{x \to 0} (\cos x)^{1/x}$ (b) $\lim_{x \to 0} \left(\frac{1}{x}\right)^x$.

15. (a) Given that $f(1) = 1$ and $f'(1) = e$, calculate $\lim_{x \to 1} \frac{[f(x)]^4 - 1}{x^2 - 1}$.

(b) Given that $\lim_{x \to 0} \frac{f(x)}{e^x - 1} = e$, where f is a continuous differentiable function, find $f'(0)$ given that $f(0) = 0$.

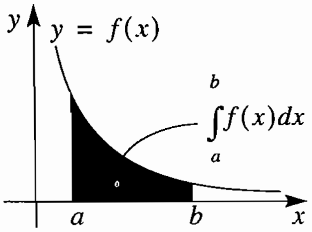
16. Evaluate (a) $\lim_{x \to 0} \frac{\sin \alpha x}{\sin \beta x}$ (b) $\lim_{x \to 0} \frac{\sin x^\alpha}{(\sin x)^\alpha}$, where $\alpha \in \mathbb{Z}^+$ and $\beta \in \mathbb{Z}^+$.

17. Evaluate (a) $\lim_{x \to 0} \frac{\cos \alpha x - \cos \beta x}{x^2}$ (b) $\lim_{x \to \beta} \frac{\sin^2 x - \sin^2 \beta}{x^2 - \beta^2}$

1.4 IMPROPER INTEGRALS

1.4.1 WHAT ARE IMPROPER INTEGRALS?

Throughout your course you have, up until now, only encountered proper integrals, that is, integrals of the form $\int_a^b f(x)dx$ where the interval of integration, $[a, b]$, is a finite interval in which the function, $f(x)$, is a continuous, bounded function.

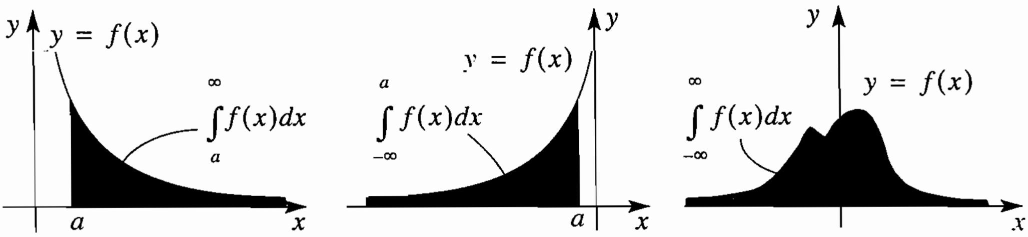


In essence, when dealing with improper integrals there are two types of 'impropriety' that occur:

Type 1. **The interval of integration may be infinite.**

e.g., $\int_1^{\infty}\frac{1}{x^2}dx$, $\int_1^{\infty}\frac{1}{\sqrt{x}}dx$, $\int_0^{\infty}\frac{1}{x^2+1}dx$, $\int_{-\infty}^{\infty}\frac{1}{x^2+1}dx$, $\int_{-\infty}^{-2}\frac{1}{x}dx$, $\int_0^{\infty}\cos x dx$.

Geometric interpretations (where $f(x) > 0$) are shown below.



A. B. C.

If the limit on the right hand side exists, then the improper integrals having the symbolic expressions

A. $\int f(x)dx = \lim \int f(x)dx$ [illegible]

B. $\int f(x)dx = \lim \int f(x)dx$ [illegible]

C. [illegible] $+ \int f(x)dx = \lim \int f(x)dx + \lim \int f(x)dx$

are said to *converge* to that value.

If the limit does not exist, the integral is said to *diverge* and no value exists.

Type 2. **The integrand may be unbounded on the interval of integration.**

e.g., $\int_0^1 \frac{1}{x^2}dx$, $\int_{-2}^2 \frac{1}{x}dx$, $\int_0^1 \ln x dx$, $\int_{-3}^2 \frac{1}{(x-1)(x+2)}dx$, $\int_0^{\frac{\pi}{2}} \tan x dx$

The reason here is that the integrand is undefined for particular values of x in the interval of integration.

For example, in the expression $\int_{-2}^2 \frac{1}{x}dx$, the integrand, $f(x) = \frac{1}{x}$ is undefined at $x = 0$.

Similarly, in the expression $\int_0^{\frac{\pi}{2}} \tan x dx$, the integrand, $f(x) = \tan x$ is undefined at $x = \frac{\pi}{2}$.

A: If the function $f(x)$ is undefined at $x = b$ and $a < b$ the symbolic expressions

$\int f(x)dx = \lim \int f(x)dx$ [illegible]

B: If the function $f(x)$ is undefined at [illegible] and [illegible] the symbolic expressions

[illegible]

C: If the function $f(x)$ is undefined at [illegible] the symbolic expressions

[illegible]

1.4.2 EVALUATING IMPROPER INTEGRALS

The improper integral $\int_a^\infty f(x)dx = \lim_{n \to \infty} \int_a^n f(x)dx$ is the one which will be of most use to us when we come to deal with infinite series later in this book. However, we now proceed with a number of examples to see how these integrals can be evaluated.

EXAMPLE 1.31

Evaluate (a) $\int_1^\infty \frac{1}{x^2}dx$ (b) $\int_1^\infty \frac{1}{x}dx$

Solution

(a) $\int_1^\infty \frac{1}{x^2}dx = \lim_{n \to \infty} \int_1^n \frac{1}{x^2}dx = \lim_{n \to \infty} \left[-\frac{1}{x}\right]_1^n = \lim_{n \to \infty} \left[-\frac{1}{n} + 1\right] = 1$

That is, the integral *converges* to 1.

(b) $\int_1^\infty \frac{1}{x}dx = \lim_{n \to \infty} \int_1^n \frac{1}{x}dx = \lim_{n \to \infty} [\ln x]_1^n = \lim_{n \to \infty} (\ln n)$

Now, as $n \to \infty$, $\ln n \to \infty$ and so the integral *diverges*, i.e., the integral doesn't exist.

EXAMPLE 1.32

Evaluate (a) $\int_0^{\infty} \frac{1}{1+x^2}dx$ (b) $\int_{-\infty}^{\infty} \frac{1}{1+x^2}dx$

Solution

(a) $$\int_0^{\infty} \frac{1}{1+x^2}dx = \lim_{n\to\infty}\int_0^n \frac{1}{1+x^2}dx = \lim_{n\to\infty}\Big[\arctan x\Big]_0^n$$
$$= \lim_{n\to\infty} \arctan n$$
$$= \frac{\pi}{2}$$

That is, the integral converges to $\frac{\pi}{2}$.

(b) $$\int_{-\infty}^{\infty} \frac{1}{1+x^2}dx = \int_{-\infty}^{0} \frac{1}{1+x^2}dx + \int_0^{\infty} \frac{1}{1+x^2}dx.$$

However, by symmetry, $\int_{-\infty}^{0} \frac{1}{1+x^2}dx = \int_0^{\infty} \frac{1}{1+x^2}dx = \frac{\pi}{2}$.

Therefore, $\int_{-\infty}^{\infty} \frac{1}{1+x^2}dx = \frac{\pi}{2} + \frac{\pi}{2} = \pi$. That is, the integral *converges* to π.

EXAMPLE 1.33

Evaluate (a) $\int_0^{\infty} \cos x\,dx$ (b) $\int_1^{\infty} \frac{1}{\sqrt{x}}dx$.

Solution

(a) $$\int_0^{\infty} \cos x\,dx = \lim_{n\to\infty}\int_0^n \cos x\,dx = \lim_{n\to\infty}\Big[\sin x\Big]_0^n = \lim_{n\to\infty} \sin n.$$

The integral *diverges* because $\sin n$ oscillates between -1 and 1 as $n \to \infty$.

(b) $$\int_1^{\infty} \frac{1}{\sqrt{x}}dx = \lim_{n\to\infty}\int_1^n \frac{1}{\sqrt{x}}dx = \lim_{n\to\infty}\Big[2\sqrt{x}\Big]_1^n = \lim_{n\to\infty}[2\sqrt{n}-2].$$

The integral *diverges* because as $n \to \infty$, $[2\sqrt{n}-2] \to \infty$.

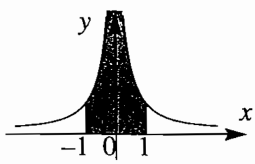
EXAMPLE 1.34

Evaluate $\int_{-1}^{1} \frac{1}{x^2}dx$

Solution

(a) We first note that the graph of $f(x) = \frac{1}{x^2}$ is discontinuous at $x = 0$ and so we need to express the integral as follows:

$$\int_{-1}^{1} \frac{1}{x^2}dx = \int_{-1}^{0} \frac{1}{x^2}dx + \int_0^{1} \frac{1}{x^2}dx = \lim_{c\to 0^-}\int_{-1}^{c} \frac{1}{x^2}dx + \lim_{c\to 0^+}\int_c^{1} \frac{1}{x^2}dx$$



We first consider $\lim_{c \to 0^-} \int_{-1}^{c} \frac{1}{x^2}dx = \lim_{c \to 0^-} \left[-\frac{1}{x}\right]_{-1}^{c} = \lim_{c \to 0^-} \left[-\frac{1}{c} + 1\right]$, which does not exist and so the integral diverges.

Similarly, $\lim_{c \to 0^+} \int_{c}^{1} \frac{1}{x^2}dx = \lim_{c \to 0^+} \left[-\frac{1}{x}\right]_{c}^{1} = \lim_{c \to 0^+} \left[-1 + \frac{1}{c}\right]$, which also doesn't exist as the integral diverges.

Therefore, the integral $\int_{-1}^{1} \frac{1}{x^2}dx$ diverges and so doesn't exist.

EXAMPLE 1.35

Evaluate $\int_{0}^{\infty} \frac{1}{x^2}dx$.

Solution

The integral has two "improprieties", one at $x = 0$ (where the function is discontinuous) and one at the upper limit (∞). We make use of Examples 1.31 and 1.34 to help us out.

We start by writing $\int_{0}^{\infty} \frac{1}{x^2}dx = \int_{0}^{1} \frac{1}{x^2}dx + \int_{1}^{\infty} \frac{1}{x^2}dx$.

Now, $\int_{1}^{\infty} \frac{1}{x^2}dx = 1$, i.e., the integral converges to 1 (Example 1.31).

However, the integral $\int_{0}^{1} \frac{1}{x^2}dx$ was shown to diverge (Example 1.34).

Therefore, the integral $\int_{0}^{\infty} \frac{1}{x^2}dx$ *diverges* and so doesn't exist.

EXAMPLE 1.36

Evaluate $\int_{0}^{1} \frac{1}{\sqrt{1-x}}dx$.

Solution

The function $f(x) = \frac{1}{\sqrt{1-x}}$ is discontinuous at $x = 1$. Therefore the integral $\int_{0}^{1} \frac{1}{\sqrt{1-x}}dx$ is an improper integral.

$$\int_{0}^{1} \frac{1}{\sqrt{1-x}}dx = \lim_{N \to 1^-} \int_{0}^{N} \frac{1}{\sqrt{1-x}}dx = \lim_{N \to 1^-} \left[-2\sqrt{1-x}\right]_{0}^{N}$$

$$= \lim_{N \to 1^-} [-2\sqrt{1-N} + 2]$$

$$= 2$$

So, the improper integral $\int_{0}^{1} \frac{1}{\sqrt{1-x}}dx$ converges to 2.

1.4.3 COMPARISON TEST FOR IMPROPER INTEGRALS

When it is difficult to evaluate an improper integral due the nature of the integrand, for example $\int_1^{\infty} e^{-x^2} dx$, and all that is required is to determine if the integral converges or diverges, we can compare the improper integral to another improper integral that is known to either converge or diverge and then make an appropriate decision. This is the essence of the comparison test for improper integrals:

If $0 \le f(x) \le g(x)$ for all $x \ge a$, we have that:

1. If $\int_a^{\infty} g(x)dx$ converges, then $\int_a^{\infty} f(x)dx$ converges.
2. If $\int_a^{\infty} f(x)dx$ diverges, then $\int_a^{\infty} g(x)dx$ diverges.

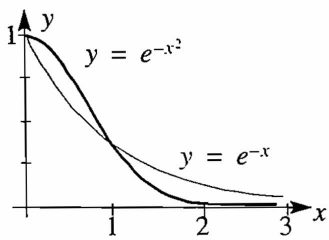
EXAMPLE 1.37

Determine if $\int_1^{\infty} e^{-x^2} dx$ converges.

Solution

We first notice that there is no immediate known result for $\int e^{-x^2} dx$. So, we make use of a second function that can be readily compared to $f(x) = e^{-x^2}$, for $x \ge 1$.

Using the function $g(x) = e^{-x}$ we have that for $x \ge 1$, $g(x) \ge f(x)$. Therefore, as $f(x) \ge 0$, if we can show that $\int_1^{\infty} g(x)dx$ converges, then by the comparison test we can conclude that $\int_1^{\infty} e^{-x^2} dx$ also converges.



$$\text{Now, } \int_1^{\infty} g(x)dx = \int_1^{\infty} e^{-x}dx = \lim_{n \to \infty} \int_1^{n} e^{-x}dx = \lim_{n \to \infty} \left[-e^{-x}\right]_1^n$$

$$= \lim_{n \to \infty} [-e^{-n} + 1]$$

$$= 1$$

Therefore, as $\int_1^{\infty} e^{-x}dx$ converges, so too does $\int_1^{\infty} e^{-x^2}dx$.

Notice then, that the first stage in the comparison process is to guess a function that behaves like the integrand, which is done by looking at the behaviour of the integrand for large values of x.

EXAMPLE 1.38

Determine if $\int_1^{\infty} \frac{1}{\sqrt{x^3+9}}dx$ converges.

Solution

We first search for a function that will 'behave' like $\frac{1}{\sqrt{x^3+9}}$ for large values of x.

As $x \to \infty$, $\frac{1}{\sqrt{x^3+9}} \sim \frac{1}{\sqrt{x^3}}$, so that for $x \geq 1$, $\frac{1}{\sqrt{x^3+9}} < \frac{1}{\sqrt{x^3}}$.

Next, $\int_1^{\infty} \frac{1}{\sqrt{x^3}}dx = \int_1^{\infty} x^{-3/2}dx = \lim_{n \to \infty} \int_1^n x^{-3/2}dx = \lim_{n \to \infty} \left[-2x^{-\frac{1}{2}}\right]_1^n$

$$= \lim_{n \to \infty} \left[-2n^{-\frac{1}{2}} + 2\right]$$

$$= 2$$

So, as $\int_1^{\infty} \frac{1}{\sqrt{x^3}}dx$ converges and $0 < \int_1^{\infty} \frac{1}{\sqrt{x^3+9}}dx \leq \int_1^{\infty} \frac{1}{\sqrt{x^3}}dx$, then $\int_1^{\infty} \frac{1}{\sqrt{x^3+9}}dx$ converges.

Next, consider the convergence or divergence of the improper integral $\int_1^{\infty} \frac{\sin x}{x^2}dx$. As the integrand, $\frac{\sin x}{x^2}$ takes on both positive and negative values we cannot use the comparison test used in Examples 1.37 and 1.38 as that test only applies to nonnegative integrands. However, we do note that $\left|\frac{\sin x}{x^2}\right| \leq \frac{1}{x^2}$ for all $x \geq 1$ and so,

$$\int_1^{\infty} \left|\frac{\sin x}{x^2}\right|dx \leq \int_1^{\infty} \frac{1}{x^2}dx = 1 \text{ (from Example 1.31 (a)).}$$

Then, as $\left|\frac{\sin x}{x^2}\right| \geq \frac{\sin x}{x^2}$ we have that $\int_1^{\infty} \frac{\sin x}{x^2}dx \leq \int_1^{\infty} \left|\frac{\sin x}{x^2}\right|dx$.

As $\int_1^{\infty} \left|\frac{\sin x}{x^2}\right|dx$ converges so too will $\int_1^{\infty} \frac{\sin x}{x^2}dx$.

This leads to the following modified comparison test – the **absolute convergence test**.

If $\int |f(x)|dx$ converges, then $\int f(x)dx$ converges.

EXERCISES 1.4

1. Evaluate, if possible, the following integrals.

(a) $\int_1^\infty \frac{1}{x+1}dx$ (b) $\int_1^\infty \frac{1}{(x+1)^2}dx$ (c) $\int_1^\infty \frac{1}{(x+1)^3}dx$

(d) $\int_1^\infty \frac{\ln x}{x}dx$ (e) $\int_1^\infty \frac{\ln x}{x^2}dx$ (f) $\int_1^\infty \frac{\ln x}{x^3}dx$

2. (a) Sketch the graph of the function $f(x) = \frac{1}{(x-2)^{2/3}}$.

(b) Evaluate $\int_1^4 \frac{1}{(x-2)^{2/3}}dx$.

3. Evaluate, if possible, the following integrals.

(a) $\int_0^1 \frac{1}{\sqrt{1-x}}dx$ (b) $\int_2^\infty \frac{1}{(x-1)^2}dx$ (c) $\int_{-\infty}^0 \frac{1}{(2x-1)^3}dx$

4. Evaluate, if possible, the following integrals.

(a) $\int_1^\infty \frac{1}{x^{4/3}}dx$ (b) $\int_{-\infty}^\infty \frac{x}{(x^2+3)^2}dx$ (c) $\int_0^4 \frac{1}{(x-2)^2}dx$

5. (a) Find $\int xe^{-x}dx$.

(b) Evaluate, if possible, the integral $\int_0^\infty xe^{x}dx$.

6. (a) Find $\int xe^{x}dx$.

(b) Evaluate, if possible, the integral $\int_{-\infty}^0 xe^{x}dx$.

7. (a) Sketch the graph of the function $f(x) = \frac{1}{(8-x)^{1/3}}$ for $x < 8$.

(b) Find the area of the region between the x-axis and the curve $y = f(x)$, $0 < x < 8$.

8. Evaluate the improper integrals

(a) $\int_a^\infty xe^{-x}dx$ (b) $\int_a^\infty xe^{-ax}dx$

9. Evaluate the improper integrals

(a) $\int_a^\infty \frac{1}{1+x^2}dx$ (b) $\int_a^\infty \frac{a}{a^2+x^2}dx$

10. Evaluate, where possible, the following integrals

(a) $\int_{-\infty}^\infty xe^{-x^2}dx$ (b) $\int_{-\infty}^\infty \frac{e^{-x}}{1+e^{-x}}dx$

11. Evaluate, where possible, the following integrals

(a) $\int_0^1 \frac{x}{\sqrt{1-x^2}}dx$ (b) $\int_0^3 (x-2)^{-4/3}dx$

12. For what value of p does the improper integral $\int_1^\infty \frac{1}{x^p}dx$ converge?

13. For what value of p does the improper integral $\int_1^\infty x^p dx$ converge?

14. For what value of p does the improper integral $\int_0^\infty x^p dx$ converge?

15. For what value of p does the improper integral $\int_0^1 x^p \ln x dx$ converge?

16. (a) Find, if it exists, the area of the region enclosed by the curve $y = \frac{1}{x}, x \geq 1$ and the x-axis.

(b) Find, if it exists, the volume of the solid of revolution when the region enclosed by the curve $y = \frac{1}{x}, x \geq 1$ and the x-axis is rotated about the x-axis.

17. Evaluate, where possible, the following

(a) $\int_2^5 \frac{1}{\sqrt{x-2}}dx$ (b) $\int_0^{\frac{\pi}{2}} \sec x dx$ (c) $\int_0^3 \frac{1}{x-1}dx$

18. (a) On the same set of axes sketch the graphs of the functions $f(x) = e^{-x}$ and $g(x) = e^{-(1+x^2)}$.

(b) Evaluate the improper integral $\int_1^\infty e^{-x}dx$.

(c) Hence, show that $\int_1^\infty e^{-(1+x^2)}dx$ converges.

19. (a) On the same set of axes sketch the graphs of the functions $f(x) = \frac{x^3}{x^5+2}$ and $g(x) = \frac{1}{x^2}$ for $x > 0$.

(b) i. Evaluate, if possible, the improper integral $\int_1^\infty \frac{1}{x^2}dx$.

ii. Hence determine if $\int_1^\infty \frac{x^3}{x^5+2}dx$ is convergent or divergent.

20. Determine if $\int_0^\infty e^{-x}\cos x dx$ converges.

21. (a) Show i. that for $x \geq 1$, $\dfrac{1}{x^2} > \dfrac{1}{\sqrt{x^4 + 2x}}$.

ii. that $\int_1^{\infty} \dfrac{1}{x^2} dx$ converges.

(b) Hence show that $\int_1^{\infty} \dfrac{1}{\sqrt{x^4 + 2x}} dx$ converges.

22. (a) Express $1 - x^4$ as a product of its real factors.

(b) Show that i. for $0 \leq x < 1$, $1 - x^4 < 4(1 - x)$

ii. $\dfrac{1}{4}\int_0^1 \dfrac{1}{1 - x} dx$ diverges.

(c) Hence show that $\int_0^1 \dfrac{1}{1 - x^4} dx$ converges.

23. (a) i. Show that for $x \geq 3$, $x^4 + x^2 + \sin x > x^4$.

ii. Hence show that $\dfrac{4x^2}{x^4 + x^2 + \sin x} < \dfrac{4}{x^2}$.

(b) Hence show that $\int_3^{\infty} \dfrac{4x^2}{x^4 + x^2 + \sin x} dx$ converges.

24. (a) On the same set of axes, sketch the graphs of the functions

i. $f(x) = \dfrac{1}{\sqrt{2x}}$ ii. $g(x) = \dfrac{x}{\sqrt{x^3 + 1}}$ iii. $h(x) = \dfrac{1}{\sqrt{x}}$

(b) Deduce what inequality exists between $f(x)$, $g(x)$ and $h(x)$ for $x \geq 1$.

(c) Evaluate, where possible, the following

i. $\int_0^{\infty} \dfrac{1}{\sqrt{2x}} dx$ ii. $\int_0^{\infty} \dfrac{1}{\sqrt{x}} dx$

(d) Using the results of part (c), what can you deduce about

i. $\int_1^{\infty} \dfrac{1}{\sqrt{2x}} dx$ ii. $\int_1^{\infty} \dfrac{1}{\sqrt{x}} dx$

(e) Hence, by making use of the comparison test and the results of (b), determine if the improper integral $\int_0^{\infty} \dfrac{x}{\sqrt{x^3 + 1}} dx$ converges.

2.1 SERIES

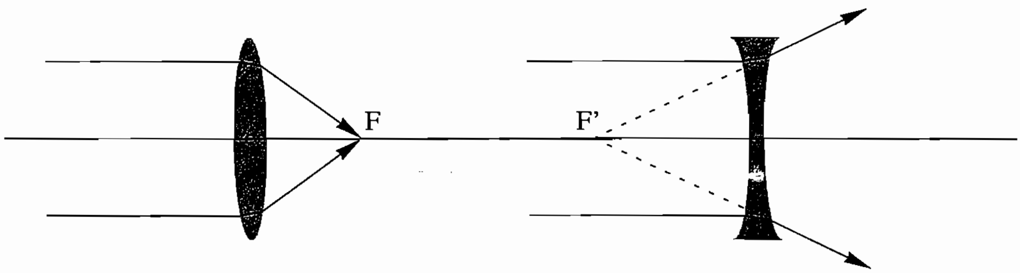
2.1.1 INTRODUCTION

We have already studied, in the core part of the syllabus, sequences and series of various types. In Chapter 1, we concerned ourselves only with **infinite** sequences, i.e. sequences that have an infinite number of terms. In general, we either knew the law giving us u_n, the general term of the sequence, or else we were given enough information to establish a general expression for the general term.

If we add all the terms of the sequence $u_1, u_2, u_3, \ldots, u_n, \ldots$ we get the expression $u_1 + u_2 + u_3 + \ldots + u_n + \ldots$ which is an infinite series. It is called an infinite series as there are an infinite number of terms.

The crucial property of series that we need to study is whether they have a finite sum or not. If the sum is finite, we shall say that the series **converges** to that particular finite value. If the sum does not exist or if it tends to ∞, we shall say that it **diverges**.

We can visualize what these two terms, converge and diverge mean. From our science studies we recall (hopefully) what converging and diverging lenses are:



A converging lens A diverging lens

In a converging lens, rays of light incident and parallel to the axis of the lens are brought together by the lens to a single point F on the axis, called the focal point. We have all played with a little magnifying-glass (which is a converging lens) and seen how it collects sunlight to a single point on a piece of paper, and how the paper starts to burn if we leave it for a while. In a diverging lens, the same rays as before would produce the opposite effect: they spread out of the lens as if they were coming from an imaginary point F' on the axis. I say "imaginary point" because, to locate this point F', the focal point of the diverging lens, we had to produce rays back to that point on the axis. F' is not on the actual path of the rays, but it serves as a convenient representation to understand how the rays diverge out of the lens.

The convergent series will have a finite sum, i.e. a single finite value, and the divergent series will have an infinite sum, or its sum will not exist.

2.1.2 SUM OF A SERIES

If we consider the **sum of the first *n* terms** of an infinite series, we can form a **partial sum** S_n such as $S_n = u_1 + u_2 + u_3 + \ldots + u_n$.

That is, we have partial sums,

$$S_1 = u_1,$$
$$S_2 = u_1 + u_2,$$
$$S_3 = u_1 + u_2 + u_3,$$
etc.

We therefore have that $S_n = S_{n-1} + u_n$. So, if we are given an expression for S_n, we can determine an expression for the general term $u_n = S_n - S_{n-1}$.

Note that these are called partial sums because there are more terms in the series after the *n*th term. Then, the sum of of all the terms in this infinite series can be quite naturally defined as:

$$S = \lim_{n \to \infty} S_n = \lim_{n \to \infty} (u_1 + u_2 + u_3 + \ldots + u_n)$$

Therefore, if this limit exists and is finite, we shall say that the series converges.

If $S_n \to \infty$ when $n \to \infty$, the series diverges and has no sum.

EXAMPLE 2.1

Consider the series $u_1 + u_1 r + u_1 r^2 + \ldots + u_1 r^{n-1} + \ldots$. We recognize this to be the sum of the terms of a geometric sequence whose first term is u_1, with common ratio r.

The sum of the first n terms of the geometric sequence, when $r \neq 1$, is $S_n = u_1 \dfrac{(1 - r^n)}{1 - r}$.

Examine the behaviour of the series in these cases:

(a) $|r| < 1$ (b) $|r| > 1$ (c) $r = +1$ (d) $r = -1$.

Solution

(a) If $|r| < 1$, when $n \to \infty$ we have $r^n \to 0$.

Therefore $S = \lim_{n \to \infty} S_n = \lim_{n \to \infty} \left(u_1 \dfrac{(1 - r^n)}{1 - r} \right) = \dfrac{u_1}{1 - r}$.

So we see that if $|r| < 1$, the series converges and its sum is the finite value $S = \dfrac{u_1}{1 - r}$.

(b) If $|r| > 1$, $|r^n| \to \infty$ for $n \to \infty$, and $S_n \to \pm\infty$ as $u_1 \dfrac{(1 - r^n)}{1 - r} \to \pm\infty$ for $n \to \infty$.

Therefore $\lim_{n \to \infty} S_n$ does not exist. Hence the series diverges.

(c) If $r = -1$, the series can be written as $u_1 - u_1 + u_1 - u_1 + \ldots$ and we have to consider whether we have an odd or an even number of terms.
If n is odd, $S_n = u_1$ and if n is even, $S_n = 0$.
Therefore S_n does not have a single limit and thus the series diverges.

(d) If $r = 1$, the series becomes $u_1 + u_1 + u_1 + \dots$.

Therefore $S_n = n \times u_1$, and so $\lim_{n \to \infty} S_n = \infty$. Thus the series diverges.

Conclusion: The series formed by adding the terms of a geometric sequence with non-zero first term u_1 converges if, and only if, the absolute value of its common ratio, $|r|$, is smaller than 1. For all other values of $|r|$, the series diverges.

2.1.3 CRITERIA FOR CONVERGENCE OF SERIES

We already said that one of the crucial questions to be established when studying a series is whether it converges or diverges. We shall therefore establish certain criteria to help us resolve this question. We shall then examine various different criteria used to test for convergence of series, and indicate some simple rules to be used according to the type of question you may encounter. Needless to say, they all involve knowing how to calculate an infinite limit of an expression.

2.1.4 NECESSARY CONDITION FOR SERIES CONVERGENCE

We first examine the **necessary** condition for convergence of series and note that this is only half of the problem. The other half of the problem will be set in the next subsection, namely the **sufficient** condition for convergence, which is needed to complete our testing for convergence of a series.

For a series to converge, we must establish that the general term $u_n \to 0$ when $n \to \infty$.

To see how this condition comes about we note that if the series $u_1 + u_2 + u_3 + \dots + u_n + \dots$ converges, we then know that $\lim_{n \to \infty} S_n = S$, where S is a fixed finite value. Equally, the $(n-1)$th sum will also tend to S, because we have $\lim_{n \to \infty} S_{n-1} = S$, and n and $(n-1)$ tend to infinity at the same rate. So if we subtract these two expressions:

$$\lim_{n \to \infty} S_n - \lim_{n \to \infty} S_{n-1} = 0 \qquad (2.1)$$

and thus $\lim_{n \to \infty} (S_n - S_{n-1}) = 0$. However, we have that $S_n - S_{n-1} = u_n$, the nth term of the series. Therefore

$$\lim_{n \to \infty} u_n = 0 \qquad (2.2)$$

By the same token we can show that if u_n does not tend to 0 as $n \to \infty$, the series diverges.

For example, in the series $1 + \frac{2}{3} + \frac{3}{5} + \dots + \frac{n}{2n-1} + \dots$, we see that $\lim_{n \to \infty} \frac{n}{2n-1} = \frac{1}{2} \neq 0$, therefore the series diverges.

What we can definitely affirm, though, is that if a series converges then $\lim u_n = 0$

In order to stress this point further, viz. that our condition (2.2) is only a **necessary** one, let us examine a further example of a series in which $u_n \to 0$ as $n \to \infty$, but the series **diverges**.

This is the well-known classical example of the harmonic series

$$1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \ldots + \frac{1}{n} + \ldots \qquad (2.3)$$

We can see that $\lim\limits_{n \to \infty} u_n = \lim\limits_{n \to \infty} \frac{1}{n} = 0$.

However we now show that the series diverges although the limit of u_n is 0 as $n \to \infty$.
For this, we shall use quite a well-known gimmick (well-known to mathematicians!), which consists of rewriting more terms of the harmonic series, and comparing them to those of another related series, chosen in a convenient way, that is already known to diverge.
Let us then rewrite more terms of the harmonic series

$$1 + \frac{1}{2} + \underbrace{\frac{1}{3} + \frac{1}{4}}_{\text{2 terms}} + \underbrace{\frac{1}{5} + \frac{1}{6} + \frac{1}{7} + \frac{1}{8}}_{\text{4 terms}} + \underbrace{\frac{1}{9} + \frac{1}{10} + \frac{1}{11} + \frac{1}{12} + \frac{1}{13} + \frac{1}{14} + \frac{1}{15} + \frac{1}{16}}_{\text{8 terms}} + \ldots \qquad (2.4)$$

Now let us use the following series for comparison of the 2 terms, 4 terms and 8 terms

$$1 + \frac{1}{2} + \underbrace{\frac{1}{4} + \frac{1}{4}}_{\text{2 terms}} + \underbrace{\frac{1}{8} + \frac{1}{8} + \frac{1}{8} + \frac{1}{8}}_{\text{4 terms}} + \underbrace{\frac{1}{16} + \frac{1}{16} + \frac{1}{16} + \frac{1}{16} + \frac{1}{16} + \frac{1}{16} + \frac{1}{16} + \frac{1}{16}}_{\text{8 terms}} + \ldots \qquad (2.5)$$

This series is constructed in the following way:

- the first two terms are 1 and $\frac{1}{2}$ just like in the harmonic series.
- the third and fourth terms are both equal to $\frac{1}{4}$
  (the last term of the corresponding group of two terms in the harmonic series),
- the next four terms are equal to $\frac{1}{8}$
  (the last term of the corresponding group of four terms in the harmonic series).
- the next eight terms are equal to $\frac{1}{16}$
  (the last term of the corresponding group of eight terms in the harmonic series),

etc…

Let us call S_n the sum of the first n terms of the harmonic series, and S'_n the sum of the first n terms of the comparison series. It can be seen that each term of the harmonic series is bigger or equal to the corresponding term in the comparison series, and in particular, for $n > 2$, $S_n > S'_n$.

Therefore, if we can prove that the comparison series diverges, it will imply that the harmonic series also diverges as the sum $S_n > S'_n$.

So our problem is now: Is it any easier to prove the divergence of the comparison series?

We see that $S'_2 = S_2 = \frac{3}{2}$, so we need only analyze terms for which $n > 2$.

If we consider the sum that will now include the next group of terms, i.e. $\frac{1}{4} + \frac{1}{4}$, we have

$$S'_4 = 1 + \frac{1}{2} + \left(\frac{1}{4} + \frac{1}{4}\right) = 1 + 2 \times \frac{1}{2}$$

Then, the sum including the next group of terms is

$$S'_8 = 1 + \frac{1}{2} + \left(\frac{1}{4} + \frac{1}{4}\right) + \left(\frac{1}{8} + \frac{1}{8} + \frac{1}{8} + \frac{1}{8}\right) = 1 + 3 \times \frac{1}{2}$$

Therefore, we can generalize the sum

$$S'_{2^k} = 1 + k \times \frac{1}{2}, \text{ where } k > 2.$$

As k can be as large as we want, it follows that $\lim_{n \to \infty} S'_n = \infty$.

As we already had concluded that $S_n > S'_n$, it is natural to assert that $\lim_{n \to \infty} S_n = \infty$.

Which proves that the harmonic series **diverges**.

This important observation means that we need another condition to prove that a series converges. That is, showing that $\lim_{n \to \infty} u_n = 0$ is simply not enough!

The only conclusions we have at this point is that:

1. If $\lim_{n \to \infty} u_n = 0$ the series might converge.
2. If $\lim_{n \to \infty} u_n \neq 0$ the series diverges.

At this point it is useful to warn the reader that although the harmonic series $\sum_{n=1}^{\infty} \frac{1}{n}$, a series with only positive terms, is divergent, the 'cousin series' $\sum_{n=1}^{\infty} (-1)^n \frac{1}{n}$ is **convergent**.

Note that the latter series is made up of positive and negative terms that alternate (hence the term **alternating series**), according to n being even or odd. We shall expand on alternating series later on and we shall show how $\sum_{n=1}^{\infty} (-1)^n \frac{1}{n}$ converges.

2.1.5 AN IMPORTANT MATHEMATICAL CURIOSITY: THE NUMBER *e*.

Leonhard Euler, the 18th century Swiss mathematician, showed how the number e (base of the natural logarithm) could be calculated out as the limit of a sequence, namely $\lim\limits_{n \to \infty}\left(1+\frac{1}{n}\right)^n$.

In the beginning this was thought to be just a mathematical curiosity, but it then became clear that 'e' was a constant of paramount importance. Euler's starting point was the sequence defined by its general term $u_n = \left(1+\frac{1}{n}\right)^n$. If we use the binomial expansion we can write the full expressionof u_n, viz.,

$$u_n = \binom{n}{0}1^n + \binom{n}{1}(1^{n-1})\left(\frac{1}{n}\right) + \binom{n}{2}(1^{n-2})\left(\frac{1}{n}\right)^2 + \ldots + \binom{n}{n}\left(\frac{1}{n}\right)^n$$

This is more conveniently written as:

$$u_n = 1 + n\cdot\frac{1}{n} + \frac{n(n-1)}{2!}\cdot\frac{1}{n^2} + \frac{n(n-1)(n-2)}{3!}\cdot\frac{1}{n^3} + \ldots + \frac{n(n-1)(n-2)\ldots[n-(n-1)]}{n!}\cdot\frac{1}{n^n}$$

$$= 1 + 1 + \frac{1}{2!}\cdot\frac{n-1}{n} + \frac{1}{3!}\cdot\frac{(n-1)(n-2)}{n^2} + \ldots + \frac{1}{n!}\cdot\frac{n(n-1)(n-2)\ldots[n-(n-1)]}{n^{n-1}}$$

and in its final form as:

$$u_n = 1 + 1 + \frac{1}{2!}\cdot\left(1-\frac{1}{n}\right) + \frac{1}{3!}\cdot\left(1-\frac{1}{n}\right)\left(1-\frac{2}{n}\right) + \ldots + \frac{1}{n!}\cdot\left(1-\frac{1}{n}\right)\left(1-\frac{2}{n}\right)\ldots\left[1-\left(\frac{n-1}{n}\right)\right]$$

Note that all these brackets are positive, therefore the sequence defined by u_n is made up of monotonically decreasing terms. Besides, all these brackets in the above expression are smaller than 1.

Therefore $u_n < 1 + 1 + \frac{1}{2!} + \frac{1}{3!} + \ldots + \frac{1}{n!} < 1 + 1 + \frac{1}{2^2} + \ldots + \frac{1}{2^{n-1}}$ as $n! > 2^{n-1}$ for $n > 2$.

The right-hand side of the above inequality, from the second term onwards, is the sum of a geometric sequence with common ratio $\frac{1}{2}$.

Thus we have $$u_n < 1 + \frac{1-\frac{1}{2^n}}{1-\frac{1}{2}} = 1 + 2\left(1-\frac{1}{2^n}\right) = 3 - \frac{1}{2^{n-1}}.$$

We can see that, for whatever large n, the value of u_n has a finite upper bound, smaller than 3. In fact

[redacted]

This can be seen to be correct by numerically calculating the value of $\left(1+\frac{1}{n}\right)^n$ for any large value of n, e.g. if $n = 10^{10}$, $\left(1+\frac{1}{10^{10}}\right)^{10^{10}} \approx 2.7182822$.

Besides being the base of the natural logarithm, the number e is the base of the exponential function $f(x) = e^x$, which happens to be the inverse of $f(x) = \ln(x)$.

The exponential function has many applications in mathematics (in real and complex analysis) and in physics (radioactive decay; charge and discharge of a capacitor in a DC circuit; as a general description of a travelling wave; in quantum mechanics it gives the interpretation to the wave function and is part of the Schrödinger formalism of QM , etc...)

2.1.6 THE SUFFICIENT CONDITION FOR SERIES CONVERGENCE

We already said that the necessary condition for series convergence, namely $\lim_{n \to \infty} u_n = 0$, is not enough to ensure convergence. We now present the other half of the problem, the **sufficient condition**. When **both** conditions are taken together, they will provide the certainty that a series converges.

As we show next, there are several criteria that can be used to test for series convergence. The choice of the particular test to be used depends on the specific problem at hand, i.e. on the series we are testing. There is no general recipe as each case is its own, however we shall indicate at the end some general features that a series must have in order to ascertain the relevance of one or the other criteria. Ultimately, experience will be the best guide as to the choice of test in each case.

TEST 1 - THE LIMIT COMPARISON TEST

This test applies only to series that have positive terms.

Let us consider two series that have positive terms

$$S_n = u_1 + u_2 + u_3 + \dots + u_n + \dots \text{ and } S'_n = u'_1 + u'_2 + u'_3 + \dots + u'_n + \dots$$

If the terms of the first series are smaller than or equal to those of the second series for all values of n, i.e., $u_n < u'_n$, and if the second series is already known to converge, then the first series will also converge. This stands to reason, and the proof is self-evident:

Let S_n and S'_n be the partial sums of the first and second series respectively.

That is, $S_n = \sum_{i=1}^{n} u_i$ and $S'_n = \sum_{i=1}^{n} u'_i$. Then, as $u_n \le u'_n$, it follows that $S_n \le S'_n$.

As the second series is known to converge, its partial sum has a finite limit $\lim_{n \to \infty} S'_n = S'$.

As the terms are positive, we can say that $S'_n < S'$ and because $S_n \le S'_n$ then $S_n < S'$.

We have thus shown that the partial sums S_n are **bounded**.

As $\lim_{n \to \infty} S_n = S$, it follows that $S \le S'$.

EXAMPLE 2.2

Investigate the convergence of the series:

$$1+\frac{1}{2^2}+\frac{1}{3^3}+\frac{1}{4^4}+\ldots+\frac{1}{n^n}+\ldots \quad (2.6)$$

Solution

Let us take as a comparison series the following: $1+\frac{1}{2^2}+\frac{1}{2^3}+\frac{1}{2^4}+\ldots+\frac{1}{2^n}+\ldots$

This latter series, from the second term onwards, is a geometric series with common ratio $\frac{1}{2}$.

As $|r| = \frac{1}{2} < 1$ the sum of the geometric series is $\frac{1}{2^2}\cdot\frac{1}{1-\frac{1}{2}} = \frac{1}{4}\cdot 2 = \frac{1}{2}$

Therefore the sum of the comparison series is $1+\frac{1}{2} = \frac{3}{2}$, i.e., the comparison series converges.

Then, as the series to be tested, (2.6), has its terms smaller than or equal to those of the comparison series (after the first two terms), it converges and its sum is less than $\frac{3}{2}$.

Note that there is a useful corollary to our limit comparison test for convergence. We can call it a limit comparison test for divergence.

If instead of the series (2.6) we have a series for which all the $u_n \geq u'_n$, and if the second series diverges, then the first series also diverges. We can see that $S_n \geq S'_n$. As the second series has only positive terms, its partial sum S'_n increases as n increases, and it is known to diverge, thus $\lim_{n\to\infty} S'_n = \infty$. Hence, as $S_n > S'_n$, $\lim_{n\to\infty} S_n = \infty$ and so the first series diverges.

EXAMPLE 2.3

Investigate the convergence of the series:

$$1+\frac{1}{\sqrt{2}}+\frac{1}{\sqrt{3}}+\ldots+\frac{1}{\sqrt{n}}+\ldots \quad (2.7)$$

Solution

Let us take as a comparison series the harmonic series $1+\frac{1}{2}+\frac{1}{3}+\frac{1}{4}+\ldots+\frac{1}{n}+\ldots$.

Comparing corresponding terms we have:

$$1 = 1;\ \frac{1}{2} < \frac{1}{\sqrt{2}};\ \frac{1}{3} < \frac{1}{\sqrt{3}};\ \ldots;\ \frac{1}{n} < \frac{1}{\sqrt{n}};\ \ldots$$

Therefore, we have that

$$1+\frac{1}{2}+\frac{1}{3}+\frac{1}{4}+\ldots+\frac{1}{n}+\ldots < 1+\frac{1}{\sqrt{2}}+\frac{1}{\sqrt{3}}+\ldots+\frac{1}{\sqrt{n}}+\ldots$$

That is, from the second term onwards, the terms of the series (2.7) are bigger than the corresponding ones of the harmonic series, which we already know diverges. Thus our series (2.7) also diverges if we apply the above corollary.

It is possible to provide an extension of the comparison test. Using this approach can sometimes make it easier to reach a conclusion as to the convergence or divergence of an infinite series.

Limit Comparison Test

Let $\sum u_n$ and $\sum v_n$ be series such that $u_n > 0$ and $v_n > 0$ for every value of n. Then, if there exists a number $c > 0$ such that $\lim \frac{u_n}{v_n} = c$ either

1. both series converge or
2. both series diverge

Note that this requires that we know that one of the series either converges or diverges.

EXAMPLE 2.4

Investigate the convergence of the series $\sum_{n=1}^{\infty} \frac{1}{3n^2 + 4n + 1}$

Solution

We use the series $\sum_{n=1}^{\infty} \frac{1}{n^2}$ which we know to be convergent as our comparison series.

With $u_n = \frac{1}{n^2}$ and $v_n = \frac{1}{3n^2 + 4n + 1}$ we have:

$$\frac{u_n}{v_n} = \frac{\frac{1}{n^2}}{\frac{1}{3n^2 + 4n + 1}} = \frac{3n^2 + 4n + 1}{n^2} = 3 + \frac{4}{n} + \frac{1}{n^2}.$$

This then gives $\lim_{n \to \infty} \frac{u_n}{v_n} = \lim_{n \to \infty} \left(3 + \frac{4}{n} + \frac{1}{n^2}\right) = 3$.

Therefore, by the limit comparison test we have that $\sum_{n=1}^{\infty} \frac{1}{3n^2 + 4n + 1}$ converges.

Again we stress that for the limit comparison test to be effective it is necessary to know that the series that is used to make the comparison is known to either converge or diverge. However, just as important is the choice of series we use to make the comparison. For example, had we used the series $\sum_{n=1}^{\infty} \frac{1}{n}$ (which is known to diverge) we would not have arrived at a concluding remark.

With $u_n = \frac{1}{n}$ we have that $\frac{u_n}{v_n} = \frac{3n^2+4n+1}{n}$ $\therefore \lim_{n\to\infty}\frac{u_n}{v_n} = \lim_{n\to\infty}\left(3n+4+\frac{1}{n}\right) = \infty$, and so, no conclusion can be made!

TEST 2 – D'ALEMBERT'S CRITERION (RATIO TEST)

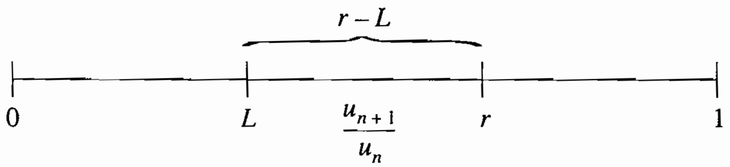
It is important that you understand the different conditions that exist between the limit comparison test and D'Alembert's ratio test.

If in a series of positive terms $u_1 + u_2 + u_3 + \ldots + u_n + \ldots$ the ratio $\frac{u_{n+1}}{u_n}$ as $n \to \infty$ is given by L, i.e. $\lim_{n\to\infty}\frac{u_{n+1}}{u_n} = L$, then:

i. if $L < 1$, the series converges;
ii. if $L > 1$ (or $L = \infty$) the series diverges;
iii. if $L = 1$, you cannot tell with the present criterion.

If you are in case (iii), it means that you shall have to use one of the other criteria that we shall study later (or use the comparison series test).

In order to first prove this statement of D'Alembert's for $L < 1$, let us consider a number r such that $L < r < 1$. This means that, for any n bigger than a certain order N, $\frac{u_{n+1}}{u_n} < r$.



This is because the difference $\frac{u_{n+1}}{u_n} - L$ can be made smaller than any positive number, in particular $r - L$, namely $\left|\frac{u_{n+1}}{u_n} - L\right| < r - L$.

As $\frac{u_{n+1}}{u_n} < r$, this allows us to write, from a certain $n = N$, the following relationships:

$$\begin{cases} u_{N+1} < ru_N \\ u_{N+2} < ru_{N+1} < r^2u_N \\ u_{N+3} < ru_{N+2} < r^2u_{N+1} < r^3u_N \\ \text{--} \end{cases}$$

Let us compare our original series

$$u_1 + u_2 + u_3 + \ldots + u_N + u_{N+1} + \ldots \qquad (2.8)$$

to the series

$$u_N + ru_N + r^2u_N + r^3u_N + \ldots \qquad (2.9)$$

We recognize that the latter is the sum of the terms of a geometric series, with common ratio $r < 1$, which we already know to converge. The terms of our original series for $n > N$, given by (2.8) are smaller than the terms of the series given by (2.9). Therefore, if the series (2.9) converges, our original series (2.8) also converges by the comparison test.

In the case of $L > 1$, we have that $\lim\limits_{n \to \infty} \frac{u_{n+1}}{u_n} = L > 1$.

That is, for a certain $n \geq N$, we have $\frac{u_{n+1}}{u_n} > 1$, which means that $u_{n+1} > u_n$ for any $n \geq N$.

Therefore the terms in the series have an ever-increasing value from $n = N + 1$ on. Hence as the general term does not tend to 0 the series diverges.

EXAMPLE 2.5

Investigate the behaviour of the series: $\frac{1}{2} + \frac{1}{5} + \frac{1}{10} + \frac{1}{17} + \ldots + \frac{1}{n^2+1} + \ldots$

Solution

Using D'Alembert's criterion, $\lim\limits_{n \to \infty} \frac{u_{n+1}}{u_n} = \lim\limits_{n \to \infty} \frac{\frac{1}{(n+1)^2+1}}{\frac{1}{n^2+1}} = \lim\limits_{n \to \infty} \frac{(n+1)^2}{(n+1)^2+1}$.

One can see that, as $n \to \infty$, this limit tends to the ratio of the monomials of highest degree, i.e. $\frac{n^2}{n^2} \to 1$, thus D'Alembert's criterion does not allow us to tell whether the series tested converges or diverges.

We may suspect the series to converge, as the limit of the general term tends to 0 as $n \to \infty$. That is, $\lim\limits_{n \to \infty} \frac{1}{n^2+1} = 0$. But for the time being this is only a conjecture as it constitutes only the necessary condition. In order to convince ourselves that this series indeed converges, we note that

$$\frac{1}{2} = 1 - \frac{1}{2};\ \frac{1}{5} = \frac{1}{2} - \frac{3}{10};\ \frac{1}{10} = \frac{3}{10} - \frac{2}{10};\ \text{etc...}$$

So, let us rewrite it as the sum of pairs of terms, viz.,

$$\begin{aligned}\frac{1}{2} + \frac{1}{5} + \frac{1}{10} + \frac{1}{17} + \ldots + \frac{1}{n^2+1} + \ldots &= \left(1 - \frac{1}{2}\right) + \left(\frac{1}{2} - \frac{3}{10}\right) + \left(\frac{3}{10} - \frac{2}{10}\right) + \left(\frac{2}{10} - \frac{24}{170}\right) + \ldots \\ &= 1 - \left(\frac{1}{2} - \frac{1}{2}\right) - \left(\frac{3}{10} - \frac{3}{10}\right) - \left(\frac{2}{10} - \frac{2}{10}\right) - \ldots \\ &= 1 - \text{small number}\end{aligned}$$

Thus, after cancellation by pairs, we can see that the sum of the series tends to a fixed, finite value, which is slightly smaller than 1. Hence we can say that the series converges.

EXAMPLE 2.6

Investigate the behaviour of the series:

$$1 + \frac{1}{1\cdot 2} + \frac{1}{1\cdot 2\cdot 3} + \dots + \frac{1}{1\cdot 2\cdot 3\cdot \dots \cdot n} + \dots$$

Solution

The general term can be written as $u_n = \frac{1}{n!}$, therefore $\lim_{n\to\infty} u_n = \lim_{n\to\infty}\frac{1}{n!} = 0$.

We suspect that the series may converge as the limit of its general term is 0, but we have to make sure by applying D'Alembert's criterion.

As $u_n = \frac{1}{n!}$ and $u_{n+1} = \frac{1}{(n+1)!}$, the D'Alembert's criterion tells us that:

$$\lim_{n\to\infty}\frac{u_{n+1}}{u_n} = \lim_{n\to\infty}\frac{\frac{1}{(n+1)!}}{\frac{1}{n!}} = \lim_{n\to\infty}\frac{n!}{(n+1)!}$$

$$= \lim_{n\to\infty}\frac{n!}{(n+1)\cdot n!}$$

$$= \lim_{n\to\infty}\frac{1}{(n+1)}$$

$$= 0$$

Therefore, as $\lim_{n\to\infty}\frac{u_{n+1}}{u_n} = 0$ (which is smaller than 1), we can now conclude that according to D'Alembert's criterion the series converges .

EXAMPLE 2.7

Investigate the behaviour of the series: $\frac{1}{2} + \frac{2}{5} + \frac{3}{10} + \frac{4}{17} + \dots + \frac{n}{n^2+1} + \dots$

Solution

In this case, the limit of the nth term u_n is $\lim_{n\to\infty}\frac{n}{n^2+1} = 0$, which indicates that the series might converge.

Next we have $u_n = \frac{n}{n^2+1}$ and $u_{n+1} = \frac{n+1}{(n+1)^2+1} = \frac{n+1}{n^2+2n+2}$.

So, using D'Alembert's criterion we have:

$$\lim_{n\to\infty}\frac{u_{n+1}}{u_n} = \lim_{n\to\infty}\frac{\frac{n+1}{n^2+2n+2}}{\frac{n}{n^2+1}} = \lim_{n\to\infty}\frac{(n+1)(n^2+1)}{n(n^2+2n+2)} = 1$$

Again, we cannot tell whether the series converges or diverges from this result. However we observe that

$$\frac{1}{2} = \left(1-\frac{1}{2}\right);\ \frac{2}{5} = \left(\frac{1}{2}-\frac{1}{10}\right);\ \frac{3}{10} = \left(\frac{1}{10}+\frac{2}{10}\right);\ \frac{4}{17} = \left(-\frac{2}{10}+\frac{74}{170}\right);\ \text{etc..}$$

Therefore, we can rewrite the series as

$$\left(1-\frac{1}{2}\right)+\left(\frac{1}{2}-\frac{1}{10}\right)+\left(\frac{1}{10}+\frac{2}{10}\right)+\left(-\frac{2}{10}+\frac{74}{170}\right)+\left(-\frac{74}{170}+\frac{2774}{4420}\right)+\dots$$

Then, after cancellation by pairs, the sum increases to be >1 and is not a fixed finite number, thus the series diverges.

EXAMPLE 2.8

Investigate the behaviour of the series: $\frac{2}{1}+\frac{2^2}{2}+\frac{2^3}{3}+\dots+\frac{2^n}{n}+\dots$

Solution

We can already tell the behaviour of this series from the limit of the general term.

As $\lim\limits_{n\to\infty}\frac{2^n}{n} = \infty$ (i.e., two to the power of n grows much faster than n) then the series diverges as the limit doesn't equal zero.

On the other hand if we use D'Alembert's criterion, we have $u_n = \frac{2^n}{n}$ and $u_{n+1} = \frac{2^{n+1}}{n+1}$.

$$\text{Therefore, } \lim_{n\to\infty}\frac{u_{n+1}}{u_n} = \lim_{n\to\infty}\frac{\frac{2^{n+1}}{n+1}}{\frac{2^n}{n}} = \lim_{n\to\infty}\frac{2^{n+1}}{2^n}\cdot\frac{n}{n+1}$$

$$= \lim_{n\to\infty}\frac{2n}{n+1}$$

Then, as $\lim\limits_{n\to\infty}\frac{2n}{n+1} = 2 > 1$, from D'Alembert's criterion, we also see that the series **diverges**.

EXAMPLE 2.9

Investigate the behaviour of the series: $\frac{1}{2}+\frac{2}{3}+\frac{3}{4}+\dots+\frac{n}{n+1}+\dots$

Solution

The limit of the general term is $\lim\limits_{n\to\infty}\frac{n}{n+1} = 1$, which is nonzero. Therefore the series diverges.

Had we used D'Alembert's criterion we have $u_n = \frac{n}{n+1}$ and $u_{n+1} = \frac{n+1}{n+2}$, so that

$$\lim_{n\to\infty}\frac{u_{n+1}}{u_n} = \lim_{n\to\infty}\frac{\frac{n+1}{n+2}}{\frac{n}{n+1}} = \lim_{n\to\infty}\frac{(n+1)^2}{n(n+2)} = 1$$

Meaning that we cannot deduce the behaviour of the series from this result. However, we note that as $u_{n+1} - u_n = \frac{n+1}{n+2} - \frac{n}{n+1} = \frac{(n+1)^2 - n(n+2)}{(n+2)(n+1)} = \frac{1}{(n+2)(n+1)} > 0$ the series **diverges**.

EXAMPLE 2.10

Study the behaviour of the series: $\frac{1}{1\cdot 2} + \frac{1}{2\cdot 3} + \frac{1}{3\cdot 4} + \ldots + \frac{1}{n(n+1)} + \ldots$

Solution

The limit of the general term is $\lim_{n\to\infty}\left[\frac{1}{n(n+1)}\right] = 0$, thus suggesting that the series might converge.

If we apply D'Alembert's criterion, we have $u_n = \frac{1}{n(n+1)}$ and $u_{n+1} = \frac{1}{(n+1)(n+2)}$.

Therefore:

$$\lim_{n\to\infty}\frac{u_{n+1}}{u_n} = \lim_{n\to\infty}\frac{\frac{1}{(n+1)(n+2)}}{\frac{1}{n(n+1)}} = \lim_{n\to\infty}\frac{n(n+1)}{(n+1)(n+2)}$$

$$= \lim_{n\to\infty}\frac{n}{(n+2)}$$

$$= 1$$

This shows that we cannot deduce the behaviour of the series from D'Alembert's criterion. However, we can rewrite the general term of the series using the **method of partial fractions**.

That is, using the fact that $\frac{1}{n(n+1)} = \frac{1}{n} - \frac{1}{n+1}$,

The series can be rewritten as $\left(1 - \frac{1}{2}\right) + \left(\frac{1}{2} - \frac{1}{3}\right) + \left(\frac{1}{3} - \frac{1}{4}\right) + \ldots + \left(\frac{1}{n} - \frac{1}{n+1}\right) + \ldots$

Cancelling by pairs, we have the partial sum

$$S_n = 1 - \frac{1}{n+1}\text{, and thus}$$

$$\lim_{n\to\infty} S_n = \lim_{n\to\infty}\left(1 - \frac{1}{n+1}\right)$$

$$= 1$$

That is, the sum tends to a fixed finite limit, 1 , which means that the series **converges**.

In some of the preceeding examples we made use of two common methods (of rewriting terms) when studying the behaviour of series, namely –

1. **Telescoping series**
2. **Partial fractions**

We consider each of these in turn:

1. Telescoping series

When the terms of series are of the form $\sum_{n=1}^{\infty}\left(\frac{1}{n}-\frac{1}{n+1}\right)$ we end up with a series that looks like

$$\sum_{n=1}^{\infty}\left(\frac{1}{n}-\frac{1}{n+1}\right) = \left(\frac{1}{1}-\frac{1}{2}\right)+\left(\frac{1}{2}-\frac{1}{3}\right)+\left(\frac{1}{3}-\frac{1}{4}\right)+\left(\frac{1}{4}-\frac{1}{5}\right)+\ldots+\left(\frac{1}{n}-\frac{1}{n+1}\right)+\ldots$$

$$= \frac{1}{1}+\left(-\frac{1}{2}+\frac{1}{2}\right)+\left(-\frac{1}{3}+\frac{1}{3}\right)+\left(-\frac{1}{4}+\frac{1}{4}\right)+\ldots+\left(-\frac{1}{n}+\frac{1}{n}\right)+\ldots$$

So, as $\sum_{n=1}^{\infty}\left(\frac{1}{n}-\frac{1}{n+1}\right) = \lim_{n\to\infty}\sum_{i=1}^{n}\left(\frac{1}{i}-\frac{1}{i+1}\right)$ we can write:

$$\lim_{n\to\infty}\sum_{i=1}^{n}\left(\frac{1}{i}-\frac{1}{i+1}\right) = \lim_{n\to\infty}\left[\frac{1}{1}+\left(-\frac{1}{2}+\frac{1}{2}\right)+\left(-\frac{1}{3}+\frac{1}{3}\right)+\left(-\frac{1}{4}+\frac{1}{4}\right)+\ldots+\left(-\frac{1}{n}+\frac{1}{n}\right)+\left(-\frac{1}{n+1}\right)\right]$$

$$= \lim_{n\to\infty}\left[1-\frac{1}{n+1}\right]$$

$$= 1$$

giving the result that $\sum_{n=1}^{\infty}\left(\frac{1}{n}-\frac{1}{n+1}\right) = 1$

The great thing about telescoping series is that not only can we determine that the series converges, but it has the advantage that it provides the limiting value of the series.

2. Partial fractions

Closely following on from telescoping series, there are expressions in which we first need to rewrite the terms as the sum of partial fractions.

For example, consider the series $\sum_{n=1}^{\infty}\frac{1}{n(n+1)}$, we would first express $\frac{1}{n(n+1)}$ as a sum of partial fractions. That is, $\frac{1}{n(n+1)} = \frac{1}{n}-\frac{1}{n+1}$.

We obtain the partial fractions by solving for A and B and assuming that the fraction can be 'broken up' as:

$\frac{1}{n(n+1)} \equiv \frac{A}{n}+\frac{B}{n+1}$, from where we get $\frac{1}{n(n+1)} \equiv \frac{A(n+1)+Bn}{n(n+1)}$.

$$\Leftrightarrow 1 = A(n+1)+Bn$$

$$\Leftrightarrow 1 = A+(A+B)n$$

Therefore, equating the coefficients we have: $A = 1, B = -A \therefore B = -1$

And so, $\sum_{n=1}^{\infty}\frac{1}{n(n+1)} = \sum_{n=1}^{\infty}\left[\frac{1}{n}-\frac{1}{n+1}\right] = 1$ (from our result of the telescoping series).

While we have already made use of these results in some of the previous examples (without acknowledging their formal existence) both these results are very useful and should become a part of your mathematical tool box.

Next, we consider another test that will help us study the behaviour of series that could not be investigated with the tools we have developed thus far. In the same way that we were able to compare the behaviour of sequences to their functional counterpart in chapter 1, this next test does the same thing except that this time we compare the sum of the series, with positive decreasing terms, to the integral of the corresponding decreasing, positive function. This result is based on the area of calculus that looks at the existence of a finite area under a curve as $x \to \infty$.

TEST 3 – THE INTEGRAL TEST

If we have a series of positive terms $u_1 + u_2 + u_3 + \ldots + u_n + \ldots$ such that they do not increase, $u_1 \ge u_2 \ge u_3 \ldots$ and if $f(x)$ is a continuous, decreasing and positive function such that $f(1) = u_1, f(2) = u_2, \ldots, f(n) = u_n$, we have that

(a) if the integral $\int_1^{\infty} f(x)dx$ converges, then the series converges;

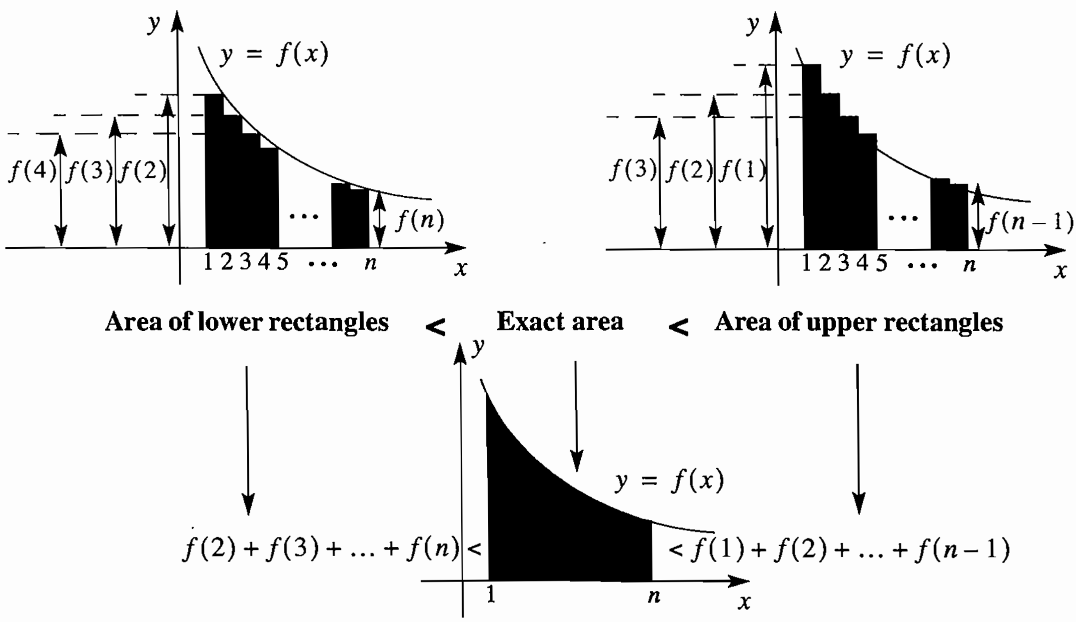
(b) if the integral $\int_1^{\infty} f(x)dx$ diverges, then the series diverges.

Why should this method work?

We are only concerned with the idea behind the proof, enabling us to 'show' how and why this test works. Comparing the areas (of the shaded regions) in the diagrams below, where $u_n = f(n)$ represents the terms of a sequence having the corresponding real valued function, $y = f(x)$, we have:

Area of lower rectangles ≤ Exact area under curve ≤ Area of upper rectangles

i.e., $f(2) + f(3) + \ldots + f(n) \le \int_1^n f(x)dx \le f(1) + f(2) + \ldots + f(n-1)$



So, if $\lim_{n \to \infty} \int_1^n f(x)dx$ is finite, then it must be the case that $f(2) + f(3) + \ldots + f(n)$ is also bounded and so the series converges.

However, if $\lim_{n \to \infty} \int_1^n f(x)dx = \infty$, then it must be the case that $f(1) + f(2) + \ldots + f(n-1)$ is unbounded and so the series diverges.

EXAMPLE 2.11

Investigate the behaviour of the series $\sum_{n=1}^{\infty} \frac{\arctan(n)}{n^2+1}$

Solution

The terms $f(n) = \frac{\arctan(n)}{n^2+1}$ in the series $\sum_{n=1}^{\infty} \frac{\arctan(n)}{n^2+1} = \frac{\arctan 1}{2} + \frac{\arctan 2}{5} + \ldots$ are decreasing and positive.

If we next consider the continuous function $f(x) = \frac{\arctan x}{x^2+1}$, which is also positive and decreasing for $x \geq 1$, then it is appropriate to use the integral test for convergence.

We also note that $f(x) = \frac{\arctan x}{x^2+1} = \arctan x \cdot \frac{1}{x^2+1}$ which means that it will be relatively straight forward to integrate:

$$\int \frac{\arctan x}{x^2+1} dx = \int \left(\arctan x \cdot \frac{1}{x^2+1} \right) dx$$

Let $u = \arctan x \Rightarrow \frac{du}{dx} = \frac{1}{x^2+1}$ so that $du = \frac{1}{x^2+1} dx$

$$\therefore \int \left(\arctan x \cdot \frac{1}{x^2+1} \right) dx = \int u \, du$$

$$= \frac{1}{2}u^2 + c$$

$$= \frac{1}{2}[\arctan x]^2 + c$$

Now, $\int_1^{\infty} \frac{\arctan x}{x^2+1} dx = \lim_{n \to \infty} \int_1^n \frac{\arctan x}{x^2+1} dx = \lim_{n \to \infty} \left[\frac{1}{2}(\arctan x)^2 \right]_1^n$

$$= \lim_{n \to \infty} \left[\frac{1}{2}(\arctan n)^2 - \frac{1}{2}(\arctan 1)^2 \right]$$

$$= \frac{1}{2} \lim_{n \to \infty} [(\arctan n)^2 - (\arctan 1)^2]$$

$$= \frac{1}{2}\left(\frac{\pi^2}{4} - (\arctan 1)^2 \right)$$

Therefore, as $\int_1^{\infty} \frac{\arctan x}{x^2+1} dx$ converges, the series $\sum_{n=1}^{\infty} \frac{\arctan n}{n^2+1}$ also converges.

EXAMPLE 2.12

Investigate the behaviour of the series $\sum_{n=2}^{\infty} \frac{2}{n\ln(n)}$.

Solution

Consider the continuous, positive and decreasing function $f(x) = \frac{2}{x\ln x}, x \geq 2$.

As this function satisfies the criteria for the integral test, we make use of it to help us determine the convergence (or otherwise) of the series $\sum_{n=2}^{\infty} \frac{2}{n\ln(n)}$.

Now, $\int_2^{\infty} \frac{2}{x\ln x}dx = \lim_{n\to\infty} \int_2^n \frac{2}{x\ln x}dx = 2\lim_{n\to\infty} \int_2^n \frac{1}{\ln x}\cdot\frac{1}{x}dx$.

So, letting $u = \ln x$ we have $\frac{du}{dx} = \frac{1}{x} \therefore \int \frac{1}{\ln x}\cdot\frac{1}{x}dx = \int \frac{1}{u}du$

$$= \ln u + c$$
$$= \ln(\ln x) + c$$

So, $2\lim_{n\to\infty} \int_2^n \frac{1}{\ln x}\cdot\frac{1}{x}dx = 2\lim_{n\to\infty} \left[\ln(\ln x)\right]_2^n$

$$= 2\lim_{n\to\infty} [\ln(\ln n) - \ln(\ln 2)]$$
$$\to \infty$$

Therefore, as $\int_2^{\infty} \frac{2}{x\ln x}dx \to \infty$, $\sum_{n=2}^{\infty} \frac{2}{n\ln(n)}$ diverges.

EXAMPLE 2.13

Investigate the behaviour of the series $\frac{1}{1^p} + \frac{1}{2^p} + \frac{1}{3^p} + \ldots + \frac{1}{n^p} + \ldots$

Solution

Let us define the function $f(x) = \frac{1}{x^p}$. Consider the integral $\int_1^N \frac{1}{x^p}dx$.

If $p \neq 1$, $\int_1^N \frac{1}{x^p}dx = \left[\frac{x^{-p+1}}{-p+1}\right]_1^N = \frac{1}{1-p}(N^{-p+1} - 1)$.

Then, if $p = 1$, $\int_1^N \frac{1}{x^p}dx = \int_1^N \frac{1}{x}dx = \ln(N)$.

If we now make $N \to \infty$, we can study the convergence of the series.

1. If $p > 1$, $\int_1^{\infty} \frac{1}{x^p}dx = \frac{1}{p-1}$ the integral is finite and thus the series **converges**.
2. If $p < 1$, $\int_1^{\infty} \frac{1}{x^p}dx = \infty$, the integral is infinite and thus the series **diverges**.
3. If $p = 1$, $\int_1^{\infty} \frac{1}{x^p}dx = \infty$, the integral is infinite and the series then **diverges**.

EXERCISES 2.1

1. Test the series $\frac{1}{7}+\frac{1}{7^2}+\frac{1}{7^3}+\ldots+\frac{1}{7^n}+\ldots$ for convergence.

2. Find the limit of the n-th sum for $n \to \infty$ and then deduce if the infinite series $\frac{1}{1\times 2}+\frac{1}{2\times 3}+\frac{1}{3\times 4}+\ldots+\frac{1}{n(n+1)}+\ldots$ converges or not.

3. Find $\lim\limits_{n\to\infty}\left(\frac{2}{3}+\frac{8}{9}+\frac{26}{27}+\ldots+\frac{3^n-1}{3^n}+\ldots\right)$ and study its convergence.

4. Study the convergence of the series $5+\sqrt{5}+\sqrt[3]{5}+\ldots+\sqrt[n]{5}+\ldots$.

5. Test the convergence of the series $\frac{1}{1\times 3}+\frac{1}{3\times 5}+\frac{1}{5\times 7}+\ldots+\frac{1}{(2n-1)(2n+1)}+\ldots$

6. Apply D'Alembert's criterion to study the convergence of the series:

(a) $\frac{1}{3\times 2}+\frac{1}{4\times 2^2}+\frac{1}{5\times 2^3}+\frac{1}{6\times 2^4}+\ldots+\frac{1}{(n+2)2^n}+\ldots$

(b) $\frac{1}{2!}+\frac{8}{3!}+\frac{27}{4!}+\frac{64}{5!}+\ldots+\frac{n^3}{(n+1)!}+\ldots$

(c) $\frac{1}{2}+\frac{3!}{2^2}+\frac{5!}{2^3}+\frac{7!}{2^4}+\ldots+\frac{(2n-1)!}{2^n}+\ldots$

(d) $\frac{1}{2}+\frac{2!}{2\times 4}+\frac{3!}{2\times 4\times 6}+\ldots+\frac{n!}{2\times 4\times\ldots\times 2n}+\ldots$

(e) $3+\frac{5}{3}\times\frac{1}{3}+\frac{7}{5}\times\frac{1}{9}+\ldots+\frac{(2n+1)}{(2n-1)}\times\frac{1}{3^{n-1}}+\ldots$

(f) $1+\frac{2^2}{2!}+\frac{3^3}{3!}+\ldots+\frac{n^n}{n!}+\ldots$

(g) $\frac{2}{3}+\left(\frac{3}{4}\right)^2+\left(\frac{4}{5}\right)^3+\ldots+\left(\frac{n+1}{n+2}\right)^n+\ldots$

(h) $1+\frac{3!}{8}+\frac{5!}{27}+\ldots+\frac{(2n-1)!}{n^3}+\ldots$

7. Use the integral test to study the series:

(a) $\frac{\ln 2}{2}+\frac{\ln 3}{3}+\ldots+\frac{\ln n}{n}+\ldots$

(b) $1+\frac{1}{4}+\frac{1}{9}+\ldots+\frac{1}{n^2}+\ldots$

(c) $\frac{1}{\sqrt{3}}+\frac{1}{\sqrt{5}}+\frac{1}{\sqrt{7}}+\ldots+\frac{1}{\sqrt{2n+1}}+\ldots$

(d) $\frac{1}{e}+\frac{2}{e^2}+\frac{3}{e^3}+\ldots+\frac{n}{e^n}+\ldots$

8. Determine if the series $\sum\limits_{n=1}^{\infty}\frac{1}{2^n}$ converges or diverges and determine its value if possible.

9. Determine if the series $\sum_{n=1}^{\infty} 2\left(\frac{1}{3}\right)^n$ converges and determine its value if it does.

10. (a) i. On the same set of axes sketch the graphs of $h(x) = \sqrt{x}$ and $g(x) = \ln x$.

ii. Find $\lim_{x \to \infty} \frac{\ln x}{\sqrt{x}}$.

(b) Determine if the series $\sum_{k=1}^{\infty} \frac{\ln k}{k^2}$ converges or diverges.

11. (a) Using the limit comparsion test, show that the series $\sum_{n=1}^{\infty} \frac{1}{an^2 + bn + c}$, where $a > 0, b > 0, c > 0$ converges.

(b) Show that $\sum_{n=1}^{\infty} \frac{1}{n^2 + 3n + 2}$ converges by making use of limit comparsion test.

(c) i. Express $\frac{1}{n^2 + 3n + 2}$ in its partial fractions, $\frac{A}{n+1} + \frac{B}{n+2}$, $A, B \in \mathbb{R}^+$.

ii. Use the integral test to show that $\sum_{n=1}^{\infty} \frac{1}{n^2 + 3n + 2}$ converges.

12. (a) i. Show that $\sum_{n=1}^{\infty} \frac{1}{2n + 20} = \frac{1}{2} \sum_{k=11}^{\infty} \frac{1}{k}$.

ii. Hence, deduce that $\sum_{n=1}^{\infty} \frac{1}{2n + 20}$ diverges.

(b) Make use of the integral test to confirm your result of (a) ii.

13. Using the limit comparison test, decide if $\sum_{n=1}^{\infty} \frac{1}{\sqrt{n^2 + 2n}}$ converges or diverges.

14. Use the integral test to decide which of the following series converge.

(a) $\sum_{n=2}^{\infty} \frac{2}{n \ln n}$ (b) $\sum_{n=2}^{\infty} \frac{2}{n(\ln n)^3}$ (c) $\sum_{n=2}^{\infty} \frac{2}{n\sqrt{\ln n}}$

15. (a) If $T(x) = \ln\left(\frac{x}{x+1}\right)^x$, show that $\lim_{x \to \infty} T(x) = -1$.

(b) Hence decide if the series $\sum_{n=1}^{\infty} \ln\left(\frac{n}{n+1}\right)^n$ converges or diverges.

16. Use the ratio test to determine the convergence or divergence of

(a) $\sum_{n=1}^{\infty} \frac{r^n}{n^r}, 0 < r < 1$ (b) $\sum_{n=1}^{\infty} \frac{n!}{n^n}$

2.2 MORE USE OF INTEGRALS

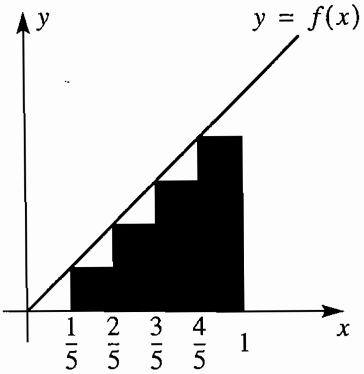
2.2.1 MORE ON COMPARING INFINITE SERIES AND THEIR INTEGRAL COUNTERPART

In the previous section we introduced the concept of comparing the limit of an infinite series to its corresponding improper integral. This was done so that we could introduce the integral test. We now briefly return to the notion of the integral as a limit of a sum – in particular, we want to have a closer look at how this correspondence comes about and introduce more thoroughly what we mean by the lower sum and the upper sum.

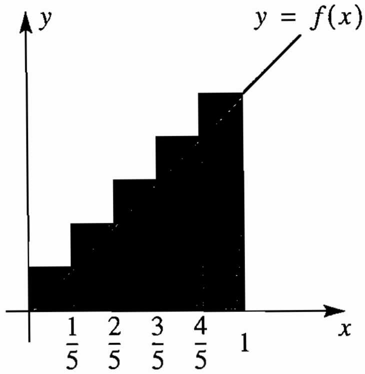
We begin by considering the definite integral $\int_0^1 x dx$. This means that we need to show that $f(x) = x$ is in fact integrable over the domain $[0, 1]$ and that this integral has a unique value.

As the function is well behaved over $[0, 1]$ we shall use partitions which give strips of equal width and which shall provide a good illustration of what we mean by a lower sum and an upper sum.

Lower Sum



Upper Sum



We can now calculate the area of the rectangular regions for both the lower sum, A_L, and the upper sum, A_U. As the areas are given by the sum of the width × height of each rectangle, then;

Lower Sum: $A_L = \left(0 \times \frac{1}{5}\right) + \left(\frac{1}{5} \times \frac{1}{5}\right) + \left(\frac{1}{5} \times \frac{2}{5}\right) + \left(\frac{1}{5} \times \frac{3}{5}\right) + \left(\frac{1}{5} \times \frac{4}{5}\right) = \frac{2}{5}$

and

Upper Sum: $A_U = \left(\frac{1}{5} \times \frac{1}{5}\right) + \left(\frac{1}{5} \times \frac{2}{5}\right) + \left(\frac{1}{5} \times \frac{3}{5}\right) + \left(\frac{1}{5} \times \frac{4}{5}\right) + \left(\frac{1}{5} \times 1\right) = \frac{3}{5}$

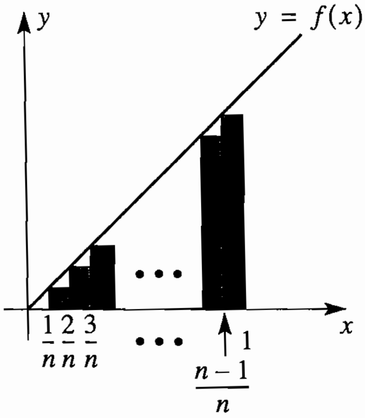
This tells us that the area represented by the integral $\int_0^1 x dx$ is such that, $\frac{2}{5} < \int_0^1 x dx < \frac{3}{5}$.

So, while this is a useful result, in the sense that we have an upper and lower bound for the area of the region defined by $\int_0^1 x dx$, we still fall short of obtaining its exact value. However, we observe that as the number of intervals used increases, the difference between the lower sum and the upper sum decreases.

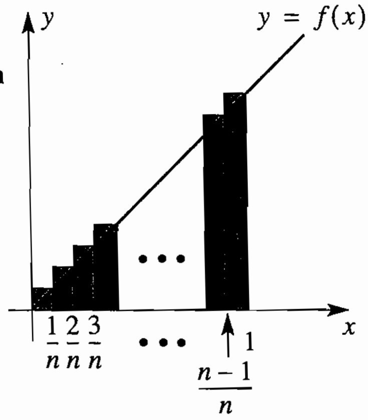
In fact, by partitioning [0, 1] into n intervals of equal width, i.e., $0, \frac{1}{n}, \frac{2}{n}, \frac{3}{n}, \ldots, \frac{n-1}{n}, 1$ and increasing the value of n indefinitely, we can determine the exact value of $\int_0^1 x\,dx$.

Using n partitions:

Lower Sum



Upper Sum



we have:

$$\text{Lower Sum, } A_L = \left(0 \times \frac{1}{n}\right) + \left(\frac{1}{n} \times \frac{1}{n}\right) + \left(\frac{1}{n} \times \frac{2}{n}\right) + \left(\frac{1}{n} \times \frac{3}{n}\right) + \ldots + \left(\frac{1}{n} \times \frac{n-1}{n}\right)$$

$$= \frac{1}{n^2}(0 + 1 + 2 + 3 + \ldots + (n-1))$$

$$= \frac{1}{n^2}\left[\frac{1}{2}n(n-1)\right]$$

$$= \frac{1}{2} \cdot \frac{n-1}{n}$$

As $n \to \infty$, $\frac{n-1}{n} = 1 - \frac{1}{n} \to 1 \therefore A_L \to \frac{1}{2}$. In fact, $A_L \to \frac{1^-}{2}$. Next, we have:

$$\text{Upper Sum, } A_U = \left(\frac{1}{n} \times \frac{1}{n}\right) + \left(\frac{1}{n} \times \frac{2}{n}\right) + \left(\frac{1}{n} \times \frac{3}{n}\right) + \ldots + \left(\frac{1}{n} \times 1\right)$$

$$= \frac{1}{n^2}(1 + 2 + 3 + \ldots + n)$$

$$= \frac{1}{n^2}\left[\frac{1}{2}n(n+1)\right]$$

$$= \frac{1}{2} \cdot \frac{n+1}{n}$$

As $n \to \infty$, $\frac{n+1}{n} = 1 + \frac{1}{n} \to 1 \therefore A_U \to \frac{1}{2}$. In fact, $A_U \to \frac{1^+}{2}$

So we have that $A_L < \int_0^1 x\,dx < A_U$, but more importantly, as the number of partitions increases and $A_L \to \frac{1^-}{2}$ and $A_U \to \frac{1^+}{2}$, it must be the case that $\int_0^1 x\,dx = \frac{1}{2}$.

We have shown that $\sum_{k=1}^{n} \left(\frac{k-1}{n}\right)\frac{1}{n} \le \int_0^1 x\,dx \le \sum_{k=1}^{n} \left(\frac{k}{n}\right)\frac{1}{n}$ and that as $n \to \infty$, $\int_0^1 x\,dx \to \frac{1}{2}$.

The reason for having spent time going through a detailed example of how the integral is evaluated via the use of an infinite series is to highlight the relationship that exists between integration and series and to provide a little more justification for the method used in Examples 2.10, 2.11 and 2.12.

The difference between the example we have just worked through and the limit of a series is that with a series, the partitions [or 'widths'] are always 1 as opposed to the partitions [or widths] of $\frac{1}{n}$, as in the example given. So, when referring to a series, we can simplify our result to:

If the function, $f(x)$ is continuous over the domain $x \geq a$, where a is an integer, the series $\sum_{i=a}^{\infty} f(i)$ can be approximated by the integral $\int_a^{\infty} f(x)dx$. That is, $\int_a^{\infty} f(x)dx \approx \sum_{i=a}^{\infty} f(i)$.

Of course, if $\int_a^{\infty} f(x)dx$ diverges, so too will the series – which was the purpose of making use of improper integrals when using the integral test!

EXAMPLE 2.14

Find an approximation to the series $\sum_{i=1}^{\infty} \frac{1}{(2i+3)^2}$.

Solution

From our result above, we have $\sum_{i=1}^{\infty} \frac{1}{(2i+3)^2} \approx \int_1^{\infty} \frac{1}{(2x+3)^2} dx$.

Now, let $u = 2x+3 \Rightarrow \frac{du}{dx} = 2 \therefore \frac{1}{2}du = dx$.

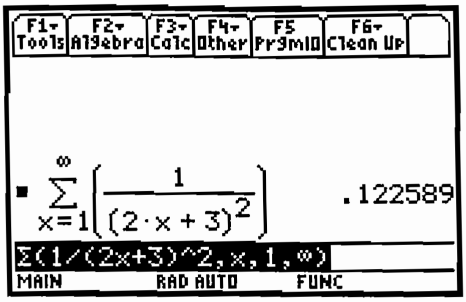
And
$$\int \frac{1}{(2x+3)^2} dx = \int \frac{1}{u^2} \cdot \frac{1}{2} du = \frac{1}{2}\int u^{-2} du$$
$$= -\frac{1}{2}u^{-1} + c$$
$$= -\frac{1}{2(2x+3)} + c$$

Therefore,
$$\int_1^{\infty} \frac{1}{(2x+3)^2} dx = \lim_{n \to \infty} \int_1^{n} \frac{1}{(2x+3)^2} dx = \lim_{n \to \infty} \left[-\frac{1}{2(2x+3)}\right]_1^n$$
$$= -\frac{1}{2} \lim_{n \to \infty} \left[\frac{1}{(2n+3)} - \frac{1}{5}\right]_1^n$$
$$= \frac{1}{10}$$

So, we have that $\sum_{i=1}^{\infty} \frac{1}{(2i+3)^2} \approx \frac{1}{10}$.

Using the TI–89, the sum is in fact 0.12259 (to 5 decimal places).

We see that in this case, our approximation of 0.1 is quite good.



EXAMPLE 2.15

Find an approximation to the series $\sum_{i=1}^{\infty}\frac{1}{i(i+1)}$.

Solution

We start with $\sum_{i=1}^{\infty}\frac{1}{i(i+1)} \approx \int_1^{\infty}\frac{1}{x(x+1)}dx = \lim_{n\to\infty}\int_1^{n}\frac{1}{x(x+1)}dx$

Next, $\int_1^{n}\frac{1}{x(x+1)}dx = \int_1^{n}\left(\frac{1}{x}-\frac{1}{x+1}\right)dx = \left[\ln x - \ln(x+1)\right]_1^n$

$$= \left[\ln\left(\frac{x}{x+1}\right)\right]_1^n$$

$$= \ln\left(\frac{n}{n+1}\right) - \ln\frac{1}{2}$$

$$= \ln\left(\frac{n}{n+1}\right) + \ln 2$$

Therefore, $\lim_{n\to\infty}\int_1^{n}\frac{1}{x(x+1)}dx = \lim_{n\to\infty}\left[\ln\left(\frac{n}{n+1}\right)+\ln 2\right] = \lim_{n\to\infty}\left[\ln\left(1-\frac{1}{n+1}\right)+\ln 2\right]$

$$= \ln 1 + \ln 2$$

$$= \ln 2$$

So, $\sum_{i=1}^{\infty}\frac{1}{i(i+1)} \approx \ln 2$ (~ 0.6931).

We note that $\sum_{i=1}^{\infty}\frac{1}{i(i+1)} = \lim_{n\to\infty}\sum_{i=1}^{n}\frac{1}{i(i+1)} = \lim_{n\to\infty}\sum_{i=1}^{n}\left[\frac{1}{i}-\frac{1}{i+1}\right]$

$$= \lim_{n\to\infty}\left[\left(\frac{1}{1}-\frac{1}{2}\right)+\left(\frac{1}{2}-\frac{1}{3}\right)+\ldots+\left(\frac{1}{n}-\frac{1}{n+1}\right)\right]$$

$$= \lim_{n\to\infty}\left[\frac{1}{1}-\frac{1}{n+1}\right]$$

$$= 1$$

This time we see that there is a significant difference between the approximation and the actual value of the limit of the series. Nonetheless, using the integral approximation did show us that the series would converge.

The question that remains is: How good an approximation is the integral?

2.2.2 ERRORS IN APPROXIMATING AN INFINITE SERIES

We have already realised that it is sometimes not possible to evaluate an infinite series exactly. For this reason we then need to be satisfied with an approximation to the infinite series. In this section we investigate the accuracy of such an approximation by using a partial sum of the series.

Let $S = \sum_{i=1}^{\infty} f(i)$ and $S_n = \sum_{i=1}^{n} f(i)$ then, $S = \sum_{i=1}^{\infty} f(i) = \sum_{i=1}^{n} f(i) + \sum_{i=n+1}^{\infty} f(i)$.

That is, $S = S_n + R_n$, where $R_n = \sum_{i=n+1}^{\infty} f(i)$ is called the **remainder** after n terms.

That is, the amount by which $S_n = \sum_{i=1}^{n} f(i)$ differs from S is given by $R_n = S - S_n$ which can be approximated as follows:

Suppose that $a_i = f(i)$ for all $i = 0, 1, 2, \ldots$ where f is a non-negative, continuous and decreasing function for all $x \geq 1$ and $\int_1^{\infty} f(x)dx$ converges, then the remainder R_n satisfies the inequality

$$0 \leq R_n = \sum_{i=n+1}^{\infty} f(i) < \int_n^{\infty} f(x)dx$$

EXAMPLE 2.16

Estimate the error in using the partial sum S_{1000} to approximate the sum of the series $\sum_{n=1}^{\infty} \frac{1}{n^3}$.

Solution

The key to answering this question is that we are required to *estimate* the error – which leads us to believe that we can make use of an appropriate integral.

From Example 2.13 the series $\sum_{n=1}^{\infty} \frac{1}{n^p}$ converges if $p > 1$ and so $\sum_{n=1}^{\infty} \frac{1}{n^3}$ converges, meaning that we can make use of the above result to determine R_n.

That is, from $0 \leq R_n = \sum_{i=n+1}^{\infty} f(i) < \int_n^{\infty} f(x)dx$ with $n = 1000$ and $f(x) = \frac{1}{x^3}$ we have:

$$0 \leq R_{1000} < \int_{1000}^{\infty} f(x)dx = \lim_{n \to \infty} \int_{1000}^{n} \frac{1}{x^3}dx$$

$$= \lim_{n \to \infty} \left[-\frac{1}{2x^2}\right]_{1000}^{n}$$

$$= \lim_{n \to \infty} \left[-\frac{1}{2n^2} + \frac{1}{2 \times 1000^2}\right]$$

$$= \frac{1}{2} \times 1000^{-2}$$

$$= 5 \times 10^{-7}$$

Therefore, the error is (approximately) 5×10^{-7}.

EXAMPLE 2.17

Determine the number of terms needed to obtain an approximation to the sum of the series $\sum_{n=1}^{\infty} \frac{1}{n^3}$ correct to within 10^{-6}.

Solution

We have already determined from Example 2.13 that the given series converges and so we are in a position to make use of the result: $0 \le R_n = \sum_{i=n+1}^{\infty} f(i) < \int_n^{\infty} f(x)dx$.

Let $n = N$ be the number of terms required to produce an error less than 10^{-6}.

Therefore, we have $$0 \le R_N = \sum_{i=N+1}^{\infty} f(i) < \int_N^{\infty} f(x)dx < 10^{-6}$$

Now, $$\int_N^{\infty} f(x)dx = \lim_{n \to \infty} \int_N^n f(x)dx = \lim_{n \to \infty} \int_N^n \frac{1}{x^3}dx$$

$$= \lim_{n \to \infty} \left[-\frac{1}{2x^2}\right]_N^n$$

$$= \lim_{n \to \infty} \left[-\frac{1}{2n^2} + \frac{1}{2N^2}\right]$$

$$= \frac{1}{2N^2}$$

That is, $0 \le R_N < \frac{1}{2N^2} < 10^{-6}$. Then, $\frac{1}{2N^2} < 10^{-6} \Leftrightarrow N^2 > \frac{1}{2} \times 10^6$

$$\Leftrightarrow N > \sqrt{\frac{1}{2} \times 10^6}$$

$$\Leftrightarrow N > 707.1068$$

Taking $N \ge 708$ will guarantee that the sum of the series $\sum_{n=1}^{\infty} \frac{1}{n^3}$ will be correct to within 10^{-6}.

EXERCISES 2.2

1. (a) Prove that $1 + 4 + 9 + 16 + \ldots + n^2 = \frac{1}{6}n(n+1)(2n+1)$.

(b) i. Sketch the graph of $f(x) = x^2, 0 \le x \le 1$.

ii. On your graph, using partitions of $\frac{1}{5}$ on the interval [0, 1], show that

$$\frac{6}{25} \le \int_0^1 f(x)dx \le \frac{11}{25}.$$

(c) i. Show that with partitions of $\frac{1}{n}$ on the interval [0, 1] we have that

$$\frac{(n-1)(2n-1)}{6n^2} \le \int_0^1 f(x)dx \le \frac{(n+1)(2n+1)}{6n^2}$$

ii. Hence, show that $\int_0^1 x^2 dx = \frac{1}{3}$.

2. (a) Prove that $1 + 8 + 27 + 64 + \ldots + n^3 = \frac{1}{4}n^2(n+1)^2$.

(b) i. Sketch the graph of $f(x) = x^3, 0 \le x \le 1$.

ii. On your graph, using partitions of $\frac{1}{5}$ on the interval [0, 1], show that

$$\frac{4}{25} \le \int_0^1 f(x)dx \le \frac{9}{25}.$$

(c) i. Show that with partitions of $\frac{1}{n}$ on the interval [0, 1] we have that

$$\frac{(n-1)^2}{4n^2} \le \int_0^1 f(x)dx \le \frac{(n+1)^2}{4n^2}$$

ii. Hence show that $\int_0^1 x^3 dx = \frac{1}{4}$.

3. (a) Evaluate i. $\sum_{n=1}^{5} \frac{4}{n^2+4}$. ii. $\sum_{n=1}^{10} \frac{4}{n^2+4}$

(b) i. Determine $\int_n^\infty \frac{4}{x^2+4}dx$.

ii. Hence estimate the error in using each of the partial sums in (a) to approximate $\sum_{n=1}^{\infty} \frac{4}{n^2+4}$.

4. (a) Evaluate i. $\sum_{n=1}^{5} \frac{n}{e^n}$ ii. $\sum_{n=1}^{9} \frac{n}{e^n}$

(b) i. Determine $\int_n^\infty \frac{x}{e^x}dx$.

ii. Hence estimate the error in using each of the partial sums in (a) to approximate $\sum_{n=1}^{\infty} \frac{n}{e^n}$.

5. (a) Evaluate i. $\sum_{n=1}^{8} \frac{1}{(n+2)^{3/2}}$. ii. $\sum_{n=1}^{16} \frac{1}{(n+2)^{3/2}}$

(b) i. Determine $\int_n^{\infty} \frac{1}{(x+2)^{3/2}} dx$.

ii. Hence estimate the error in using each of the partial sums in (a) to approximate $\sum_{n=1}^{\infty} \frac{1}{(n+2)^{3/2}}$.

6. Estimate the error in using the partial sum S_n to approximate the sum of the series

(a) S_{100}; $\sum_{n=1}^{\infty} \frac{3}{n^4}$. (b) S_{50}; $\sum_{n=2}^{\infty} \frac{2}{n(\ln n)^3}$ (c) S_{30}; $\sum_{n=0}^{\infty} ne^{-n^2}$

7. Find how many terms would be required to ensure that the partial sum below is accurate to within the specified error for its corresponding infinite series.

(a) $\sum_{n=1}^{n} \frac{1}{n(\ln n)^2}$; error $= 5 \times 10^{-2}$.

(b) $\sum_{n=1}^{n} \frac{1}{n^2}$; error $= 5 \times 10^{-5}$.

(c) $\sum_{n=1}^{n} \frac{2n}{(1+2n^2)^3}$; error $= 1 \times 10^{-6}$.

8. (a) Evaluate i. $\sum_{n=1}^{10} \frac{1}{n^3}$ ii. $\int_n^{\infty} \frac{1}{x^3} dx$

(b) Hence, estimate the error in using $\sum_{n=1}^{10} \frac{1}{n^3}$ to approximate $\sum_{n=1}^{\infty} \frac{1}{n^3}$.

(c) How many terms would be needed to ensure that a partial sum is accurate to within 5×10^{-5}.

9. (a) Evaluate i. $\sum_{n=1}^{10} ne^{-n^2}$ ii. $\int_n^{\infty} xe^{-x^2} dx$

(b) Hence estimate the error in using $\sum_{n=1}^{10} ne^{-n^2}$ to approximate $\sum_{n=1}^{\infty} ne^{-n^2}$.

(c) How many terms would be needed to ensure that a partial sum is accurate to within 5×10^{-5}.

10. (a) Evaluate i. $\sum_{n=0}^{9} \frac{\arctan(n)}{1+n^2}$ ii. $\int_n^{\infty} \frac{\arctan(x)}{1+x^2} dx$

(b) Hence estimate the error in using $\sum_{n=0}^{9} \frac{\arctan(n)}{1+n^2}$ to approximate $\sum_{n=0}^{\infty} \frac{\arctan(n)}{1+n^2}$.

(c) How many terms would be needed to ensure that a partial sum is accurate to within 5×10^{-4}.

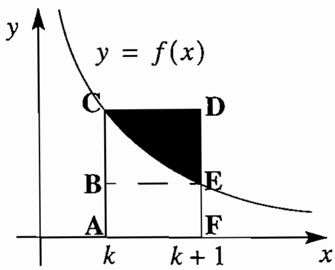
11. (a) i. Show that $\frac{1}{\sqrt{n(n+1)}} = \frac{1}{n} \cdot \frac{1}{\sqrt{1+\frac{1}{n}}}$

ii. Hence show that

$$\frac{1}{\sqrt{n^2}} + \frac{1}{\sqrt{n(n+1)}} + \frac{1}{\sqrt{n(n+2)}} + \ldots + \frac{1}{\sqrt{n(2n-1)}} = \sum_{i=0}^{n-1} \left(\frac{1}{n} \cdot \left(\frac{1}{\sqrt{1+\frac{i}{n}}} \right) \right)$$

(b) Use an appropriate integral to find an approximation to $\lim_{n \to \infty} \sum_{i=0}^{n-1} \left(\frac{1}{n} \cdot \left(\frac{1}{\sqrt{1+\frac{i}{n}}} \right) \right)$.

12. Consider the graph of the function $f(x) = \frac{1}{x}, x > 0$.



(a) i. Using the figure alongside show that

$\frac{1}{k+1} < \int_k^{(k+1)} \frac{1}{x} dx < \frac{1}{k}$ where $k \in \mathbb{Z}^+$.

ii. Hence, show that $\ln(n) \le u_n \le 1 + \ln(n)$

where $u_n = 1 + \frac{1}{2} + \frac{1}{3} + \ldots + \frac{1}{n} = \sum_{i=1}^{n} \frac{1}{i}$.

(b) Show that $\lim_{n \to \infty} \frac{u_n}{\ln(n)} = 1$.

(c) Let $A(k)$ denote the area of the shaded region shown in the figure above.

i. Show that $\frac{1}{2}\left(\frac{1}{k} - \frac{1}{k+1}\right) < A(k) < \frac{1}{k} - \frac{1}{k+1}$, for $1 \le k \le n$.

ii. Show that $u_n - \ln(n) = \left[\sum_{k=1}^{n-1} A(k)\right] + \frac{1}{n}$.

(d) i. Show that the sequence $v_n = u_n - \ln(n)$ is monotonic.

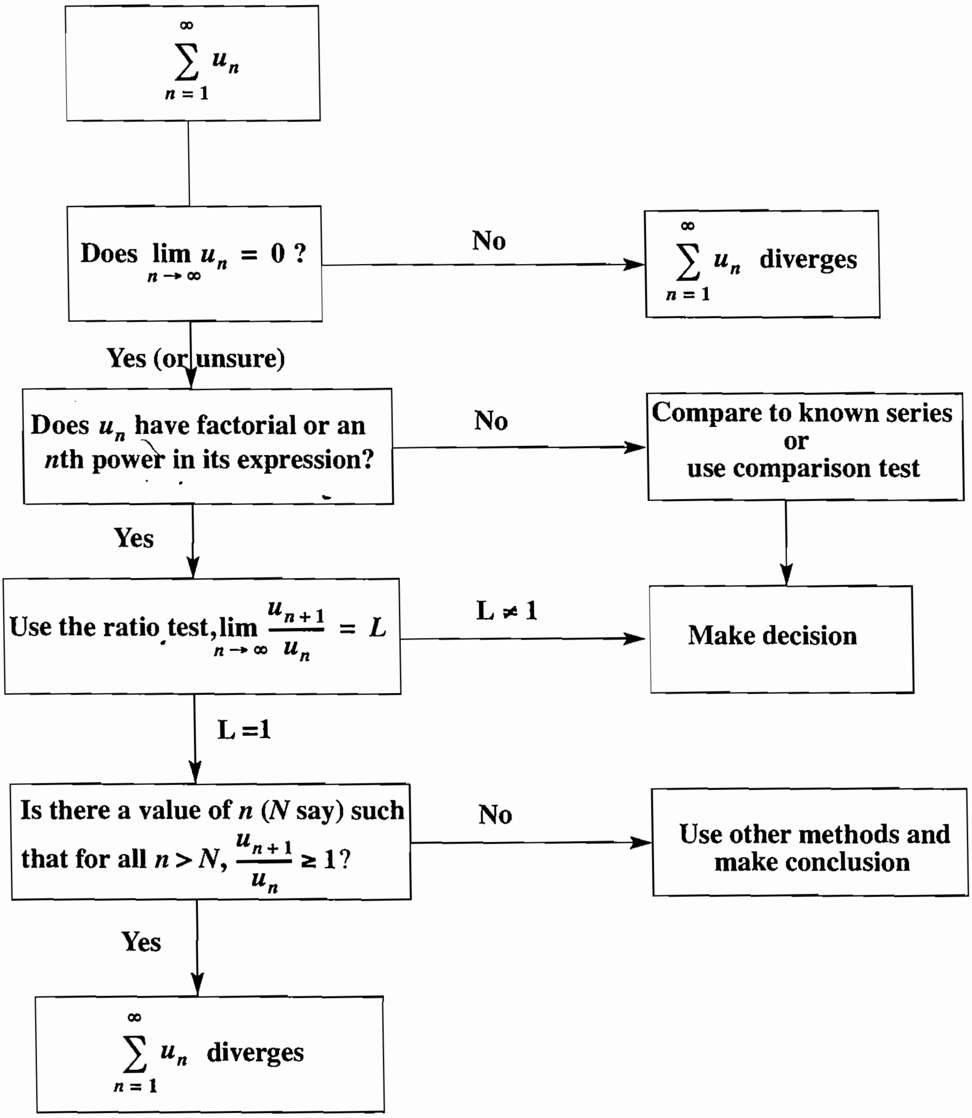
ii. Hence show that $\lim_{n \to \infty} v_n$ exists.

iii. Hence show that $\frac{1}{2} < \lim_{n \to \infty} v_n < 1$

2.3 WHICH TEST TO USE?

Even though we have covered a number of tests in this chapter, this list is not exhaustive. Given that, it is natural to ask which test should be used in any one particular situation. The answer, as unsatisfactory as it seems, is to use the test that, based on your experiences in dealing with infinite series, works best.

Having said that, we present a flow chart that only applies to series with non-negative terms. Now, this flowchart is a guide. While it does help, it is not always successful and you may need to resort to a mixture of other skills and methods you have at your disposal. Note that in this flow chart no mention is made of the integral test – that is one of the 'other methods' that can be considered when you get stuck.



Next we provide a summary of the laws and results that are relevant to our study of infinite series.

1. Convergence

Necessary condition: If $\sum_{n=1}^{\infty} u_n$ converges, then $\lim_{n \to \infty} u_n = 0$.

Not sufficient condition: If $\lim_{n \to \infty} u_n = 0$ it does not guarantee that $\sum_{n=1}^{\infty} u_n$ converges.

However, if $\lim_{n \to \infty} u_n \neq 0$ then $\sum_{n=1}^{\infty} u_n$ diverges.

Results:

1. If $\sum_{n=1}^{\infty} u_n$ converges to U and $\sum_{n=1}^{\infty} v_n$ converges to V, the series $\sum_{n=1}^{\infty} (u_n \pm v_n)$ converges to U ± V

2. If $\sum_{n=1}^{\infty} u_n$ converges to U and $\sum_{n=1}^{\infty} v_n$ diverges then series $\sum_{n=1}^{\infty} (u_n \pm v_n)$ diverges.

2. Tests (when $u_n \geq 0$)

Geometric series: $\sum_{n=1}^{\infty} r^n$ converges to $\frac{1}{1-r}$, if $|r| < 1$ and diverges if $|r| > 1$.

***p*–series:** $\sum_{n=1}^{\infty} \frac{1}{n^p}$ converges if $p > 1$ and diverges if $0 \leq p \leq 1$.

Comparison test:

	[redacted]	[redacted]
[redacted]	$\sum_{n=1}^{\infty} u_n$ converges	no valid conclusion
[redacted]	no valid conclusion	$\sum_{n=1}^{\infty} u_n$ diverges

Note that it is not necessary that $u_n \leq v_n$ or $u_n \geq v_n$ for all n, it only needs to be true for some $n \geq k$ where k is some integer. This is because the convergence or divergence of a series is not affected by the values of the first few terms.

Limit comparison test: If $u_n > 0$ and $v_n > 0$ and there is a number c such that

$\lim_{n \to \infty} \frac{u_n}{v_n} = c$, then either both series converge or both diverge.

Note: Use this test when a series $\sum_{n=1}^{\infty} v_n$ is known to converge or diverge and when it appears that $\lim_{n \to \infty} \frac{u_n}{v_n}$ is easily computed.

Ratio test: If $u_n > 0$ and $\lim_{n \to \infty} \frac{u_{n+1}}{u_n} = L$, then $\sum_{n=1}^{\infty} u_n$

1. converges if $L < 1$.
2. diverges if $L > 1$.

Note: (a) If $L = 1$ the series may either diverge or converge so another approach will be needed.
(b) This is often the easiest test to use.

Integral test: If $u_n = f(n)$ where $f(n) \geq 0$ and is decreasing, then $\sum_{n=1}^{\infty} u_n$

1. converges if $\int_1^{\infty} f(x)dx$ converges.

2. diverges if $\int_1^{\infty} f(x)dx$ diverges.

Note: Use this test whenever $f(x)$ can be easily integrated.

3.1 ALTERNATING SERIES

3.1.1 INTRODUCTION

In chapter 2 we have only considered series of positive terms. We shall now study series in which the signs of the terms **alternate**, viz.

$$u_1 - u_2 + u_3 - u_4 + \dots$$

with all the u_i's being positive.

Our aim is to find the conditions under which alternating series **converge**. For this purpose we shall use the criterion first proposed by Leibnitz, which states:

If an alternating series $u_1 - u_2 + u_3 - u_4 + \dots$ is such that the terms are decreasing, i.e. $u_1 > u_2 > u_3 > \dots$ and if $\lim_{n \to \infty} u_n = 0$, the series converges.

As a corollary to Leibnitz's theorem, we will also prove that the sum of the series is positive and always smaller than the first term.

PROOF:

Let us consider groups of pairs of terms and write the sum of the first $n = 2N$ terms of the series

$$S_{2N} = (u_1 - u_2) + (u_3 - u_4) + (u_5 - u_6) + \dots + (u_{2N-1} - u_{2N})$$

As $u_1 > u_2 > u_3 > \dots$, then, $(u_1 - u_2) > 0$, $(u_3 - u_4) > 0$, ... That is, each bracket gives a positive term. Hence, $S_{2N} > 0$ and increases as N increases.

We next show that $S_{2N} < u_1$. We start by rewriting S_{2N} in the form

$$S_{2N} = u_1 - (u_2 - u_3) - (u_4 - u_5) - \dots - (u_{2N-2} - u_{2N-1}) - u_{2N}$$

Again, each bracket gives a positive term. Therefore,

$$S_{2N} = u_1 - [(u_2 - u_3) + (u_4 - u_5) + \dots + (u_{2N-2} - u_{2N-1}) + u_{2N}]$$

$$= u_1 - [\text{positive term}]$$

The final result gives $S_{2N} < u_1$.

So far we have shown that S_{2N} increases with N and has an upper limit **S** such that

$$\lim_{n \to \infty} S_{2N} = \mathbf{S}, \text{ with } 0 < \mathbf{S} < u_1 .$$

Next we show that the partial sums of an odd order will also tend to **S**.

The sum of the first $2N + 1$ terms is given by

$$S_{2N+1} = u_1 - u_2 + u_3 - u_4 + \ldots - u_{2N} + u_{2N+1}$$
$$= S_{2N} + u_{2N+1}$$

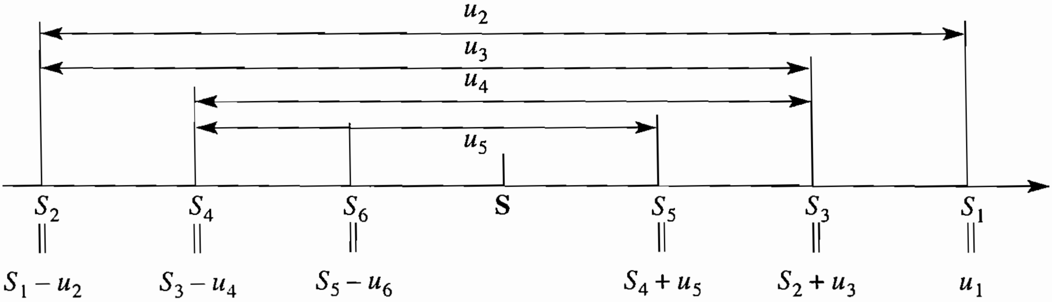
As $\lim_{n \to \infty} u_n = 0$, we have $\lim_{N \to \infty} u_{2N+1} = 0$.

Thus we are left with $\lim_{N \to \infty} S_{2N+1} = \lim_{N \to \infty} S_{2N} + \lim_{N \to \infty} u_{2N+1} = \lim_{N \to \infty} S_{2N} = \mathbf{S}$.

Therefore, $\lim_{N \to \infty} S_N = \mathbf{S}$ with $0 < \mathbf{S} < u_1$ for both N even and N odd, hence the series **converges.**

END OF PROOF

Leibnitz himself represented graphically, on an axis, the values of the partial sums and was able to show that **even** sums will tend to **S** from the **left** (as their values increase), whereas **odd** sums tend to **S** from the **right** (as their values decrease).



In the same way, we can show that the absolute value of the **truncation error,** $|\mathbf{S} - S_n|$, is less than the next term in the series.

If we approximate the sum **S** of the series by a partial sum S_n, we are neglecting all terms from u_{n+1} onwards. But these terms also form an alternating series, and the absolute value of their sum is **smaller** than the first term neglected, i.e., u_{n+1}. Therefore replacing **S** by S_n produces an error which must be smaller than the first term neglected, i.e., $\mathbf{S} - S_n < u_{n+1}$.

In summary:

If $\sum_{n=1}^{\infty} (-1)^{n+1} u_n$ is a convergent alternating series, then the absolute value of the remainder after n terms is less than the absolute value of the $(n+1)$th term

i.e., $|R_n| = |\mathbf{S} - S_n| < |(-1)^{n+2} u_{n+1}| = u_{n+1}$

Let us now see some examples in order for these ideas to become clearer. In some of these examples we shall show how many terms there should be in the series in order for the truncation error to be smaller than a certain quantity, typically 10^{-6}.

EXAMPLE 3.1

Show that the alternating harmonic series $1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \ldots$ converges.

Solution

We first note that $1 > \frac{1}{2} > \frac{1}{3} > \ldots$, i.e., $u_1 > u_2 > u_3 > \ldots$.

Next, we have that $\lim_{n \to \infty} u_n = \lim_{n \to \infty} \left(\frac{1}{n}\right) = 0$. Therefore, the series converges.

We also note that if $S_n = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \ldots + (-1)^{n+1} \cdot \frac{1}{n}$, the error made in taking S_n to approximate the sum of the series is smaller than $\frac{1}{n+1}$.

EXAMPLE 3.2 What is the least number of terms in the series

$$\frac{1}{2} - \frac{1}{2^2} + \frac{1}{2^3} - \frac{1}{2^4} + \ldots + \frac{1}{2^n} - \frac{1}{2^{n+1}} + \ldots$$

that must be taken in order for the error to be smaller than 10^{-6}.

Solution

We need to determine the value n such that $u_{n+1} < 10^{-6}$.

We have that $u_{n+1} = \frac{1}{2^{n+1}} < 10^{-6} \Leftrightarrow 10^6 < 2^{n+1}$.

We can now take logarithms of both sides of this inequality:

$$6 < \log_{10} 2^{n+1}$$

$$\Leftrightarrow (n+1)\log_{10} 2 > 6$$

$$\Leftrightarrow n + 1 > \frac{6}{\log_{10} 2}$$

$$\Leftrightarrow n + 1 > 19.93$$

$$\Leftrightarrow n > 18.93$$

That is, $n = 19$.

EXAMPLE 3.3 How many terms of the series below must be taken in order for the error to be smaller than 10^{-6}?

$$\frac{1}{2} - \frac{1}{2 \times 3} + \frac{1}{2 \times 3 \times 4} - \frac{1}{2 \times 3 \times 4 \times 5} + \ldots + \frac{1}{n!} - \frac{1}{(n+1)!} + \ldots$$

Solution

Again we need to determine the value n such that $u_{n+1} < 10^{-6}$.

In this case we must have $\frac{1}{(n+1)!} < \frac{1}{10^6}$

$$\Leftrightarrow (n+1)! > 10^6$$

Using trial and error with a calculator we have that $9! < 10^6 < 10!$.
That is, $n + 1 \geq 10$, and therefore $n \geq 9$.

EXERCISES 3.1

1. (a) Show that the alternating harmonic series $1 - \frac{1}{2^2} + \frac{1}{3^2} - \frac{1}{4^2} + \ldots$ converges.

(b) How many terms of the series must be taken for the error to be less than 10^{-6} ?

2. (a) Show that the alternating harmonic series $1 - \frac{1}{2^3} + \frac{1}{3^3} - \frac{1}{4^3} + \ldots$ converges.

(b) How many terms of the series must be taken for the error to be less than 10^{-5} ?

3. Find the least number of terms in the series $\frac{1}{3} - \frac{1}{3^2} + \frac{1}{3^3} - \frac{1}{3^4} + \ldots + (-1)^{n+1} \cdot \frac{1}{3^n} + \ldots$ that must be taken in order for the error to be smaller than 10^{-7} .

4. (a) Show that the series $\sum_{n=2}^{\infty} (-1)^n \frac{1}{n!}$ converges.

(b) How many terms of the series must be taken for the error to be less than 10^{-4} ?

5. Investigate the convergence of the series $\sum_{n=1}^{\infty} (-1)^{n+1} \frac{n+1}{n}$.

6. Show that the alternating series $\sum_{n=2}^{\infty} (-1)^n \frac{\ln(n)}{n}$ converges.

7. Find the difference between the sum of the first 5 terms of the series $\sum_{n=2}^{\infty} (-1)^{n+1} \frac{1}{10^n}$, and the actual sum.

3.2 OTHER TYPES OF CONVERGENCE

3.2.1 CONDITIONAL AND ABSOLUTE CONVERGENCE

We begin by providing some definitions and a theorem relating to conditional and absolute convergence.

Definition:

The series $\sum_{n=1}^{\infty} u_n$ is absolutely convergent if and only if the series $\sum_{n=1}^{\infty} |u_n|$ is convergent

This is followed by the theorem:

If the series $\sum_{n=1}^{\infty} |u_n|$ is convergent, then the series $\sum_{n=1}^{\infty} u_n$ converges.

That is, if a series converges absolutely, then it converges. This idea of **absolute convergence** for alternating series can be restated as:

An alternating series $u_1 - u_2 + u_3 - u_4 + \ldots + u_n - \ldots$ is said to be **absolutely convergent** if the series formed with the absolute values of these terms is convergent, i.e. if $|u_1| + |u_2| + |u_3| + |u_4| + \ldots + |u_n| + \ldots$ converges.

This allows us to strengthen our arguments when considering series such as $\sum_{n=1}^{\infty} (-1)^{n+1}\frac{1}{n^2}$.

That is, the series $\sum_{n=1}^{\infty} (-1)^{n+1}\frac{1}{n^2}$ is absolutely convergent since $\sum_{n=1}^{\infty} \left|(-1)^{n+1}\frac{1}{n^2}\right| = \sum_{n=1}^{\infty} \frac{1}{n^2}$ is convergent.

An alternating series is called **conditionally convergent** if and only if it is convergent but not absolutely convergent.

It is important to realise that this definition applies only to convergent series. Which in essence is Leibnitz's criterion from section 3.1.

It is equally important to realise that in determining the convergence or divergence of an alternating series the comparison test cannot be used. However, the comparison test can be used to determine absolute convergence of an alternating series because $\sum_{n=1}^{\infty} |u_n|$ is a series of non-negative terms.

So, what is the relationship between *convergence* and *absolute convergence*? Basically, **every absolutely convergent series is also convergent** – however, **the reverse is not necessarily true**; there are many series that are convergent, but not absolutely convergent. These are called **conditionally convergent series.**

This idea will also be usefully applied to a new type of series that we shall study in the next section, the **power series**, where we will also look at what we mean by conditionally convergent via a number of examples that involve power series.

EXERCISES 3.2

1. Determine if the series $\sum_{n=1}^{\infty} (-1)^{n+1}\frac{2}{3^n}$ is absolutely convergent.

2. Show that the following series converge absolutely.

(a) $\sum_{n=1}^{\infty}(-1)^{n+1}\frac{1}{n^3}$ (b) $\sum_{n=1}^{\infty}(-1)^{n+1}\frac{n}{e^n}$ (c) $\sum_{n=1}^{\infty}(-1)^{n+1}\frac{1}{1+n^2}$

3. Show that the following series converge absolutely.

(a) $\sum_{n=1}^{\infty}\frac{1}{n^2}\cos\frac{n\pi}{3}$ (b) $\sum_{n=1}^{\infty}\frac{1}{n^4}\sin(n)$ (c) $\sum_{n=2}^{\infty}(-1)^{n+1}\frac{1}{n(\ln n)^2}$

4. Determine if the following series converge absolutely, converge conditionally or diverge, giving reasons for your answer.

(a) $\sum_{n=1}^{\infty}(-1)^{n+1}\frac{2^n}{n!}$ (b) $\sum_{n=1}^{\infty}(-1)^{n+1}\frac{n!}{2^{n+1}}$ (c) $\sum_{n=1}^{\infty}(-1)^{n+1}\frac{n^n}{n!}$

5. Prove which of the following series are
(a) absolutely convergent;
(b) conditionally convergent;
(c) divergent.

i. $\sum_{n=2}^{\infty}(-1)^n\frac{1}{n\ln n}$ ii. $\sum_{n=1}^{\infty}(-1)^n\frac{n!}{3^n}$ iii. $\sum_{n=1}^{\infty}\frac{(-1)^n n^2}{1+n^2}$

6. (a) Approximate the sum of the series $S = \sum_{n=1}^{\infty}(-1)^{n+1}\frac{1}{n!}$ by calculating S_{10}.

(b) Estimate the level of the error involved in this approximation.

7. Consider the series $S = \sum_{n=1}^{\infty}(-1)^{n+1}\frac{1}{n^4}$.

(a) i. Approximate the sum of the series by using the 20th partial sum.
ii. Estimate the error involved in this approximation.

(b) How many terms would be needed to guarantee that the partial sum is within 10^{-8} of S?

8. Estimate the sum of each convergent series to within 0.01.

(a) $S = \sum_{n=1}^{\infty}(-1)^{n+1}\frac{4}{n^3}$ (b) $S = \sum_{n=0}^{\infty}(-1)^n\frac{3}{n!}$

9. How many terms are needed to estimate $S = \sum_{n=0}^{\infty}(-1)^n\frac{2^n}{n!}$ to within 0.0001?

10. Determine if (a) $\sum_{n=1}^{\infty}(-1)^{n+1}\frac{1}{2^n}$ (b) $\sum_{n=1}^{\infty}(-1)^{n+1}\frac{1}{n}$

is absolutely convergent.

11. Investigate the convergence of (a) $\sum_{n=1}^{\infty}(-1)^n\frac{n}{2^n}$ (b) $\sum_{n=0}^{\infty}(-1)^n\frac{n!}{e^n}$

3.3 POWER SERIES

3.3.1 POWER SERIES AND RADIUS OF CONVERGENCE

A power series is a series of the form $a_0 + a_1x + a_2x^2 + a_3x^3 + \dots + a_nx^n + \dots$ (3.1)

where $a_0, a_1, a_2, \dots$ are the coefficients of the increasing powers of x. In general, we shall see that the set of points of convergence for a power series is an interval which, in some cases, reduces to a single point.

We must first give the criteria of convergence for power series, and from there we shall define the concept of the radius of convergence.

If a power series converges for a nonzero value of $x = x_0$, it will have absolute convergence for any value of x such that $|x| < |x_0|$

We shall also prove that if the power series **diverges** for a value of $x = x_0$, then it will diverge for all values of x such that $|x| > |x_0|$.

PROOF:

If the power series converges for $x = x_0$ the series $a_0 + a_1x + a_2x^2 + a_3x^3 + \dots + a_nx^n + \dots$ converges.

Its general term $a_nx^n \to 0$ when $n \to \infty$. Therefore there must be a positive number M such that the absolute values of all terms in the series are smaller than M. We can rewrite the series as

$$a_0 + a_1x_0\left(\frac{x}{x_0}\right) + a_2(x_0)^2\left(\frac{x}{x_0}\right)^2 + a_3(x_0)^3\left(\frac{x}{x_0}\right)^3 + \dots + a_n(x_0)^n\left(\frac{x}{x_0}\right)^n + \dots \quad (3.2)$$

If we consider the series made up of the absolute values of these terms, namely

$$|a_0| + |a_1x_0|\left|\frac{x}{x_0}\right| + |a_2(x_0)^2|\left|\frac{x}{x_0}\right|^2 + |a_3(x_0)^3|\left|\frac{x}{x_0}\right|^3 + \dots + |a_n(x_0)^n|\left|\frac{x}{x_0}\right|^n + \dots \quad (3.3)$$

we can see that the terms of this series are **smaller** than the corresponding terms of the series

$$M + M\left|\frac{x}{x_0}\right| + M\left|\frac{x}{x_0}\right|^2 + M\left|\frac{x}{x_0}\right|^3 + \dots + M\left|\frac{x}{x_0}\right|^n + \dots \quad (3.4)$$

If $|x| < |x_0|$, the above expression (3.4) is a geometric series of common ratio $\frac{x}{x_0} < 1$, thus it converges. As the terms of the series (3.3) are smaller than those of series (3.4), it then follows that series (3.3) converges also.

Therefore we can say that our original power series $a_0 + a_1x + a_2x^2 + a_3x^3 + \ldots + a_nx^n + \ldots$ has **absolute convergence** .

We can now show that, if our original power series diverges at x_0 , it will also diverge for any x such that $|x| > |x_0|$. Assume that the power series converges at a value of x satisfying this inequality. The absolute convergence criterion proved previously would tell us that it would also converge at x_0 , since $|x_0| < |x|$. But this would contradict the initial assumption that the series diverges at x_0 . Thus the series also diverges at point x.

We can also say that, if x_0 is a point for which the series converges, all points that belong to the interval $]-|x_0|, |x_0|[$ are points of absolute convergence. Conversely, if x_0 were a point for which the series diverges, then for any $x > |x_0|$ or $x < -|x_0|$ the series will diverge.

Note: the case where $|x| = |x_0|$ needs to be dealt with separately.

END OF PROOF

3.3.2 CONVERGENCE INTERVAL OF A POWER SERIES

A natural consequence of the previous section would be that, as all points belonging to the interval $]-|x_0|, |x_0|[$ are points of absolute convergence, the set of points of convergence of a power series is an interval about the origin, i.e., centered at $x = 0$. If we call R, the **radius of convergence**, then the **interval of convergence** is the set of points x belonging to the interval $]-R, R[$ [1] such that the series converges.

The case where $|x| = R$ has to be treated separately.

Sometimes the convergence interval reduces to a point, i.e., $R = 0$; at other times, it can be the whole of the half-axis Ox if $R = \infty$.

There is a **method for determining the radius of convergence of a power series** that makes use of D'Alembert's ratio test.

If we consider the power series (3.1) , we can form the series of the absolute values of these terms, as we did before

$$|a_0| + |a_1||x| + |a_2||x|^2 + |a_3||x|^3 + \ldots + |a_n||x|^n + \ldots \qquad (3.5)$$

This latter series is a series of positive terms, so we can apply to it the ratio test in order to determine whether it converges or not. We therefore calculate the limit

$$\lim_{n \to \infty} \frac{u_{n+1}}{u_n} = \lim_{n \to \infty} \left| \frac{a_{n+1}x^{n+1}}{a_nx^n} \right| = \lim_{n \to \infty} \left| \frac{a_{n+1}}{a_n} \right| |x| = k|x|$$

---

1. $]-R, R[$ is also written as $x \in]-R, R[$ or $|x| < R$

If we apply D'Alembert's criterion, the series converges if $k|x| < 1$, i.e., $|x| < \frac{1}{k}$, and it diverges if $k|x| > 1$, i.e., $|x| > \frac{1}{k}$. Therefore the convergence interval of (3.1) is $\left]-\frac{1}{k}, \frac{1}{k}\right[$ and the radius of convergence is $R = \frac{1}{k} = \lim_{n \to \infty}\left|\frac{a_n}{a_{n+1}}\right|$.

Let us now see how it works with some chosen examples.

EXAMPLE 3.4

Find the interval convergence of the series $1 + x + x^2 + x^3 + \dots + x^n + \dots$

Solution

If we apply D'Alembert's rule we have

$$\lim_{n \to \infty}\left|\frac{u_{n+1}}{u_n}\right| = \lim_{n \to \infty}\left|\frac{x^{n+1}}{x^n}\right|$$

$$= \lim_{n \to \infty}\left|\frac{x}{1}\right|$$

$$= |x|$$

The series converges for $|x| < 1$, so the interval of convergence is $-1 < x < 1$. The radius of convergence is $R = 1$.

EXAMPLE 3.5

Find the values of x for which the series $1 + \frac{x}{2} + \frac{x^2}{4} + \dots + \frac{x^n}{2^n} + \dots$ converges.

Solution

D'Alembert's criterion shows that $\lim_{n \to \infty}\left|\frac{u_{n+1}}{u_n}\right| = \lim_{n \to \infty}\left|\frac{\frac{x^{n+1}}{2^{n+1}}}{\frac{x^n}{2^n}}\right|$

$$= \lim_{n \to \infty}\left|\frac{x}{2}\right|$$

$$= \left|\frac{x}{2}\right|$$

Then, with $\left|\frac{x}{2}\right| < 1$ we have $|x| < 2$. That is, we have the condition $-2 < x < 2$.

Note that the radius of convergence is 2 and the interval of convergence is $]-2, 2[$.

EXAMPLE 3.6

Investigate further the interval of convergence of the series in Example 3.5.

Solution

The series in Example 3.5 is given by $1 + \frac{x}{2} + \frac{x^2}{2^2} + \frac{x^3}{2^3} + \ldots + \frac{x^n}{2^n} + \ldots$ where we obtained the interval of absolute convergence as $-2 < x < 2$.

This means that the series diverges for $|x| > 2$.

The case $|x| = 2$ needs to be treated separately, but first we write our series as

$$1 + \frac{x}{2} + \frac{x^2}{2^2} + \frac{x^3}{2^3} + \ldots + \frac{x^n}{2^n} + \ldots = \sum_{n=1}^{\infty} \frac{x^n}{2^n}$$

Case 1 ($x = 2$): $\quad \sum_{n=1}^{\infty} \frac{x^n}{2^n} = \sum_{n=1}^{\infty} \frac{2^n}{2^n} = \sum_{n=1}^{\infty} 1^n$ which diverges.

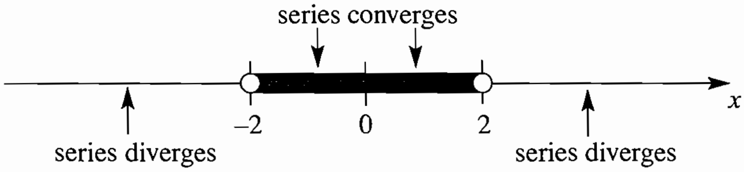
Case 2 ($x = -2$): $\quad \sum_{n=1}^{\infty} \frac{x^n}{2^n} = \sum_{n=1}^{\infty} \frac{(-2)^n}{2^n} = \sum_{n=1}^{\infty} (-1)^n$ which also diverges.

Summary: For the series $\sum_{n=1}^{\infty} \frac{x^n}{2^n}$, the following results hold;

1. series converges for $|x| < 2$.
2. series diverges for (a) $|x| = 2$
   (b) $|x| > 2$

Note that we could restate/combine (a) and (b) to read as $|x| \geq 2$.

Using the real number line we can visualise this as:



EXAMPLE 3.7

Find the interval of convergence for the series

$$1 - \frac{x^2}{2^2} + \frac{x^3}{3^2} - \frac{x^4}{4^2} + \ldots + (-1)^{n+1}\frac{x^n}{n^2} + \ldots$$

Solution

Applying D'Alembert's rule we have

$$\lim_{n \to \infty}\left|\frac{u_{n+1}}{u_n}\right| = \lim_{n \to \infty}\left|\frac{\dfrac{(-1)^{n+2}x^{n+1}}{(n+1)^2}}{\dfrac{(-1)^{n+1}x^n}{n^2}}\right| = \lim_{n \to \infty}\left|\frac{xn^2}{(n+1)^2}\right|$$

$$= \lim_{n \to \infty}\frac{n^2}{(n+1)^2}|x|$$

As $\lim\limits_{n \to \infty} \frac{n^2}{(n+1)^2} = 1$ our condition for absolute convergence is $|x| < 1$.

This also means that the series diverges for $|x| > 1$. Also, when $x = +1$ and $x = -1$ the series still converges (see Example 3.1), so that the convergence interval is $-1 \le x \le 1$.

EXAMPLE 3.8

Study the interval of convergence of the series

$$\sin x + 2\sin\frac{x}{3} + 4\sin\frac{x}{9} + \ldots + 2^{n-1}\sin\frac{x}{3^{n-1}} + \ldots$$

Solution

Again, we apply D'Alembert's criterion in the form of

$$\lim_{n \to \infty}\left|\frac{u_{n+1}}{u_n}\right| = \lim_{n \to \infty}\left|\frac{2^n \sin\left(\frac{x}{3^n}\right)}{2^{n-1}\sin\left(\frac{x}{3^{n-1}}\right)}\right| = \lim_{n \to \infty} 2\left|\frac{\sin\left(\frac{x}{3^n}\right)}{\sin\left(\frac{x}{3^{n-1}}\right)}\right| = 0$$

NB: The limit equals 0 because as $n \to \infty$ the numerator reaches 0 before the denominator does. Thus, as $R = 0$ (< 1), the series converges for all values of x.

EXAMPLE 3.9

Determine the interval of absolute convergence for the series

$$x + \frac{2^2}{4!}x^2 + \frac{(1\cdot 2\cdot 3)^2}{6!}x^3 + \ldots + \frac{(n!)^2}{(2n)!}x^n + \ldots$$

Solution

We again use D'Alembert's criterion for convergence, i.e., $\lim\limits_{n \to \infty}\left|\frac{u_{n+1}}{u_n}\right| < 1$.

$$\lim_{n \to \infty}\left|\frac{u_{n+1}}{u_n}\right| = \lim_{n \to \infty}\left|\frac{\frac{[(n+1)!]^2 x^{n+1}}{(2n+2)!}}{\frac{[n!]^2 x^n}{(2n)!}}\right| = \lim_{n \to \infty}\left|\frac{[(n+1)!]^2}{(2n+2)!} \times \frac{(2n)!}{[n!]^2}\right||x|$$

First we need to simplify the expression $\frac{[(n+1)!]^2}{(2n+2)!} \times \frac{(2n)!}{[n!]^2}$.

$$\frac{[(n+1)!]^2}{(2n+2)!} \times \frac{(2n)!}{[n!]^2} = \frac{[(n+1)n!]^2}{(2n+2)(2n+1)(2n)!} \times \frac{(2n)!}{[n!]^2}$$

$$= \frac{[(n+1)]^2}{(2n+2)(2n+1)}$$

$$= \frac{(n+1)^2}{2(n+1)(2n+1)}$$

$$= \frac{n+1}{2(2n+1)}$$

Therefore, as $\lim\limits_{n \to \infty} \frac{n+1}{2(2n+1)} = \frac{1}{4} \Rightarrow \lim\limits_{n \to \infty}\left|\frac{[(n+1)!]^2}{(2n+2)!} \times \frac{(2n)!}{[n!]^2}\right| = \frac{1}{4}$

Note: $\lim_{n \to \infty} \frac{n+1}{2(2n+1)} = \lim_{n \to \infty} \frac{1 + \frac{1}{n}}{2\left(2 + \frac{1}{n}\right)} = \frac{1}{4}$ (after dividing through by n).

Which means that the condition for x is $\frac{1}{4}|x| < 1 \Leftrightarrow |x| < 4$.

And so we have that as $n \to \infty$, $|x| < 4$. That is, the series converges for $-4 < x < 4$.

EXAMPLE 3.10

Study the convergence interval of the series

$$x + \frac{2^k}{2!}x^2 + \frac{3^k}{3!}x^3 + \ldots + \frac{n^k}{n!}x^n + \ldots$$

Solution

Applying D'Alembert's criterion, we have

$$L = \lim_{n \to \infty}\left|\frac{u_{n+1}}{u_n}\right| = \lim_{n \to \infty}\left|\frac{\frac{(n+1)^k}{(n+1)!}x^{n+1}}{\frac{n^k}{n!}x^n}\right| = \lim_{n \to \infty}\left|\frac{(n+1)^k}{(n+1)n^k}\right||x|$$

$$= \lim_{n \to \infty}\left|\frac{(n+1)^{k-1}}{n^k}\right||x|$$

$$= \lim_{n \to \infty}\left|\frac{(n+1)^{k-1}}{n^{k-1} \times n}\right||x|$$

$$= \lim_{n \to \infty}\left|\frac{1}{n}\left(\frac{n+1}{n}\right)^{k-1}\right||x|$$

$$= \lim_{n \to \infty}\left|\frac{1}{n}\left(1 + \frac{1}{n}\right)^{k-1}\right||x|$$

$$= 0 \times |x|$$

$$= 0$$

Then, as $L = 0$, meaning that it is always less than one (i.e., $L < 1$ for all real values of x), the series converges for all real values of x. That is, $R = \infty$.

EXAMPLE 3.11

Find the interval of absolute convergence for the series

$$x + \frac{2!}{2^2}x^2 + \frac{3!}{3^3}x^3 + \ldots + \frac{n!}{n^n}x^n + \ldots$$

Solution

Again the D'Alembert criterion is the most appropriate in this case, viz.

$$L = \lim_{n \to \infty}\left|\frac{u_{n+1}}{u_n}\right| = \lim_{n \to \infty}\left|\frac{\frac{(n+1)!}{(n+1)^{n+1}}x^{n+1}}{\frac{n!}{n^n}x^n}\right| = \lim_{n \to \infty}\left|\frac{(n+1)!n^n}{(n+1)^{n+1}n!}\right||x|$$

Next, we need to simplify $\dfrac{(n+1)!n^n}{(n+1)^{n+1}n!}$:

$$\frac{(n+1)!n^n}{(n+1)^{n+1}n!} = \frac{(n+1)n!}{n!} \times \frac{n^n}{(n+1)(n+1)^n}$$
$$= \frac{n^n}{(n+1)^n}$$
$$= \left(\frac{n}{n+1}\right)^n$$
$$= \left(1 - \frac{1}{n+1}\right)^n$$

Then, $\lim\limits_{n \to \infty}\left(1 - \dfrac{1}{n+1}\right)^n = \lim\limits_{n \to \infty}\dfrac{\left(1 - \dfrac{1}{n+1}\right)^{n+1}}{\left(1 - \dfrac{1}{n+1}\right)} = \dfrac{e^{-1}}{1} = e^{-1}$

So, we have that $\lim\limits_{n \to \infty}\left|\left(\dfrac{n}{n+1}\right)^n\right||x| < 1 \Leftrightarrow e^{-1}|x| < 1$. Thus we obtain $|x| < e$.

Therefore the interval of absolute convergence is $-e < x < e$. ($R = e$)

EXAMPLE 3.12 Determine the interval of absolute convergence of the series

$$\frac{x}{1+\sqrt{1}} + \frac{x^2}{2+\sqrt{2}} + \ldots + \frac{x^n}{n+\sqrt{n}} + \ldots$$

Solution

According to D'Alembert's criterion we have

$$L = \lim_{n \to \infty}\left|\frac{u_{n+1}}{u_n}\right| = \lim_{n \to \infty}\left|\frac{\dfrac{x^{n+1}}{(n+1)+\sqrt{n+1}}}{\dfrac{x^n}{n+\sqrt{n}}}\right| = \lim_{n \to \infty}\left|\frac{n+\sqrt{n}}{(n+1)+\sqrt{n+1}}\right||x|$$

So, with $L < 1$ this becomes $|x| < \lim\limits_{n \to \infty}\left|\dfrac{(n+1)+\sqrt{n+1}}{n+\sqrt{n}}\right|$

Next, we need to simplify $\dfrac{(n+1)+\sqrt{n+1}}{n+\sqrt{n}} = \dfrac{(n+1)+\sqrt{n+1}}{n+\sqrt{n}} \times \dfrac{n-\sqrt{n}}{n-\sqrt{n}}$

$$= \frac{[(n+1)+\sqrt{n+1}](n-\sqrt{n})}{n^2-n}$$
$$< \frac{n^2+n+n\sqrt{n+1}}{n^2-n}$$

As $\lim\limits_{n \to \infty}\dfrac{n^2+n+n\sqrt{n+1}}{n^2-n} = 1$ (after dividing by n^2) we have $|x| < \lim\limits_{n \to \infty}\left|\dfrac{(n+1)+\sqrt{n+1}}{n+\sqrt{n}}\right| = 1$.

That is, $|x| < 1$. Therefore the interval of absolute convergence is $-1 < x < 1$. That is, $R = 1$.

Note that the lower limit **can be** equal to -1, as this would make the denominator nonzero. On the other hand, the upper limit **cannot** be equal to 1 as this would make the denominator equal to 0, and thus diverge. This is where the notion of **conditional convergence** comes into play.

So far we have 'interchanged' the notion of the radius of convergence and the interval of convergence of a power series. We now provide a clearer distinction between the two.

3.3.3 RADIUS OF CONVERGENCE OR INTERVAL OF CONVERGENCE OF A POWER SERIES?

We quickly review the discussion of the previous section. First, what we mean by convergence for a power series:

If $\sum_{n=0}^{\infty} a_n x^n$

(a) converges at x_0, $x_0 \neq 0$, then it converges absolutely at all x such that $-|x_0| < x < |x_0|$.

(b) diverges at x_0, $x_0 \neq 0$, then it diverges at all x such that $x < -|x_0|$ or $x > |x_0|$.

Next, the definition of the **radius of convergence**, for which we identify 3 categories:

Category 1: $\sum_{n=0}^{\infty} a_n x^n$ converges only at $x = 0$. We say that the radius of convergence is 0.

Category 2: $\sum_{n=0}^{\infty} a_n x^n$ converges for all $x \in \mathbb{R}$. We say that the radius of convergence is ∞.

Category 3: $\sum_{n=0}^{\infty} a_n x^n$ i. converges if $|x| < R$

ii. diverges if $|x| > R$

iii. may converge or diverge at $x = R$ and $x = -R$

R is called the radius of convergence.

Category 3., iii., is where **conditional convergence** is addressed.
Having defined the radius of convergence, we need an approach to help us determine it. We do this as follows:

For the power series $\sum a_n x^n$ we calculate (if it exists) $\lim_{n \to \infty} \left| \frac{a_{n+1}}{a_n} \right| = L$. Then, if

i. [illegible] R [illegible] and so the series is a type that falls into Category 1

ii. [illegible] R [illegible] and so the series is a type that falls into Category 2

iii. [illegible] R [illegible] and so the series is a type that falls into Category 3

Having derived how we are to determine the radius of convergence we now define the interval of convergence:

> The interval of convergence of the power series $\sum_{n=0}^{\infty} a_n x^n$ is the set of values of x for which the power series converges.

So, while the radius of convergence is a value R (if it exists), the interval of convergence is given by the set of values of x (i.e., an interval) based on the value of R. The results can be summarised in the table below.

	$\sum_{n=0}^{\infty} a_n x^n$
If $R = 0$	converges only at $x = 0$.
If $R = \infty$	interval of convergence is $]-\infty, \infty[$
If $0 < R < \infty$	Step 1: Check the series at $x = \pm R$. Then depending on the convergence or divergence of the series at $x = \pm R$ the possible intervals of convergence are: Step 2: $]-R, R[$ or $]-R, R]$ or $[-R, R[$ or $[-R, R]$

EXAMPLE 3.13

For the series $\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n}\left(\frac{2x}{3}\right)^n$ determine its

(a) radius of convergence (b) interval of convergence

Solution

(a) For the given series we have $u_n = \frac{(-1)^{n+1}}{n} \cdot \left(\frac{2x}{3}\right)^n$ and $u_{n+1} = \frac{(-1)^{n+2}}{n+1} \cdot \left(\frac{2x}{3}\right)^{n+1}$.

Using the ratio
$$\lim_{n\to\infty}\left|\frac{u_{n+1}}{u_n}\right| = \lim_{n\to\infty}\left|\frac{\frac{(-1)^{n+2}}{n+1}\cdot\left(\frac{2x}{3}\right)^{n+1}}{\frac{(-1)^{n+1}}{n}\cdot\left(\frac{2x}{3}\right)^{n}}\right| = \lim_{n\to\infty}\left|-\frac{n}{n+1}\cdot\left(\frac{2x}{3}\right)\right|$$
$$= \frac{2}{3}|x| \lim_{n\to\infty}\left|\frac{n}{n+1}\right|$$
$$= \frac{2}{3}|x| \lim_{n\to\infty}\left|1 - \frac{1}{n+1}\right|$$
$$= \frac{2}{3}|x|$$

Therefore, the radius of convergence, R, is given by $R = \frac{1}{L}$ where $L = \frac{2}{3}$. That is, $R = \frac{3}{2}$.

(b) The power series is absolutely convergent when $\frac{2}{3}|x| < 1 \Leftrightarrow |x| < \frac{3}{2}$

Next we check to see if the series converges at $x = \pm R$, i.e., when $\frac{2}{3}|x| = 1 \Leftrightarrow |x| = \frac{3}{2}$.

Case 1: When $x = \frac{3}{2}$ the series is given by $\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} \cdot \left(\frac{2}{3} \times \frac{3}{2}\right)^n = \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n}$.

That is, $\frac{1}{1} - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \ldots + (-1)^{n+1} \cdot \frac{1}{n} + \ldots$ which we know is convergent from Example 3.1.

Case 2: When $x = -\frac{3}{2}$ the series is given by $\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} \cdot \left(\frac{2}{3} \times -\frac{3}{2}\right)^n = \sum_{n=1}^{\infty} \frac{(-1)^{2n+1}}{n}$.

That is, $-\frac{1}{1} - \frac{1}{2} - \frac{1}{3} - \frac{1}{4} - \ldots - \frac{1}{n} - \ldots = -\left(\frac{1}{1} + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \ldots + \frac{1}{n} + \ldots\right)$ which we know to be divergent.

Therefore we conclude that the series is convergent for $-\frac{3}{2} < x \le \frac{3}{2}$, so that the interval of convergence is given by $\left]-\frac{3}{2}, \frac{3}{2}\right]$.

Note that this means that the power series is absolutely convergent when $-\frac{3}{2} < x < \frac{3}{2}$ and is **conditionally convergent** when $x = \frac{3}{2}$.

Example 3.13 highlights the work required to produce a detailed investigation in determining the radius of convergence and interval of convergence. The point should also be made that the process in determining the value of L for the power series $\sum_{n=0}^{\infty} a_n x^n$ is given by evaluating the limit $\lim_{n \to \infty} \left|\frac{a_{n+1}}{a_n}\right| = L$. However, in Example 3.13 we made use of $\lim_{n \to \infty} \left|\frac{u_{n+1}}{u_n}\right|$!

Had we wanted to use $\lim_{n \to \infty} \left|\frac{a_{n+1}}{a_n}\right|$ we would have had to first rewrite the power series as

$$\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} \cdot \left(\frac{2x}{3}\right)^n = \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} \cdot \left(\frac{2}{3}\right)^n \cdot x^n \text{ giving } a_n = \frac{(-1)^{n+1}}{n} \cdot \left(\frac{2}{3}\right)^n.$$

The difference being that when using $\lim_{n \to \infty} \left|\frac{u_{n+1}}{u_n}\right|$ the '$|x|$–term' is part of the result whereas using $\lim_{n \to \infty} \left|\frac{a_{n+1}}{a_n}\right|$ produces a numerical value and hence the reason why $\lim_{n \to \infty} \left|\frac{a_{n+1}}{a_n}\right| = L$.

In short,
$$\lim_{n \to \infty} \left|\frac{u_{n+1}}{u_n}\right| = \lim_{n \to \infty} \left|\frac{a_{n+1}}{a_n}\right| |x| = L|x|.$$

3.3.4 POWER SERIES IN $(x-k)$ WHERE $k \neq 0$

Consider the series

$$\sum_{n=0}^{\infty} a_n(x-k)^n = a_0 + a_1(x-k) + a_2(x-k)^2 + \dots + a_n(x-k)^n + \dots \qquad (3.6)$$

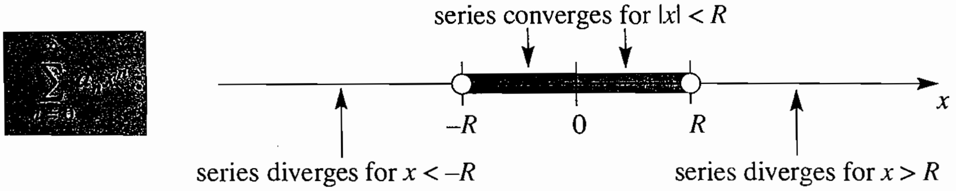
This is a series in increasing powers of $(x-k)$ with coefficients $a_0, a_1, a_2, \dots, a_n, \dots$.
As $k \neq 0$, we can determine the interval of convergence of this series by making a change of variable. If we let $z = (x-k)$, our original series (3.5) now becomes

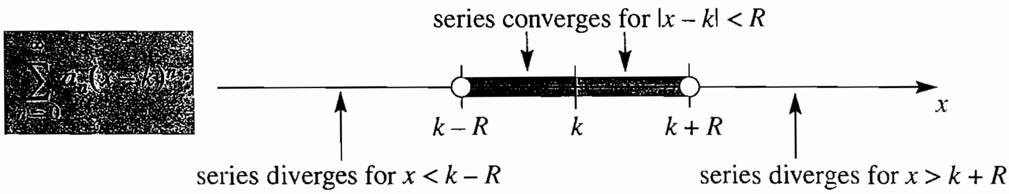
$$\sum_{n=0}^{\infty} a_n z^n = a_0 + a_1 z + a_2 z^2 + \dots + a_n z^n + \dots \qquad (3.7)$$

We already know that the convergence interval of series (3.7) is $-R < z < R$.

Therefore series (3.6) will have a convergence interval $-R < (x-k) < R$, or $k-R < x < k+R$, centered at $x = k$.

We can visualise the similarity between the interval of convergence for the power series $\sum_{n=0}^{\infty} a_n x^n$ and $\sum_{n=0}^{\infty} a_n(x-k)^n$ as follows:





The series $\sum_{n=0}^{\infty} a_n(x-k)^n$ is a translation of the series $\sum_{n=0}^{\infty} a_n x^n$ so, the interval of convergence for $\sum_{n=0}^{\infty} a_n(x-k)^n$ is simply a translation of the interval of convergence for $\sum_{n=0}^{\infty} a_n x^n$.

EXAMPLE 3.14

Find the convergence interval of the series

$$(x-3) + (x-3)^2 + (x-3)^3 + \dots + (x-3)^n + \dots \qquad (3.8)$$

Solution

Letting $z = x - 3$, we obtain the series

$$\sum_{n=1}^{\infty} z^n = z + z^2 + z^3 + \ldots + z^n + \ldots \qquad (3.9)$$

a power series in the new variable z.

We recognise this as an infinite geometric series and therefore it has the convergence interval of this series, namely, $-1 < z < 1$.

Thus the original series (3.7) will converge for values of x such that $-1 < x - 3 < 1 \Leftrightarrow 2 < x < 4$.

EXERCISES 3.3

1. (a) Show that the series $\sum_{n=0}^{\infty} 2^n x^n$ converges for $|x| < \frac{1}{2}$.

(b) Show that the series $\sum_{n=1}^{\infty} n x^{n-1}$ converges for $|x| < 1$.

2. For what value(s) of x does the series $\sum_{n=0}^{\infty} n! x^n$ converge?

3. Find the value(s) of x for which the following power series converge.

(a) $\sum_{n=0}^{\infty} \frac{x^n}{3^n}$ (b) $\sum_{n=0}^{\infty} \frac{x^n}{n+1}$ (c) $\sum_{n=0}^{\infty} \frac{x^n}{n!}$

4. Find i. the radius of convergence
ii. interval of convergence
for each of the following power series:

(a) $\sum_{n=0}^{\infty} \frac{x^n}{5^n}$ (b) $\sum_{n=0}^{\infty} (4x)^n$ (c) $\sum_{n=0}^{\infty} \frac{(x-1)^n}{2^n}$

(d) $\sum_{n=0}^{\infty} x^{2n}$ (e) $\sum_{n=0}^{\infty} (-1)^n \frac{nx^n}{\sqrt{n+1}}$ (f) $\sum_{n=0}^{\infty} \frac{(2x-1)^n}{3^n}$

5. Investigate the convergence of the power series $\sum_{n=0}^{\infty} \frac{(x+5)^n}{(n+1) \cdot 2^n}$.

6. Find the radius and interval of convergence of the power series

(a) $\sum_{n=1}^{\infty} \frac{(3x)^n}{n}$ (b) $\sum_{n=1}^{\infty} \frac{(3x)^n}{n!}$ (c) $\sum_{n=1}^{\infty} \frac{(x+1)^n}{n}$

7. Determine (a) the radius of convergence (b) the interval of convergence

of the series $\sum_{n=0}^{\infty} \frac{(x-3)^n}{(n+1)\cdot 2^n}$.

8. Determine the values of x for which the following power series converge:

(a) $\sum_{n=0}^{\infty} \frac{n}{3^{n+1}} \cdot x^n$ (b) $\sum_{n=0}^{\infty} \frac{5^n}{n!} \cdot (x-1)^n$ (c) $\sum_{n=1}^{\infty} \frac{1}{n \cdot 4^n} \cdot x^n$

9. Find the interval of convergence of the power series $\sum_{n=0}^{\infty} \frac{(ax-b)^n}{c^n}$, $a > 0, c > 0$.

10. Find the radius of convergence of the following power series:

(a) $\sum_{n=1}^{\infty} \frac{x^n}{n \cdot 2^n}$ (b) $\sum_{n=1}^{\infty} \frac{x^n}{n^2+1}$

11. The interval of convergence of the power series $\sum_{n=1}^{\infty} \frac{(ax)^n}{n^2}$ is $\left[-\frac{1}{3}, \frac{1}{3}\right]$, find a.

12. The power series $\sum_{n=2}^{\infty} \frac{(x-a)^{2n}}{n^4}$ has radius of convergence 1 and interval of convergence $[2, 4]$. Find the value of a.

13. The power series $\sum_{n=0}^{\infty} a_n(x-b)^n$ converges only if $x \in [-9, 19[$.

Determine (a) the radius of convergence for this series;
(b) the value of b.

14. The interval of convergence for the series $\sum_{n=1}^{\infty} \frac{(x-a)^n}{n \cdot b^n}$ is $[-3, 5[$. Find a and b.

15. The power series $\sum_{n=0}^{\infty} a_n x^n$ converges only if $-2 \le x \le 2$. Determine the interval of convergence of the series

(a) $\sum_{n=0}^{\infty} a_n(x-2)^n$. (b) $\sum_{n=0}^{\infty} a_n(x-3)^n$

16. Find the radius and interval of convergence for the power series $\sum_{n=0}^{\infty} \frac{2^n \cdot (x-1)^n}{\ln(n+2)}$.

3.4 CALCULUS WITH POWER SERIES

This post-scriptum to chapter 3 is a good example of how to connect various parts of the syllabus:

1. the core of the course (integration – chapters 22 and 23 of the Higher Level (Core) textbook by Ibid Press) and
2. the option (power series and Maclaurin's series – also a power series that will be discussed at great length in chapter 4).

While it might appear at times that the syllabus of a course is presented in a "pigeonholed" fashion, i.e. as totally distinct and unconnected parts, we make use of this section to highlight the connections between different parts of the syllabus and to illustrate how we can achieve synthesis as well as cross-fertilization between those parts and, indeed, other understanding already acquired knowledge. This in turn will enable us to go further in our mathematical trek.

3.4.1 POWER SERIES AS FUNCTIONS

Any power series can be thought of as a function of x, i.e., we can write

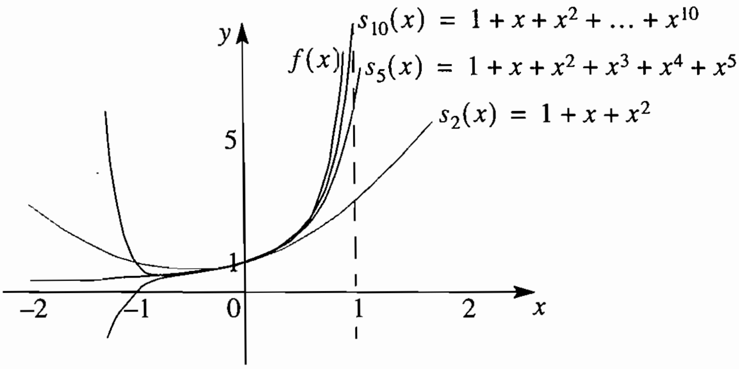
$$f(x) = \sum_{n=0}^{\infty} a_n x^n = a_0 + a_1 x + a_2 x^2 + a_3 x^3 + \dots$$

where the domain of f corresponds to the interval of convergence of the series.

We start with an example we have dealt with on many ocassions, the infinite geometric series:

$$\frac{1}{1-x} = \sum_{n=0}^{\infty} x^n, |x| < 1 = 1 + x + x^2 + x^3 + \dots, |x| < 1 .$$

Letting $f(x) = \frac{1}{1-x}$ we see that $f(x) = \lim_{n \to \infty}(1 + x + x^2 + \dots + x^n)$ is the limit of the nth partial sum, $s_n(x) = 1 + x + x^2 + \dots + x^n$. So, as n increases, $s_n(x)$ becomes a better approximation to $f(x)$ for $|x| < 1$. The figure below shows the cases $n = 2$, $n = 5$ and $n = 10$.



The graph clearly shows that for $|x| \geq 1$ the approximation no longer holds, but within the interval $|x| < 1$ the approximation improves as n increases.

So, are there other functions that can be represented by a power series? The answer is yes, and while we will be providing a specific approach as to how to obtain a power series for a given function in the next chapter, we want to continue with our informal approach and see how other functions can be expressed as a power series by making use of simple algebraic manipulation.

EXAMPLE 3.15

Express the function $f(x) = \frac{1}{3+x}$ as a power series, stating the values of x for which this approximation is valid.

Solution

We can rewrite $f(x) = \frac{1}{3+x}$ in the form $f(x) = \frac{1}{3\left(1+\frac{x}{3}\right)} = \frac{1}{3} \cdot \frac{1}{\left(1-\left(-\frac{x}{3}\right)\right)}$

Based on the result that $\frac{1}{1-x} = \sum_{n=0}^{\infty} x^n, |x| < 1$, we have: $f(x) = \frac{1}{3}\sum_{n=0}^{\infty}\left(-\frac{x}{3}\right)^n, \left|-\frac{x}{3}\right| < 1$.

That is, $f(x) = \sum_{n=0}^{\infty} \frac{1}{3} \cdot \frac{(-x)^n}{3^n}, \left|\frac{x}{3}\right| < 1$

$$= \sum_{n=0}^{\infty} \frac{(-1)^n}{3^{n+1}} x^n, |x| < 3.$$

EXAMPLE 3.16

Express the function $f(x) = \frac{x^2}{3+x}$ as a power series, stating the values of x for which this approximation is valid.

Solution

We can rewrite $f(x) = \frac{x^2}{3+x}$ in the form $f(x) = \frac{x^2}{3\left(1+\frac{x}{3}\right)} = x^2 \cdot \frac{1}{3} \cdot \frac{1}{\left(1-\left(-\frac{x}{3}\right)\right)}$

Using the results of Example 3.15 we can then write

$$f(x) = x^2 \sum_{n=0}^{\infty} \frac{(-1)^n}{3^{n+1}} x^n, |x| < 3$$

$$= \sum_{n=0}^{\infty} \frac{(-1)^n}{3^{n+1}} x^{n+2}, |x| < 3$$

$$= \frac{1}{3}x^2 - \frac{1}{9}x^3 + \frac{1}{27}x^5 + \ldots, |x| < 3.$$

3.4.2 DIFFERENTIATING & INTEGRATING POWER SERIES

We recall that if the power series $\sum_{n=0}^{\infty} a_n(x-k)^n$ has a radius of convergence $R > 0$, then the series converges absolutely for all values of $x \in \;]k-R, k+R[$ and might converge at one or both of the endpoints. So, if we define a function $f(x) = \sum_{n=0}^{\infty} a_n(x-k)^n$ it too will be well defined on the interval $x \in \;]k-R, k+R[$. Such a function is then continuous and differentiable on the interval $x \in \;]k-R, k+R[$. With this is mind the following results hold true and are known as term-by-term differentiation and integration (the proof of which lies beyond the scope of this course).

Term-by-term differentiation:

$$f(x) = \sum_{n=0}^{\infty} a_n(x-k)^n = a_0 + a_1(x-k) + a_2(x-k)^2 + a_3(x-k)^3 + \ldots$$

$$f'(x) = \sum_{n=0}^{\infty} na_n(x-k)^{n-1} = a_1 + 2a_2(x-k) + 3a_3(x-k)^2 + \ldots$$

Term-by-term antidifferentiation:

$$f(x) = \sum_{n=0}^{\infty} a_n(x-k)^n = a_0 + a_1(x-k) + a_2(x-k)^2 + a_3(x-k)^3 + \ldots$$

$$\int f(x)dx = \sum_{n=0}^{\infty} \frac{a_n}{n+1}(x-k)^{n+1} + c$$

$$= a_0(x-k) + \frac{a_1}{2}(x-k)^2 + \frac{a_2}{3}(x-k)^3 + \frac{a_3}{4}(x-k)^4 + \ldots$$

Note that while the radius of convergence remains the same when a power series is differentiated or integrated, this does not imply that the interval of convergence remains the same. By this we mean that while the original series might converge at an endpoint, the differentiated series might then diverge at that endpoint.

EXAMPLE 3.17

Find the power series for the function $f(x) = \frac{1}{(1+x)^2}$

Solution

We start by noting that if $g(x) = \frac{1}{1+x}$ then $g'(x) = -\frac{1}{(1+x)^2}$.

That is, $f(x) = \frac{1}{(1+x)^2} = -\left[-\frac{1}{(1+x)^2}\right] = -g'(x)$.

As we know the power series for $g(x) = \frac{1}{1+x}$, all we need to do is differentiate this series term-by-term and its result will give the power series for $f(x)$.

So, let $g(x) = \frac{1}{1+x} = \sum_{n=0}^{\infty}(-x)^n = 1 - x + x^2 - x^3 + x^4 - \ldots$

On differentiating term-by-term we have

$$g'(x) = 0 - 1 + 2x - 3x^2 + 4x^3 - \ldots$$

$$= \sum_{n=0}^{\infty}(-1)^n n x^{n-1}$$

Therefore, as $f(x) = -g'(x)$ we have $f(x) = -\sum_{n=0}^{\infty}(-1)^n n x^{n-1} = \sum_{n=0}^{\infty}(-1)^{n+1} n x^{n-1}$.

So, $\frac{1}{(1+x)^2} = \sum_{n=0}^{\infty}(-1)^{n+1} n x^{n-1}$, $|x| < 1$.

EXAMPLE 3.18

Find the power series for the function $f(x) = \ln(1-x)$ and hence find the power series for $\ln 2$.

Solution

We start by noting that $\int \frac{1}{1-x}dx = -\ln(1-x) + c$ so that $\ln(1-x) = -\int \frac{1}{1-x}dx + C$.

However, $\frac{1}{1-x} = \sum_{n=0}^{\infty} x^n = 1 + x + x^2 + x^3 + \ldots$.

Therefore, $\int \frac{1}{1-x}dx = \int\left(\sum_{n=0}^{\infty} x^n\right)dx = \int(1 + x + x^2 + x^3 + \ldots)dx$

$$= x + \frac{1}{2}x^2 + \frac{1}{3}x^3 + \frac{1}{4}x^4 + \ldots$$

$$= \sum_{n=1}^{\infty}\frac{x^n}{n}$$

Note that $\sum_{n=1}^{\infty}\frac{x^n}{n}$ can also be expressed as $\sum_{n=0}^{\infty}\frac{x^{n+1}}{n+1}$. Therefore $\int \frac{1}{1-x}dx = \sum_{n=0}^{\infty}\frac{x^{n+1}}{n+1}$, $|x| < 1$.

Then, as $\ln(1-x) = -\int \frac{1}{1-x}dx + c$ we have that $\ln(1-x) = -\sum_{n=0}^{\infty}\frac{x^{n+1}}{n+1} + c$.

However, when $x = 0$, $\ln(1-0) = -\sum_{n=0}^{\infty}\frac{0^{n+1}}{n+1} + c \Leftrightarrow c = 0$.

Therefore, $\ln(1-x) = -\sum_{n=0}^{\infty}\frac{x^{n+1}}{n+1}$, $|x| < 1$.

Next, with $x = \frac{1}{2}$ we have $\ln(1-x) = \ln\left(1 - \frac{1}{2}\right) = \ln\left(\frac{1}{2}\right) = -\ln 2$.

Therefore, using the result obtained, we have that

$$-\ln 2 = -\sum_{n=0}^{\infty} \frac{1}{n+1}\left(\frac{1}{2}\right)^{n+1} \text{ or } \ln 2 = \sum_{n=0}^{\infty} \frac{1}{n+1}\left(\frac{1}{2}\right)^{n+1} = \frac{1}{2}+\frac{1}{8}+\frac{1}{24}+\frac{1}{64}+\ldots$$

Note, we could also have expressed $\ln 2$ as $\sum_{n=1}^{\infty} \frac{1}{n \cdot 2^n}$.

In Example 3.18 we could have made immediate use of the summation expression as follows:

$$\int \frac{1}{1-x}dx = \int\left(\sum_{n=0}^{\infty} x^n\right)dx = \sum_{n=0}^{\infty}\left(\int x^n dx\right) = \sum_{n=0}^{\infty} \frac{x^{n+1}}{n+1} + c$$

$$= \sum_{n=1}^{\infty} \frac{x^n}{n} + c$$

That is, $-\ln(1-x) = \sum_{n=1}^{\infty} \frac{x^n}{n} + c$ so that when $x = 0$, $-\ln(1) = \sum_{n=1}^{\infty} \frac{0^n}{n} + c \Leftrightarrow c = 0$.

So, $-\ln(1-x) = \sum_{n=1}^{\infty} \frac{x^n}{n} \therefore \ln(1-x) = -\sum_{n=1}^{\infty} \frac{x^n}{n}$, $|x| < 1$.

EXAMPLE 3.19

Find the power series for the function $f(x) = \frac{1}{1+x^2}$ and hence find the power series for $\arctan x$.

Solution

We note the relationship between $\frac{1}{1+x^2}$ and $\arctan x$, namely, $\int \frac{1}{1+x^2}dx = \arctan x$.

By replacing 'x' with 'x^2' in the power series $\frac{1}{1+x} = \sum_{n=0}^{\infty} (-1)^n x^n$ the function

$f(x) = \frac{1}{1+x^2}$ can be written as the power series $\sum_{n=0}^{\infty} (-1)^n (x^2)^n = \sum_{n=0}^{\infty} (-1)^n x^{2n}$,

where $|x^2| < 1 \Leftrightarrow -1 < x < 1$

Therefore, integrating both sides of the equation $\frac{1}{1+x^2} = \sum_{n=0}^{\infty} (-1)^n x^{2n}$ we have:

$$\int \frac{1}{1+x^2} = \int \sum_{n=0}^{\infty} (-1)^n x^{2n} = \sum_{n=0}^{\infty} \int (-1)^n x^{2n} dx$$

$$= \sum_{n=0}^{\infty} \frac{(-1)^n}{2n+1} x^{2n+1} + c$$

Therefore, $\arctan x = \sum_{n=0}^{\infty} \frac{(-1)^n}{2n+1} x^{2n+1} + c$, $-1 < x < 1$.

However, when $x = 0$, $\arctan 0 = \sum_{n=0}^{\infty} \frac{(-1)^n}{2n+1} 0^{2n+1} + c \Leftrightarrow 0 = 0 + c \Leftrightarrow c = 0$.

Therefore, $\arctan x = \sum_{n=0}^{\infty} \frac{(-1)^n}{2n+1} x^{2n+1}, -1 < x < 1$

$$= x - \frac{1}{3}x^3 + \frac{1}{5}x^5 - \frac{1}{7}x^7 + \dots, -1 < x < 1 .$$

Note that from Example 3.19 we have the result that $\frac{\pi}{4} = 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \dots$.

This is achieved by substituting $x = 1$ into the series $\arctan x = x - \frac{1}{3}x^3 + \frac{1}{5}x^5 - \frac{1}{7}x^7 + \dots$.

It should be noted that this series also holds true for $x = 1$.

3.4.3 USE OF POWER SERIES TO SOLVE SOME TRICKY INTEGRALS

Let us turn our attention to some examples which illustrate this fact.

EXAMPLE 3.20

Calculate the integral $\int_0^a e^{-x^2} dx$.

Solution

This is an apparently tricky definite integral to calculate as all the methods you know involve calculating the indefinite integral first, and there is no obvious method to calculate this indefinite integral.

Now the lower limit of this integral is 0, so you could say that you are doing the calculation in the vicinity of 0, up to a value $x = a$. If a is near enough to 0, there is a straightforward method to perform the integration.

If you can bear postponing the proof of the method to chapter 4, just for the purposes of the present calculation. I will give you the result of the application of the method (which is fully proved in chapter 4). It relates to Maclaurin's expansion.
The Maclaurin's expansion around $x = 0$ of e^{-x^2} is

$$e^{-x^2} = 1 - x^2 + \frac{x^4}{2!} - \frac{x^6}{3!} + \dots + (-1)^n \cdot \frac{x^{2n}}{n!} + \dots \qquad (3.10)$$

Let us see where this expression arises from: the Maclaurin's expansion is nothing but a power series in x with, as coefficients, the successive derivatives of $f(x)$ calculated at $x = 0$ divided by successive factorials. The presence in those coefficients of the factorials gives it a rather familiar flavour as we have seen these appear already in the binomial expansions.

In fact, the full expression of the Maclaurin's expansion for the function $f(x)$ is

$$f(x) = f(0) + f'(0)\cdot\frac{x}{1!} + f''(0)\cdot\frac{x^2}{2!} + f'''(0)\cdot\frac{x^3}{3!} + f^{iv}(0)\cdot\frac{x^4}{4!} + \ldots + f^{(n)}(0)\cdot\frac{x^n}{n!} + \ldots$$

Where the $f^{(n)}$ terms represent the first, second, third, ..., derivatives of the function considered, evaluated at $x = 0$. In our case we have,

$$f(x) = e^{-x^2} \text{ and thus } f(0) = 1$$
$$f'(x) = -2xe^{-x^2} \text{ and } \therefore f'(0) = 0$$
$$f''(x) = -2e^{-x^2} + 4x^2e^{-x^2} \text{ and } \therefore f''(0) = -2$$
$$f'''(x) = (12x - 8x^3)e^{-x^2} \text{ and } \therefore f'''(0) = 0$$
$$f^{iv}(x) = (12 - 48x^2 + 16x^4)e^{-x^2} \text{ and } \therefore f^{iv}(0) = 12$$

Therefore the full expression becomes

$$e^{-x^2} = 1 - 2\cdot\frac{x^2}{2!} + 12\cdot\frac{x^4}{4!} - \frac{x^6}{3!} + \ldots + (-1)^n\cdot\frac{x^{2n}}{n!} + \ldots$$

That is, $$e^{-x^2} = 1 - x^2 + \frac{x^4}{2!} - \frac{x^6}{3!} + \ldots + (-1)^n\cdot\frac{x^{2n}}{n!} + \ldots \qquad (3.11)$$

which is identical to the expression (3.10) found previously.

You can recognize here an alternating series, which also happens to be a power series. The only peculiarity is that only the **even** powers of x are present, which only reflects a property of the successive derivatives of the function $f(x) = e^{-x^2}$ when evaluated at $x = 0$.

We can now substitute the power series 3.11 into the integrand of the integral that we had to calculate, giving

$$\int_0^a e^{-x^2}dx = \int_0^a\left(1 - x^2 + \frac{x^4}{2!} - \frac{x^6}{3!} + \ldots\right)dx.$$

Now this is a much more familiar type of integration which you can do by direct integration as it is of the form $\int_0^a x^n dx$.

Therefore the result is $$\int_0^a e^{-x^2}dx = \left[x - \frac{x^3}{3} + \frac{1}{5}\cdot\frac{x^5}{2!} - \frac{1}{7}\cdot\frac{x^7}{3!} + \quad\right]_0^a$$

$$= a - \frac{a^3}{3} + \frac{a^5}{10} - \frac{a^7}{42} + \ldots \qquad (3.12)$$

EXAMPLE 3.21

Calculate the integral $\int_0^a \frac{\sin x}{x} dx$

Solution

Here again we shall use the Maclaurin's expansion of $\sin x$, which is given by

$$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \ldots$$

You can already see that this is an alternating power series where, this time, **only** the odd powers of x appear. The proof of this expansion will be given in chapter 4.

Therefore the integrand can be written as

$$\frac{\sin x}{x} = 1 - \frac{x^2}{3!} + \frac{x^4}{5!} - \frac{x^6}{7!} + \ldots \qquad (3.13)$$

By the way, this alternating power series converges for all x. It is not difficult to prove it, try it out and you will see!

So we can now perform the calculation of our definite integral as

$$\int_0^a \frac{\sin x}{x} dx = \int_0^a \left(1 - \frac{x^2}{3!} + \frac{x^4}{5!} - \frac{x^6}{7!} + \ldots\right) dx$$

$$= \left[x - \frac{1}{3} \cdot \frac{x^3}{3!} + \frac{1}{5} \cdot \frac{x^5}{5!} - \frac{1}{7} \cdot \frac{x^7}{7!} + \ldots\right]_0^a$$

$$= a - \frac{a^3}{18} + \frac{a^5}{600} - \frac{a^7}{35490} + \ldots \qquad (3.14)$$

"Et le tour est joué" (and the trick has been achieved!)

As we have mentioned, these two integrals have applications in other related areas.

The first one comes from the world of probabilities, in fact of continuous probability functions. $\int_0^a e^{-x^2} dx$, resembles the normal distribution function $\int_0^a \frac{1}{\sqrt{2\pi}\sigma} e^{-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2} dx$, where μ and σ are the mean and the standard deviation of the normal distribution function respectively.

By making the change of variable $z = \frac{x-\mu}{\sigma\sqrt{2}}$ so that $dz = \frac{dx}{\sigma\sqrt{2}}$, the well-known bell-shaped distribution can be converted to the form $\frac{1}{2\sqrt{2\pi}\sigma}\int_0^b e^{-x^2} dx$, which is, apart from the constant outside the integral sign, identical to our Example 3.20.

As for the integral that appears in the Example 3.21, $\int_0^a \frac{\sin x}{x}dx$, it frequently appears in calculations of diffraction amplitude in wave optics. The diffraction intensity is a function of the modulus of the amplitude squared, $\frac{\sin^2 x}{x^2}$, and thus the total light intensity can be seen as the infinite sum of the contributions of each diffraction order to the total intensity. This infinite sum can be regarded as

$$\lim_{a \to \infty} \int_0^a \frac{\sin^2 x}{x^2}dx \qquad (3.15)$$

which can be solved exactly as our example, by simply squaring (3.13).
The last step is just a simple integration.

Many more examples exist in nature, and physics is one of the most prolific fields in which we can find applications for our mathematical knowledge. Theoretical physics uses mathematics as its major tool to solve its problems, giving mathematics its physical interpretation and contents. But if you cannot first solve the mathematical problem, then there is nothing to interpret, is there?

EXERCISES 3.4

1. Express the following functions as power series giving the interval of convergence.

(a) $f(x) = \frac{1}{2-x}$ (b) $f(x) = \frac{2}{1+4x}$ (c) $f(x) = \frac{3}{3+4x}$

2. Express the following functions as power series giving the interval of convergence.

(a) $f(x) = \frac{1}{1-x^2}$ (b) $f(x) = \frac{1}{4+x^2}$ (c) $f(x) = \ln(5-x)$

3. Express the following functions as power series giving the interval of convergence.

(a) $f(x) = \frac{x}{1-x^2}$ (b) $f(x) = \frac{x^2}{x+2}$ (c) $f(x) = \frac{x^2}{1+x^2}$

4. (a) i. Differentiate the function $g(x) = \frac{1}{2+x}$.

ii. Hence find a power series for the function $f(x) = \frac{1}{(2+x)^2}$, $-R < x < R$.

iii. Determine the value of R.

(b) i. Using part (a), find a power series for the function $h(x) = \frac{1}{(2+x)^3}$.

ii. Hence, find a power series for $q(x) = \frac{x^2}{(2+x)^3}$.

5. Evaluate the indefinite integral $\int \frac{1}{1+x^7}dx$ as a power series.

6. (a) i. Show that a power series of $\ln(3+x)$ can be expressed in the form

$$\ln a + \sum_{n=1}^{\infty} (-1)^{n-1} \cdot \frac{1}{nb^n} x^n .$$

ii. Determine the value of a and b.

iii. Determine the interval of convergence for this series.

(b) Let the partial sum of this power series be $s_k(x) = \ln a + \sum_{n=1}^{k} (-1)^{n-1} \cdot \frac{1}{nb^n} x^n$.

i. Sketch, accurately, the graph of $f(x) = \ln(x+3)$.

ii. On the same set of axes, sketch the graphs of $s_2(x)$, $s_5(x)$ and $s_9(x)$.

7. Use a power series to determine

(a) $\int \frac{\arctan x}{x} dx$ (b) $\int \frac{\arctan(x^2)}{x} dx$ (c) $\int x^2 \arctan(x^4) dx$

8. Find a power series for $f(x) = \ln(x+1)$ and use it to estimate $\ln(1.5)$ with an error of less than 0.0001.

9. Given that the power series for $\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \frac{x^9}{9!} - \ldots$

(a) determine the series for $f(x) = \sin(x^2)$.

(b) estimate $\int_0^1 \sin(x^2) dx$ to within an error of $\frac{1}{19 \times 9!}$.

10. Let the function f be defined by the power series $f(x) = \sum_{n=0}^{\infty} \frac{x^{n+1}}{(n+1)^3}$.

Find the intervals of convergences for (a) $f'(x)$ (b) $f''(x)$ (c) $f'''(x)$

11. Let $f(x) = e^x$,

(a) show that $e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!}$ for all real values of x.

(b) estimate the error involved calculating e^1 if the series is evaluated using the first 10 terms.

12. (a) Determine a power series for $\frac{1}{1-x^2}$.

(b) Find a power series representing the function $f(x) = \frac{x}{(1-x^2)^2}$.

13. Determine a power series for the function $f(x) = \ln\left(\frac{1+x}{1-x}\right)$, $|x| < 1$

14. Consider the power series $f(x) = \sum_{n=1}^{\infty} \frac{x^n}{n^2}$. Determine the intervals of convergence for

(a) $f'(x)$ (b) $f''(x)$ (c) $f'''(x)$

15. (a) Show that $\cos x = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{(2n)!}$ for all real values of x.

(b) Hence, determine a power series for $f(x) = \cos\sqrt{x}$, $x \geq 0$.

(c) Evaluate the definite integral $\int_0^{0.1} \cos\sqrt{x}dx$ correct to nine decimal places.

16. Given the power series $f(x) = e^{-x} = \sum_{n=0}^{\infty} (-1)^n \frac{x^n}{n!}$, $g(x) = \cos x = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{(2n)!}$ and

$$h(x) = \ln x = \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n}(x-1)^n$$

(a) Show that the interval of convergence for $f(x)$, $g(x)$ and $h(x)$ are $]-\infty, \infty[$, $]-\infty, \infty[$ and $]0, 2]$ respectively.

(b) Determine a power series, including the interval of convergence for $u(x) = e^{-x}\cos x$.

(c) Using the power series, evaluate $\int_1^{1.25} (\ln x)^2 dx$ correct to 5 decimal places.

4.1 SERIES EXPANSION

4.1.1 INTRODUCTION – POLYNOMIALS

We have already studied power series in general in Chapter 3. This chapter will be devoted to a most important application of power series, the Taylor series, and a special case of the latter, the Maclaurin series.

The problem that we have at hand is that of calculating as accurately as possible the value of a function in the vicinity of a given point $x = a$. For this we shall use the Taylor series, or in the vicinity of $x = 0$, we shall use the Maclaurin series.

The idea behind this powerful method is conceptually simple: we shall attempt to approximate a function by a polynomial. This was already introduced in section 3.4, but now we will formalise the process. A polynomial can be viewed as a special type of power series, whose real coefficients are to be determined: what is special about it is that, if the power series is **truncated**, i.e. if it terminates at a certain term, it becomes a polynomial.

Therefore, if we are attempting to approximate a function by a polynomial, which is itself a truncated power series, we can see that the smaller the difference between the remainder of the truncated series and the function, the better the approximation will be.

Let us now examine the formal aspect of our statements above.

A power series can be written as

[redacted] (4.1)

where $a_0, a_1, a_2, \ldots$, the coefficients of the powers of x, are constants (for our purposes, these constants will always belong to the set of real numbers). At the moment, as the constants have all non-zero values, we have an infinite series because there is an infinite number of terms in the series.

If, from a certain term onwards the constants are all zero, we have what is called a "truncated" power series as the series terminates at a certain term.

Let us assume that, for $i > n$, all $a_i = 0$, i.e., $a_{n+1} = a_{n+2} = \ldots = 0$.
Our power series becomes

$$a_0 + a_1x + a_2x^2 + a_3x^3 + \ldots + a_nx^n \qquad (4.2)$$

Now this is nothing more than a polynomial of degree n (where n is a positive integer) which we can call $P_n(x)$, representing the fact that the highest power of x is n. Thus

$$P_n(x) = a_0 + a_1x + a_2x^2 + a_3x^3 + \ldots + a_nx^n \qquad (4.3)$$

At this point we see the correspondence between polynomials of degree n and functions.
A polynomial of degree 0 (i.e., $n = 0$) is $P_0(x) = a_0$. This represents a function that is constant,

its graph being a line that is parallel to the x-axis.

A polynomial of degree 1 (i.e., $n = 1$) is $P_n(x) = a_0 + a_1 x$, which is a linear function whose graph is a straight line with slope a_1 and y-intercept a_0.

A polynomial of degree 2 (i.e., $n = 2$) is $P_2(x) = a_0 + a_1 x + a_2 x^2$, which is a quadratic function whose graph is a parabola.

A polynomial of degree 3 is therefore $P_3(x) = a_0 + a_1 x + a_2 x^2 + a_3 x^3$, which is a cubic function. The process can be continued to any finite degree.

From eq. (4.3) we observe that a polynomial of degree n contains $(n+1)$ coefficients.

4.1.2 COEFFICIENTS OF A POLYNOMIAL WRITTEN IN TERMS OF ITS DERIVATIVES

In order to determine these coefficients we need $(n+1)$ pieces of information – remember that we need $(n+1)$ equations to solve a system of $(n+1)$ variables. In order to anticipate somewhat what will come next so that the reader can get familiar with the result, and without giving too much away yet, we shall give an example of how to find the coefficients of a cubic polynomial if we were given, for example, the values of the polynomial and of its first three derivatives when $x = 0$ (that makes 4 pieces of information).

That is, we want to find $P_3(x) = a_0 + a_1 x + a_2 x^2 + a_3 x^3$ given that $P_3(0) = 3$, $P'_3(0) = 1$, $P''_3(0) = -2$ and $P'''_3(0) = 4$.

When $x = 0$, $P_3(0) = 3$. However, $P_3(0) = a_0 \therefore a_0 = 3$

Next, $P'_3(x) = 0 + a_1 + 2a_2 x + 3a_3 x^2$ so that $P'_3(0) = a_1$. Then, $P'_3(0) = 1 \Leftrightarrow a_1 = 1$.

Differentiating again, $P''_3(x) = 2a_2 + 6a_3 x$ so that $P''_3(0) = -2 \Leftrightarrow 2a_2 = -2 \Leftrightarrow a_2 = -1$

Lastly, $P'''_3(x) = 6a_3$ and as $P'''_3(0) = 4$ we have that $6a_3 = 4 \Leftrightarrow a_3 = \frac{2}{3}$.

As we now have all 4 coefficients we can write $P_3(x)$ as $P_3(x) = 3 + x - x^2 + \frac{2}{3}x^3$. (4.4)

The same could have been achieved if we were given the values of the polynomial and its derivatives when $x = 1$. As we want to make life simple, being lazy as we are, we will write the polynomial not in terms of powers of x but of powers of $(x - 1)$, so as to have the same kind of easy cancellations as we had previously. In other words, we now write

$$Q_3(x) = b_0 + b_1(x-1) + b_2(x-1)^2 + b_3(x-1)^3 \quad (4.5)$$

where the coefficients are now b_0, b_1, b_2, b_3 (and are not the same as those for $P_3(x)$!)

Next we determine these coefficients using the following conditions:

$Q_3(1) = \frac{11}{3}, Q_3'(1) = 1, Q_3''(1) = 2$ and $Q_3'''(1) = 4$.

Now, $Q_3(1) = b_0 + b_1(1-1) + b_2(1-1)^2 + b_3(1-1)^3 = b_0 \therefore b_0 = \frac{11}{3}$.

Next, $Q_3'(x) = b_1 + 2b_2(x-1) + 3b_3(x-1)^2$ as $Q_3'(1) = b_1$ then $b_1 = 1$.

Again, $Q_3''(x) = 2b_2 + 6b_3(x-1)$ so that $Q_3''(1) = 2b_2 \therefore 2b_2 = 2 \Leftrightarrow b_2 = 1$.

Lastly, $Q_3'''(x) = 6b_3$ so that $6b_3 = 4 \Leftrightarrow b_3 = \frac{2}{3}$.

We can now write our polynomial as

$$Q_3(x) = \frac{11}{3}(x-1) + (x-1)^2 + \frac{2}{3}(x-1)^3. \qquad (4.6)$$

We shall leave it to the keen reader to prove that, in fact, $Q_3(x) = P_3(x)$. They look different only because they have been written in terms of powers of $(x-1)$ and x respectively.

So basically we can write **any** polynomial (not only of degree 3 but of any degree n) as one whose coefficients, $a_0, a_1, a_2, \dots$ can be expressed in terms of the derivatives of the polynomial $P_n, P'_n, P''_n, \dots, P_n^{(n)}$. This applies to our knowing the $P_n^{(n)}$ derivatives at any $x = a$, including $x = 0$.

Going back to $P_3(x)$, we see that we can write it in terms of its derivatives, as opposed to the coefficients a_i, i.e., $P_3(0) = a_0, P'_3(0) = a_1, P''_3(0) = 2!a_2, P'''_3(0) = 6a_3 = 3!a_3$.

Therefore, we can write that $a_2 = \frac{P''_3(0)}{2!}, a_3 = \frac{P''_3(0)}{3!}$ so that we then have

$$P_3(x) = P_3(0) + P'_3(0)\cdot x + P''_3(0)\cdot\frac{x^2}{2!} + P'''_3(0)\cdot\frac{x^3}{3!}. \qquad (4.7)$$

We can extend (4.7) to any degree n as

$$P_n(x) = P_n(0) + P'_n(0)\cdot x + \frac{1}{2!}P''_n(0)\cdot x^2 + \frac{1}{3!}P'''_n(0)\cdot x^3 + \dots + \frac{1}{n!}P^{(n)}{}_n(0)\cdot x^n \qquad (4.8a)$$

As we have seen, $P_n(x)$ can be expressed in powers of x or powers of $(x-a)$, so, we can write a similar expression for $P_n(x)$ in terms of powers of $(x-a)$:

$$P_n(x) = P_n(a) + P'_n(a)(x-a) + \frac{P''_n(a)}{2!}(x-a)^2 + \frac{P'''_n(a)}{3!}(x-a)^3 \dots + \frac{P^{(n)}{}_n(a)}{n!}(x-a)^n \qquad (4.8b)$$

The expressions in the two boxes labelled equations (4.8a) and (4.8b), give the polynomial $P_n(x)$ as power series to the n-th term, in the neighbourhood of $x = 0$ and $x = a$ respectively. Note the coefficients of the power series are written in terms of the n derivatives of $P_n(x)$ evaluated respectively at $x = 0$ and $x = a$.

At this point, the reader might ask; "Why do we go on with all this at such length?" At first, one might notice the striking formal similarity between eqs. (4.8) and the binomial expansion. Next, there is (hopefuly) a sense in which the transition from here to the final expression of the Taylor series (and then the Maclaurin series) will be smoother and clearer.

4.1.3 ADVANCING TO APPROXIMATING A FUNCTION BY A POLYNOMIAL

1. Approximation in the vicinity of x = 0

We now try to find a polynomial $P_n(x)$ that best approximates to a function $f(x)$ that is continuous and differentiable, and that fits it best in the neighbourhood of the point $x = 0$.

For this, we shall use the results of the previous section, namely we need to find an approximating polynomial $P_n(x)$ such that its value and that of its derivatives at $x = 0$, i.e. $P_n(0)$, $P'_n(0)$, $P''_n(0)$, ..., $P_n^{(n)}(0)$ are, inasmuch as is possible, equal to the values of $f(0)$, $f'(0)$, $f''(0)$, . . . , $f^{(n)}(0)$ respectively. We expect that, when x is in the vicinity of and close to 0, this approximation will be a reasonable one.

EXAMPLE 4.1 Find a polynomial that fits the graph of $f(x) = \ln(x+1)$.

Solution

We shall calculate the polynomial that fits best to the graph of $f(x) = \ln(x+1)$ in the vicinity of the point (0, 0), for when $x = 0$, $f(x) = 0$.

This we do by taking the value of the function and its derivatives for $x = 0$. Later on, we shall extend this method to any point $x = a$.

As $f(x) = \ln(x+1)$ then $f(0) = 0$.

Now $f'(x) = \dfrac{1}{x+1}$, and at $x = 0$, $f'(0) = 1$. This, by the way, is the gradient of the function $f(x)$ at $x = 0$.

Next, $f''(x) = -\dfrac{1}{(x+1)^2}$, which at $x = 0$ gives $f''(0) = -1$. This is in fact the rate of change of the gradient at $x = 0$.

Finally, $f'''(x) = \dfrac{2}{(x+1)^3}$, and at $x = 0$, $f'''(0) = 2$.

We can then summarise these results as follows.

$$f(x) = \ln(x+1) \quad : f(0) = 0$$

$$f'(x) = \frac{1}{x+1} \quad : f'(0) = 1$$

$$f''(x) = -\frac{1}{(x+1)^2} \quad : f''(0) = -1$$

$$f'''(x) = \frac{2}{(x+1)^3} \quad : f'''(0) = 2$$

We observe that for any n, $f^{(n)}(x) = \dfrac{(-1)^{n+1}(n-1)!}{(1+x)^n}$ and $f^{(n)}(0) = (-1)^{n+1}(n-1)!$.

The **linear** function that fits best has gradient 1 and passes through (0, 0), thus it is $f(x) = x$, which is in fact the tangent to $f(x) = \ln(x+1)$ at (0, 0).

The **quadratic** function that best fits is such that $f(x) = a_0 + a_1x + a_2x^2$, which must have $f(0) = 0$, $f'(0) = 1$ and $f''(0) = -1$.

Using our results from the previous section we obtain $f(x) = x + (-1)\cdot\dfrac{x^2}{2!} = x - \dfrac{x^2}{2}$.

The **cubic** function that fits best is such that $f(x) = a_0 + a_1x + a_2x^2 + a_3x^3$ with $f(0) = 0$, $f'(0) = 1$, $f''(0) = -1$ and $f'''(0) = 2$.

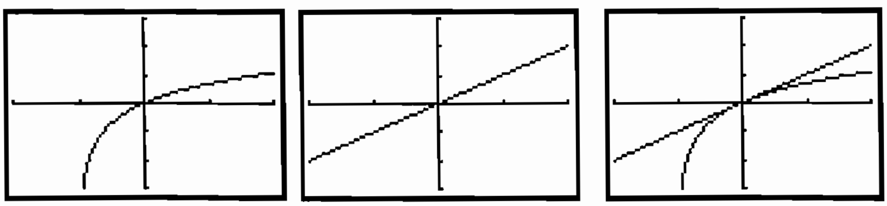
Again, using our results from the previous section we have $f(x) = x - \dfrac{x^2}{2} + \dfrac{x^3}{3}$.

This can then be extended to degree n: $f(x) = x - \dfrac{x^2}{2} + \dfrac{x^3}{3} + \ldots + (-1)^{n+1}\cdot\dfrac{x^n}{n}$.

Note that each polynomial arises from the previous one by adding just one more term. This means that by adding a further term, we can refine the accuracy of the fit between the original function $f(x)$ and the approximating function.

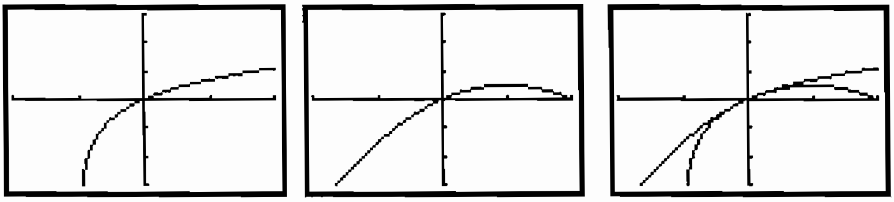
The following graphs display how this works, Window size is [–2, 2] by [–4, 4] & a scale of 1.
Linear function:

$f(x) = \ln(x+1)$: $f(x) = x$:



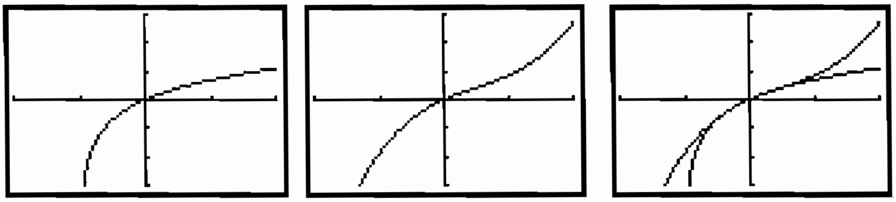
Quadratic function:

$f(x) = \ln(x+1)$: $f(x) = x - \frac{x^2}{2}$:



Cubic function:

$f(x) = \ln(x+1)$: $f(x) = x - \frac{x^2}{2} + \frac{x^3}{3}$:



The graph of the cubic approximation shows that the fit with $f(x)$ is excellent in the vicinity of $x = 0$. At $x = 0.5$, $f(x) = x - \frac{x^2}{2} + \frac{x^3}{3} = 0.417$, and $f(x) = \ln(x+1) = 0.405$. For larger values of x, i.e. x–values no longer in the vicinity of $x = 0$, the departure becomes considerable and the fit no longer works as we are outside the domain in which it is valid.

To summarize our result so far,

If $f(x)$ is a continuous and differentiable function, the polynomial of degree n that best fits the function in the vicinity of $x = 0$ is

$$P_n(x) = f(0) + f'(0) \cdot x + f''(0) \cdot \frac{x^2}{2!} + f'''(0) \cdot \frac{x^3}{3!} + \ldots + f^{(n)}(0) \cdot \frac{x^n}{n!} \quad (4.9)$$

2. Extension for approximating polynomials near x = a

We can extend our result of equation (4.9) to the case of the best approximating polynomial to a function $f(x)$, continuous and differentiable, in the vicinity of $x = a$. Equation (4.9) naturally becomes

$$P_n(x) = f(a) + f'(a) \cdot (x-a) + f''(a) \cdot \frac{(x-a)^2}{2!} + \ldots + f^{(n)}(a) \cdot \frac{(x-a)^n}{n!} \quad (4.10)$$

EXAMPLE 4.2

Find the cubic polynomial that fits best the function $f(x) = \ln(x+1)$ in the neighbourhood of $x = a = 1$.

Solution

Note that we choose this peculiar value of x so that calculations will become simpler. We can draw up the table as shown below

$$f(x) = \ln(x+1) \quad : f(e-1) = \ln e = 1$$

$$f'(x) = \frac{1}{x+1} \quad : f'(e-1) = \frac{1}{(e-1)+1} = \frac{1}{e}$$

$$f''(x) = -\frac{1}{(x+1)^2} \quad : f''(e-1) = -\frac{1}{((e-1)+1)^2} = -\frac{1}{e^2}$$

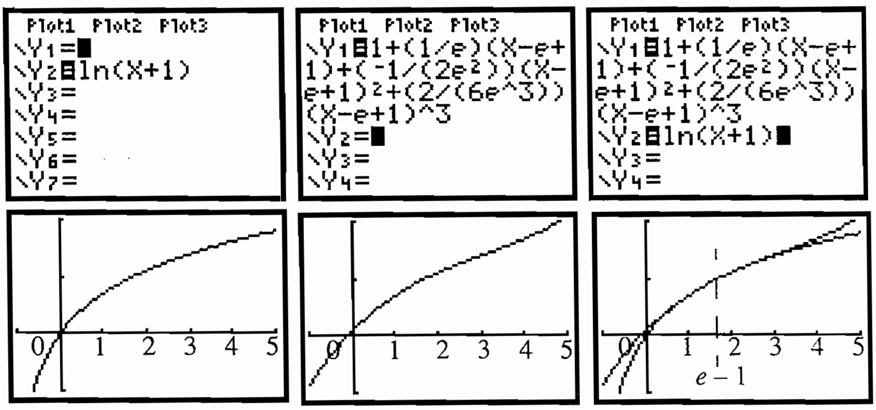
$$f'''(x) = \frac{2}{(x+1)^3} \quad : f'''(e-1) = \frac{2}{((e-1)+1)^3} = \frac{2}{e^3}$$

Therefore the cubic polynomial that best fits the function $f(x) = \ln(x+1)$ near $x = e-1$ is

$$P_3(x) = 1 + \frac{1}{e}(x-e+1) + \left(-\frac{1}{e^2}\right)\frac{(x-e+1)^2}{2!} + \left(\frac{2}{e^3}\right)\frac{(x-e+1)^3}{3!}$$

$$= 1 + \frac{1}{e}(x-e+1) - \frac{1}{e^2}\frac{(x-e+1)^2}{2!} + \frac{2}{e^3}\frac{(x-e+1)^3}{3!}.$$

Again, we can sketch both the function and it's cubic approximation to see the close fit in the neighbourhood of $x = e-1$. The operative word is 'neighbourhood'. The closer we are to $x = e-1$ the better the fit and away from this neighbourhood the polynomial and the function differ drastically!



4.1.4 FROM THE POLYNOMIAL APPROXIMATION TO THE EXPRESSION OF A FUNCTION AS AN INFINITE SERIES. TAYLOR AND MACLAURIN SERIES

1. Accuracy and the Maclaurin series

From our results in the previous sections, we have seen, equations (4.8) and (4.9), how we obtained successively better fits to our function as we added terms to the polynomial we found.

In particular, for our example of $f(x) = \ln(x+1)$ we found our cubic polynomial to be

$P_3(x) = x - \frac{x^2}{2} + \frac{x^3}{3}$ as an approximation near the value $x = 0$. How near is "near zero"? Until what value will our approximation be good enough?

Taking $x = 0.5$, the true value of the function would be $f(0.5) = \ln(0.5 + 1) = 0.40546$. At the moment, i.e. with our cubic polynomial, the value we have is

$$\begin{aligned} P_3(0.5) &= 0.5 - \frac{(0.5)^2}{2} + \frac{(0.5)^3}{3} \\ &= 0.5 - 0.125 + \frac{0.125}{3} \\ &= 0.4167 \end{aligned}$$

If we now took the next approximating polynomial, i.e. the quartic, using $f^{(iv)}(0) = -6$ would give the quartic polynomial $P_4(x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4}$.

Then, $P_4(0.5) = (0.5) - \frac{(0.5)^2}{2} + \frac{(0.5)^3}{3} - \frac{(0.5)^4}{4} = 0.4010$, which is now a closer value to the true value of the function.

As we can see, adding extra terms to the polynomial refines the approximation. Ultimately, we would get an exact value for $f(0.5)$ if we added an infinite number of terms to the polynomial, i.e. if it became an infinite series. Thus

$$\ln(x + 1) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \ldots$$

provided that x is sufficiently near to $x = 0$. We have already seen that, in the vicinity of $x = 0$, the polynomial of degree n,

$$P_n(x) = f(0) + f'(0) \cdot x + f''(0) \cdot \frac{x^2}{2!} + f'''(0) \cdot \frac{x^3}{3!} + \ldots + f^{(n)}(0) \cdot \frac{x^n}{n!}$$

gave us the best fit to our function $f(x)$. Therefore, for x sufficiently small,

$$f(x) \approx f(0) + f'(0) \cdot x + f''(0) \cdot \frac{x^2}{2!} + f'''(0) \cdot \frac{x^3}{3!} + \ldots + f^{(n)}(0) \cdot \frac{x^n}{n!}.$$

As n can be made as large as we want, the bigger n is, the better our approximation will be.

So we can conclude that the **exact** value of $f(x)$ will be given by the infinite series

$$f(x) = f(0) + f'(0) \cdot x + f''(0) \cdot \frac{x^2}{2!} + f'''(0) \cdot \frac{x^3}{3!} + \ldots + f^{(n)}(0) \cdot \frac{x^n}{n!} + \ldots \qquad (4.11)$$

And this is what we call the **Maclaurin series** or **Maclaurin's expansion**.

2. Two useful examples

As an example we will find the Maclaurin's expansion for $f(x) = \sin x$:

$$f(x) = \sin x \qquad : f(0) = 0$$

$$f'(x) = \cos x \qquad : f'(0) = 1$$

$$f''(x) = -\sin x \qquad : f''(x) = 0$$

$$f'''(x) = -\cos x \qquad : f'''(x) = -1$$

$$f^{(iv)}(x) = \sin x \qquad : f^{(iv)}(0) = 0$$

$$f^{(v)}(x) = \cos x \qquad : f^{(v)}(0) = 1$$

Therefore, $$f(x) = 0 + 1 \cdot x + 0 + (-1) \cdot \frac{x^3}{3!} + 0 + (1) \cdot \frac{x^5}{5!}$$

$$\therefore \sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \ldots \qquad (4.12)$$

Note: Only the **odd** powers of x are present in this Maclaurin series and the signs alternate.

If we take, for example, $x = 0.15$ and we substitute it in the above expression (4.12), we get

$$\begin{aligned} \sin(0.15) &= 0.15 - \frac{(0.15)^3}{3!} + \frac{(0.15)^5}{120} - \\ &= 0.15 - 5.625 \times 10^{-4} + 6.328 \times 10^{-7} - \\ &= 0.14943813\ldots \end{aligned}$$

that is, $\sin(0.15) = 0.14944$ to 5 decimal places.

On the other hand, the actual value of $\sin(0.15) = 0.1494381325...$, that is 0.14944 to 5 decimal places. In fact, we can see that the two expressions are equal to even 9 decimal places, which is not bad at all!!!

Another interesting example is that of the function $f(x) = (1 + x)^n$. This function is particularly useful in physical applications.

For this function, n can be an integer or a fraction. Be it positive or negative, there is no restriction on it.

If we apply Maclaurin's expansion, equation (4.11), to $f(x) = (1 + x)^n$ we obtain

$$f(x) = (1 + x)^n = f(0) + f'(0) \cdot x + f''(0) \cdot \frac{x^2}{2!} + f'''(0) \cdot \frac{x^3}{3!} + \ldots$$

The derivatives are as follows:

$$f(x) = (1+x)^n \quad : f(0) = 1$$

$$f'(x) = n(1+x)^{n-1} \quad : f'(0) = n$$

$$f''(x) = n(n-1)(1+x)^{n-2} \quad : f''(0) = n(n-1)$$

$$f'''(x) = n(n-1)(n-2)(1+x)^{n-3} \quad : f'''(0) = n(n-1)(n-2)$$

$$(x+1)^n = 1 + n \cdot x + n(n-1) \cdot \frac{x^2}{2!} + n(n-1)(n-2) \cdot \frac{x^3}{3!} + \ldots \qquad (4.13)$$

Many times we read in a physics book that "*to the linear approximation* $(1+x)^2 \cong 1+2x$ ".

If we look at equations (4.13) and put $n = 2$, $(1+x)^2 \cong 1+2x$ if we keep only up to the linear term in the Maclaurin expansion, i.e. if we neglect the powers of x higher than 1. So now you know where this comes from!!!

Alternatively, we read "neglecting nonlinear terms, $(1+x)^2 \cong 1+2x$ ". It is just more of the same. We neglect terms with powers of x from x^2 on.

4.1.5 TRIGONOMETRIC FUNCTIONS

We have already worked out, equations (4.12), the expansion for $f(x) = \sin x$. It is interesting to compare it to that of $f(x) = \cos x$. In this case,

$$f(x) = \cos x \quad : f(0) = 1$$

$$f'(x) = -\sin x \quad : f'(0) = 0$$

$$f''(x) = -\cos x \quad : f''(0) = -1$$

$$f'''(x) = \sin x \quad : f'''(0) = 0$$

$$f^{(iv)}(x) = \cos x \quad : f^{(iv)}(0) = 1$$

So we get

$$\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} + \ldots \qquad (4.14)$$

Here only the **even** powers of x appear, as opposed to the odd powers for $\sin x$, and the signs also alternate.

In a similar way, but with more care, we can show that

$$\tan x = x + \frac{2x^3}{3!} + \frac{16x^5}{5!} + \ldots = x + \frac{x^3}{3} + \frac{2x^5}{15} + \ldots \qquad (4.15)$$

Again, just as in the case for $\sin x$, only the **odd** powers of x are present.

Notice, that, within the **linear approximation** we have;

$$\cos x \approx 1, \ \tan x \approx x, \ \sin x \approx x .$$

If this seems a rather unexpected result, you can convince yourselves by calculating $\sin(0.1)$ and $\tan(0.1)$: $\sin(0.1) = 0.0998$ and $\tan(0.1) = 0.1003$
which are near enough to 0.1 (in fact, to 3 d.p., they are both equal to 0.100)

As an added bonus, you can even verify that the equality $\tan x = \dfrac{\sin x}{\cos x}$ is also valid using a Maclaurin approximation. Let us take equations (4.12), (4.14) and (4.15) to the third term, so we are left to prove that

$$\frac{x - \frac{x^3}{3!} + \frac{x^5}{5!}}{1 - \frac{x^2}{2!} + \frac{x^4}{4!}} = x + \frac{x^3}{3} + \frac{2x^5}{15} \text{ or that } \left(1 - \frac{x^2}{2!} + \frac{x^4}{4!}\right) \cdot \left(x + \frac{x^3}{3} + \frac{2x^5}{15}\right) = x - \frac{x^3}{3!} + \frac{x^5}{5!} \qquad (4.16)$$

If you multiply out the two brackets on the left-hand side of this equation and keep only the terms up to x^5 , you can see that the left-hand side and the right-hand side are equal. Quite a satisfactory exercise! This anticipates the last section of this chapter – the Maclaurin and Taylor series by multiplication.

To end this subsection on trigonometric functions, we shall show a very elegant proof of a well-known result , viz. that the derivative of $\sin x$ is $\cos x$, worked out from first principles and making use of the Maclaurin approximation.

Differentiating $f(x) = \sin x$ from first principles is found by using $\lim\limits_{\delta x \to 0} \dfrac{\sin(x + \delta x) - \sin x}{\delta x}$.
Remember that δx is a very small quantity compared to x, i.e., $\delta x \ll x$.

Bearing in mind that $\sin(a + b) = \sin(a)\cos(b) + \sin(b)\cos(a)$, we get

$$f'(x) = \lim_{\delta x \to 0} \frac{\sin x \cos \delta x + \sin \delta x \cos x - \sin x}{\delta x}$$
$$= \lim_{\delta x \to 0} \frac{\sin x(\cos \delta x - 1) + \sin \delta x \cos x}{\delta x}$$

As $\delta x \to 0$, $\cos \delta x \to 1$ and thus the first term $\to 0$. So we are left with the expression

$$f'(x) = \lim_{\delta x \to 0} \frac{\sin \delta x \cos x}{\delta x}$$
$$= \left(\lim_{\delta x \to 0} \frac{\sin \delta x}{\delta x}\right) \cdot \cos x \qquad (4.17)$$

At this stage, before being acquainted with the Maclaurin expansion, you had to resort to the "brute force" method: you had to fiddle with your calculator, enter several small angles and their sines (0.1, 0.01, 0.001,...) and convince yourselves that the sines of all these angles were nearly equal to the angles, and that the approximations were better the smaller the angles became.

It is much more satisfying to be elegant, this being one of the key features of Mathematics, and therefore to say that for δx in the vicinity of 0 (a small angle), Maclaurin tells you, from (4.12) that

$$\sin\delta x \approx \delta x - \frac{(\delta x)^3}{3!} + \frac{(\delta x)^5}{5!} - \ldots$$

Then, dividing by δx,

$$\frac{\sin\delta x}{\delta x} \approx 1 - \frac{(\delta x)^2}{3!} + \frac{(\delta x)^4}{5!} - \ldots$$

So that $\lim\limits_{\delta x \to 0} \frac{\sin\delta x}{\delta x} = 1$ as all other terms tend to 0 when $\delta x \to 0$!!

Therefore, if we substitute this result in (4.17), we obtain, finally, that $f'(x) = \cos x$.

4.1.6 MORE MACLAURIN EXPANSIONS

1. Exponential functions

In this section we consider the function $f(x) = e^x$.

As all derivatives are equal to the function itself, $f(0) = f'(0) = f''(0) = \ldots = 1$, therefore, the Maclaurin expansion is

$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \ldots + \frac{x^n}{n!} + \ldots \qquad (4.18)$$

It is interesting to see here that this series in fact converges. If we use the D'Alembert criterion for testing the convergence of a series - also called the **ratio test** (which we have already discussed in the previous chapter), we have;

$$u_n = \frac{x^n}{n!} \text{ and } u_{n+1} = \frac{x^{n+1}}{(n+1)!}$$

Therefore,

$$\lim_{n \to \infty}\left|\frac{u_{n+1}}{u_n}\right| = \lim_{n \to \infty}\left|\frac{x^{n+1}}{(n+1)!} \cdot \frac{n!}{x^n}\right|$$

$$= \lim_{n \to \infty}\left|\frac{x}{(n+1)}\right|$$

$$= \lim_{n \to \infty}\frac{|x|}{(n+1)}$$

$$= 0$$

So the series converges for all x with a radius of convergence, $R = \infty$.

We can say that, for any real number x, $\lim\limits_{n \to \infty}\left|\frac{x^n}{n!}\right| = 0$.

2. Logarithmic functions

Here we look at the function $f(x) = \ln(x+1)$.

Remember that you cannot expand $f(x) = \ln x$ in a Maclaurin series as this function is only defined for $x > 0$. We have already calculated the derivatives of $f(x) = \ln(x+1)$ in section 4.1.3, as well as its values at $x = 0$. Therefore the Maclaurin expansion is

$$\ln(x+1) = x - \frac{x^2}{2} + \frac{x^3}{3} + \ldots + (-1)^{n+1}\frac{x^n}{n} + \ldots$$

3. Inverse Trigonometric Relation

A. The inverse sine function

We first consider the **inverse sine function**, $f(x) = \sin^{-1}x$ (also written as $\arcsin x$).

We know from the core of the course that $f'(x) = \dfrac{1}{\sqrt{1-x^2}}$, so that $f'(0) = 1$.

Writing $f'(x) = (1-x^2)^{-1/2}$, we have, $f''(x) = -\frac{1}{2}\cdot -2x\cdot(1-x^2)^{-3/2}$, so that $f''(0) = 0$.

Next, with a little effort, we have that $f'''(x) = \dfrac{1+2x^2}{(1-x^2)^{5/2}}$, so that $f'''(0) = 1$.

With more effort, we find that $f^{(iv)}(x) = \dfrac{9x+6x^3}{(1-x^2)^{7/2}}$, and so, $f^{(iv)}(0) = 0$.

With even more effort, we have $f^{(v)}(x) = \dfrac{9+72x^2+24x^4}{(1-x^2)^{9/2}}$ so that $f^{(v)}(0) = 9$.

So, the expansion is given by

$$\sin^{-1}x = x + \frac{x^3}{3!} + \frac{9x^5}{5!} + \ldots = x + \frac{x^3}{6} + \frac{3x^5}{40} + \ldots$$

where only the odd powers of x appear, just as in the expansion of its cousin, $f(x) = \sin x$.

B. The inverse tangent function

Next we consider the **inverse tangent function**, $f(x) = \tan^{-1}x$ (also written as $\arctan x$).

Again, from the result in the core of the course we have that $f'(x) = \dfrac{1}{1+x^2}$, so that $f'(0) = 1$.

Next, $f''(x) = -1\cdot 2x\cdot(1+x^2)^{-2} = -\dfrac{2x}{(1+x^2)^2}$. Giving $f''(0) = 0$.

With some effort we find $f'''(x) = \dfrac{6x^2-2}{(1+x^2)^3}$, so that $f'''(0) = -2$.

Finally, the following expansion results:

$$\tan^{-1}x = x - \frac{2x^3}{3!} + \frac{24x^5}{5!} - \ldots = x - \frac{x^3}{3} + \frac{x^5}{5} - \ldots$$

4.1.7 Generalising the Maclaurin series for a value of a: the Taylor series

We are now going to generalize equation (4.11), i.e. the Maclaurin series, for the approximation of a function $f(x)$, continuous and differentiable n times, in the neighbourhood of a value $x = a$. We write it as

$$f(x) = f(a) + f'(a)(x-a) + f''(a)\cdot\frac{(x-a)^2}{2!} + \ldots + f^{(n)}(a)\cdot\frac{(x-a)^n}{n!} + \ldots \qquad (4.19)$$

Instead of the little dots after the n-th term, we shall call what remains of the infinite seris the **remainder** R_n, quite an original name! Thus the previous expression now becomes

$$f(x) = f(a) + f'(a)(x-a) + f''(a)\cdot\frac{(x-a)^2}{2!} + R_n \qquad (4.20)$$

As the Taylor series is an infinite series, it will give **exactly** the value of $f(x)$. If we only use the first n terms, this will produce an error, which will be given by R_n.

You are required to know the expression for R_n (though not a proof of it). We shall give it as

$$R_n = \frac{f^{(n+1)}(c)}{(n+1)!}(x-a)^{n+1} \quad \textbf{or} \quad R_n = \frac{1}{n!}\int_a^x f^{(n+1)}(z)(x-z)^n dz$$

for some number or value c in the interval $]a, x[$.

One of the most useful applications of the Taylor's series is that of calculating approximations to certain expressions within a certain degree of accuracy.

If we are in the **vicinity** of $x = a$, it is clear that $x - a = h$, where $|h|$ can be as small as we like. In other words, we want to work out an approximation to $f(a+h)$ from $f(a)$, provided $|h|$ is small. All we need to do is substitute $a+h$ for x in equation (4.20), viz.

$$f(a+h) = f(a) + f'(a)h + f''(a)\cdot\frac{h^2}{2!} + \ldots + f^{(n)}(a)\cdot\frac{h^n}{n!} + R_n \qquad (4.21)$$

where our $R_n = \dfrac{f^{(n+1)}(a+c)}{(n+1)!}h^{n+1}$ and now c is between 0 and h. As already said, the error in the approximation of $f(a+h)$ to a certain degree of accuracy will be given by R_n.

EXAMPLE 4.3

Calculate $\cos\left(\frac{\pi}{3} + 0.1\right)$ for $n = 4$ to five decimal places.

Solution

We shall take $f(x) = \cos x$, $a = \frac{\pi}{3}$ and $h = 0.1$, with $0 \le c \le 0.1$.

After repeated differentiation we have $f'(x) = -\sin x$, $f''(x) = -\cos x$, $f'''(x) = \sin x$, $f^{(iv)}(x) = \cos x$. From (3.18) we have

$$\cos\left(\frac{\pi}{3} + 0.1\right) = \frac{1}{2} - \sin\left(\frac{\pi}{3}\right)\cdot(0.1) - \cos\left(\frac{\pi}{3}\right)\cdot\frac{(0.1)^2}{2} + \sin\left(\frac{\pi}{3}\right)\cdot\frac{(0.1)^3}{6} + R_4$$

where $R_4 = \left|\cos\left(\frac{\pi}{3} + c\right)\cdot\frac{(0.1)^4}{24}\right|$ with $0 \le c \le 0.1$.

Therefore, $\cos\left(\frac{\pi}{3}+0.1\right) = \frac{1}{2}-0.05\sqrt{3}-\frac{1}{4}\times 0.01+\frac{\sqrt{3}}{2}\times\frac{0.001}{6}+R_4$

$= 0.41104180+R_4$

As $0 \le c \le 0.1$, $0 \le R_4 \le \frac{(0.1)^4}{24}\cos\left(\frac{\pi}{3}\right) = 0.0000021$. Then, $\cos\left(\frac{\pi}{3}+0.1\right) = 0.41104$ to 5 d.p.

An interesting example of an unusual function is that of $f(x) = x^x$.

EXAMPLE 4.4

Expand the function $f(x) = x^x$ about $x = e$ to the second degree term.

Solution

We have $f(e) = e^e$, which is a numerical constant.
In order to find the derivatives of the function, we must first take logarithms on both sides of the expression of the function, i.e.

$$f(x) = x^x \Leftrightarrow \ln f(x) = \ln x^x$$
$$\Leftrightarrow \ln f(x) = x\ln x$$

Then, after differentiating both sides with respect to x we have:

$$\frac{1}{f(x)}\cdot f'(x) = \ln x+1 \Leftrightarrow f'(x) = (\ln x+1)f(x)$$

Therefore, we have that $f'(x) = x^x(\ln x+1)$. So, $f'(e) = e^e(1+\ln e) = 2e^e$.

Next, $f''(x) = (\ln x+1)f'(x)+\frac{1}{x}\cdot f(x) = (\ln x+1)\times x^x(\ln x+1)+\frac{1}{x}\cdot x^x$

$= x^x(\ln x+1)^2+x^{x-1}$

Thus we get the final expression $f''(e) = 4e^e+e^{e-1} = e^{e-1}(4e+1)$.

Therefore, we have that $x^x = e^e+2e^e(x-e)+e^{e-1}(4e+1)\cdot\frac{(x-e)^2}{2!}$ to the 2nd order term.

EXAMPLE 4.5

Evaluate the definite integral $\int_0^1 e^{-x^2}dx$ to within an error of 10^{-3}.

Solution

In this example we make use of the result of (4.18), for which we take $a = 1$.

$$\int_0^1 e^{-x^2}dx = 1-\frac{1}{3}+\frac{1}{10}-\frac{1}{42}\approx 0.7429.$$

So the error involved in this approximation would be less than the next term in the alternating series, i.e., smaller than $\frac{1}{216}$. We see that this *is not* smaller than 10^{-3}, therefore we must take one more term in the approximation, i.e. add one more term to the series:

$$\int_0^1 e^{-x^2}dx = 1 - \frac{1}{3} + \frac{1}{10} - \frac{1}{42} + \frac{1}{210} \approx 0.7475 \qquad (4.22)$$

Now the error involved would be less than the next term, which is $\frac{1}{5! \times 11} = \frac{1}{1320} < 10^{-3}$

As the last inequality is correct, the result obtained in (4.22) is correct to 10^{-3}.

EXAMPLE 4.6

Evaluate e to four decimal places, i.e., to an error of less than 5×10^{-5}.

Solution

Using the series for e^x where $x = 1$, i.e., $e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \frac{x^5}{5!} + \ldots$, we have

$$e^1 = 1 + 1 + \frac{1^2}{2!} + \frac{1^3}{3!} + \frac{1^4}{4!} + \frac{1^5}{5!} + \ldots$$

For comparison, we can say that this series is smaller than

$$1 + 1 + \frac{1}{2^2} + \frac{1}{2^3} + \frac{1}{2^4} + \frac{1}{2^5} + \ldots = 1 + \frac{1}{1 - \frac{1}{2}} = 3$$

since, from the second term onwards, we have a geometric series with common ratio 0.5.
So our first result is that $e < 3$. For our function e^x, using Taylor's remainder, we want the error to be smaller than 5×10^{-5}, with $x = 1$. In our present case, we have

$$|R_n(1)| = \left|\frac{f^{(n+1)}(c)}{(n+1)!}\right| = \frac{e^c}{(n+1)!}.$$

As e^x is an increasing function, $e^c < e^1 < 3$. Therefore $|R_n(1)| < \frac{3}{(n+1)!}$.

As we want the error to be smaller than 5×10^{-5} it is sufficient to have $\frac{3}{(n+1)!} < 5 \times 10^{-5}$,

which means $(n+1)! \geq 6 \times 10^4$.

Although never a satisfying way to do it, we have to resort to trial and error to see that, in this case, we must have $n \geq 8$.
Therefore, we can use the approximation to $n = 8$ and get a value of e of

$$e \approx 1 + 1 + \frac{1}{2} + \frac{1}{3} + \ldots + \frac{1}{8} = 2.7183.$$

4.1.8 SUMMARY

1. Maclaurin series

If the function f possesses infinitely many derivatives at $x = 0$ and is such that it can be expressed as a power series in x in its interval of convergence, then the series

$$f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!}x^n = f(0) + f'(0) \cdot x + \frac{f''(0)}{2!}x^2 + \ldots + \frac{f^{(n)}(0)}{n!}x^n + \ldots$$

is called the *Maclaurin series*.

Functions such as $f(x) = \ln x$ and $f(x) = \sqrt{x}$ do not have Maclaurin's expansions at $x = 0$ as $\ln x$ is not defined at $x = 0$ and $\sqrt{x}$ is not differentiable at $x = 0$. However, such functions can often be represented by a series expanded about a point other than $x = 0$.

2. Taylor series

If the function f possesses infinitely many derivatives at $x = a$ and is such that it can be expressed as a power series in x in its interval of convergence $]a - R, a + R[$, then the series

$$f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!}(x-a)^n$$

$$= f(a) + f'(a) \cdot (x-a) + \frac{f''(a)}{2!}(x-a)^2 + \ldots + \frac{f^{(n)}(a)}{n!}(x-a)^n + \ldots$$

is called the *Taylor series* expanded about $x = a$.

3. Taylor polynomial

The partial sums of a Taylor series are simply polynomials. The polynomial of degree n,

$$P_n(x) = \sum_{r=0}^{n} \frac{f^{(r)}(a)}{r!}(x-a)^r$$

$$= f(a) + f'(a) \cdot (x-a) + \frac{f''(a)}{2!}(x-a)^2 + \ldots + \frac{f^{(n)}(a)}{n!}(x-a)^n$$

is called the *Taylor polynomial of degree n for f* expanded about $x = a$.

4. Remainder term

Using the expressions for the Taylor polynomial and series we have:

$$f(x) = \sum_{r=0}^{n} \frac{f^{(r)}(a)}{r!}x^r + \sum_{r=n+1}^{\infty} \frac{f^{(r)}(a)}{r!}x^r = P_n(x) + R_n(x)$$

The term $R_n(x)$, the remainder term, is s.t. $R_n(x) = \frac{f^{(n+1)}(c)}{(n+1)!}(x-a)^{n+1}$ for $a < c < x$

or $\quad R_n(x) = \frac{1}{n!}\int_a^x f^{(n+1)}(z)(x-z)^n dz$

Note:

If $\lim_{n \to \infty} R_n(x) = 0$ then $f(x) = \lim_{n \to \infty} [P_n(x) + R_n(x)] = \lim_{n \to \infty} P_n(x) + 0 = \lim_{n \to \infty} P_n(x)$.

Realise then that when determining the errors involved when using Taylor expansions, we have that the error in approximating $f(x)$ by $P_n(x) = |R_n(x)| = \frac{|f^{(n+1)}(c)|}{(n+1)!}|(x-a)^{n+1}|$ for some value of c between a and x. We can often estimate the error by replacing $|f^{(n+1)}(c)|$ with its maximum for $a \le c \le x$ if $x > a$ and its maximum for $x \le c \le a$ if $x < a$.

We conclude this section with one more example:

EXAMPLE 4.7

Use the third degree Taylor polynomial for $\ln x$ to approximate the value of $\ln(1.1)$ and determine the error involved in this approximation.

Solution

We start by determining the third degree Taylor polynomial approximation of $\ln x$ centered at $x = 1$:

$$P_3(x) = f(1) + f'(1)\cdot(x-1) + \frac{1}{2!}f''(1)\cdot(x-1)^2 + \frac{1}{3!}f'''(1)\cdot(x-1)^3$$

Now, $f(x) = \ln x$, $f'(x) = \frac{1}{x}$, $f''(x) = -\frac{1}{x^2}$ and $f'''(x) = \frac{2}{x^3}$.

So, $f(1) = \ln 1 = 0$, $f'(1) = \frac{1}{1} = 1$, $f''(1) = -\frac{1}{1^2} = -1$, $f'''(1) = \frac{2}{1^2} = 2$.

Therefore, $P_3(x) = 0 + 1\cdot(x-1) + \frac{1}{2}(-1)\cdot(x-1)^2 + \frac{1}{6}(2)\cdot(x-1)^3$

$$= (x-1) - \frac{1}{2}(x-1)^2 + \frac{1}{3}(x-1)^3.$$

Then, $P_3(1.1) = (1.1-1) - \frac{1}{2}(1.1-1)^2 + \frac{1}{3}(1.1-1)^3 = (0.1) - \frac{1}{2}(0.1)^2 + \frac{1}{3}(0.1)^3$

$$\approx 0.09533333 .$$

Next, $R_3(x) = \frac{1}{4!}f^{(4)}(c)(x-1)^4 = \frac{1}{4!}\cdot\left(-\frac{6}{c^4}\right)\cdot(x-1)^4$, where c is such that $1 \le c \le 1.1$.

This give $R_3(1.1) = -\frac{1}{4c^4}\cdot 0.1^4$, $1 \le c \le 1.1$.

We now need to determine the maximum value this can be. The maximum will occur when c = 1 and so the maximum value of $R_3(1.1)$ is $\left|-\frac{1}{4(1)^4}\cdot 0.1^4\right| = \frac{1}{4} \times \frac{1}{10^4} = 0.000025$.

That is, the error $|R_3(1.1)|$ is less than 0.000025.

So, we have that $\ln(1.1) \approx 0.09533$ with an error of less than 0.000025.

Note that $\ln(1.1) - 0.09533 = 0.00002 < 0.000025$ (as was expected).

EXERCISES 4.1

1. (a) Show that the Maclaurin series expansion of $f(x) = e^x$ is given by $\sum_{n=0}^{\infty} \frac{x^n}{n!}$.

(b) Let $s_n(x)$ be the partial sum $s_n(x) = \sum_{k=0}^{n} \frac{x^k}{k!}$.

On the same set of axes and over the domain $-2 < x < 2$, sketch the graphs

i. $y = f(x)$ ii. $y = s_2(x)$ iii. $y = s_3(x)$ iv. $y = s_4(x)$

(c) What do you observe as n increases?

2. (a) Show that the Maclaurin series expansion of $f(x) = \sin x$ is given by

$$\sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{(2n+1)!}.$$

(b) Let $s_n(x)$ be the partial sum $\sum_{k=0}^{n} (-1)^k \frac{x^{2k+1}}{(2k+1)!}$.

On the same set of axes and over the domain $-4 < x < 4$, sketch the graphs

i. $y = f(x)$ ii. $y = s_2(x)$ iii. $y = s_6(x)$ iv. $y = s_8(x)$

(c) What do you observe as n increases?

3. (a) Show that the Maclaurin series expansion of $f(x) = \cos x$ is given by

$$\sum_{n=0}^{\infty} (-1)^n \frac{x^{2n}}{(2n)!}.$$

(b) Let $s_n(x)$ be the partial sum $\sum_{k=0}^{n} (-1)^k \frac{x^{2k}}{(2k)!}$.

On the same set of axes and over the domain $-4 < x < 4$, sketch the graphs

i. $y = f(x)$ ii. $y = s_2(x)$ iii. $y = s_6(x)$ iv. $y = s_8(x)$

(c) What do you observe as n increases?

4. Find the first four non-zero terms of the Maclaurin series for :

(a) $f(x) = e^{2x}$ (b) $f(x) = \ln(1+x)$ (c) $f(x) = \sin(2x)$

(d) $f(x) = \tan x$ (e) $f(x) = \sqrt{1-x^2}$ (f) $f(x) = 2^x$

5. Find the first three non-zero terms of the Maclaurin series for :

(a) $f(x) = e^x \cos x$ (b) $f(x) = e^{-x^2}$ (c) $f(x) = e^{-x} \sin x$

(d) $f(x) = \arcsin x$ (e) $f(x) = \sec x$ (f) $f(x) = x\ln(x+2)$

6. Find the Maclaurin series for the function

(a) $f(x) = a^x$ (b) $f(x) = \log_a(1+x)$

7. Find the Taylor series for

(a) e^x about $x = 1$ (b) $\ln x$ about $x = 2$

(c) e^{-x} about $x = -2$ (d) $\sqrt{x}$ about $x = 1$

8. (a) Using the Taylor polynomial, $P_n(x)$, for $\sin x$ about $x = 0$ deduce the Taylor polynomial of $\sin(x^2)$ about $x = 0$.

(b) If $f(x) = \sin(x^2)$, find the expression for $f(x) = P_n(x^2) + R_n(x^2)$, with $n = 7$.

(c) Using part (b), show that $\int_0^1 \sin x^2 dx = \frac{1}{3} - \frac{1}{7 \cdot 3!} + \frac{1}{11 \cdot 5!} - \frac{1}{15 \cdot 7!} + \text{Error}$, where

Error $= \int_0^1 R_7(x^2)dx$.

(d) Show that

i. the error involved in (c) is approximately 0.0000015.

ii. correct to at least 4 decimal places, $\int_0^1 \sin(x^2)dx = 0.3102682 \pm 0.00000015$.

9. Use a fourth degree Taylor polynomial for $\ln(x)$ to approximate the value of ln(1.2) and determine the error involved in this approximation.

10. (a) Find the third degree Taylor polynomial for $x^{5/2}$ expanded about $x = 4$.

(b) Using your polynomial in (a), approximate the value of $6^{5/2}$.

(c) Estimate the error using the third degree Taylor polynomial in approximating $6^{5/2}$.

11. Find an approximation to the integral $\int_0^{\frac{\pi}{2}} \sqrt{1 - k^2 \sin^2 x}dx$, $0 \le k < 1$ using the binomial expansion with four terms. Determine the error involved in this approximation.

4.2 SERIES EXPANSION OF COMBINED EXPRESSIONS

4.2.1 EXPANSION OF COMPOSITE FUNCTIONS

We have already produced a series expansion for the composite function $f(x) = e^{-x^2}$. This was done by using the series expansion of $f(x) = e^x$, i.e., $e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \ldots + \frac{x^n}{n!} + \ldots$, and then making the substitution $z = -x^2$, so that

$$\begin{aligned} e^{-x^2} = e^z &= 1 + z + \frac{z^2}{2!} + \frac{z^3}{3!} + \ldots + \frac{z^n}{n!} + \ldots \\ &= 1 + (-x^2) + \frac{(-x^2)^2}{2!} + \frac{(-x^2)^3}{3!} + \ldots + \frac{(-x^2)^n}{n!} + \ldots \\ &= 1 - x^2 + \frac{x^4}{2!} - \frac{x^6}{3!} + \frac{x^8}{4!} + \ldots + (-1)^n \cdot \frac{x^{2n}}{n!} + \ldots \end{aligned}$$

A similar approach can be used for other expressions that involve the composition of functions.

EXAMPLE 4.8 Find the Maclaurin expansion of $\log(2+\sin x)$.

Solution

Bearing in mind the expansion of $\sin x$, i.e., $\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \dots$.

We first rewrite the original expression as

$$\log(2+\sin x) = \log\left(2 + x - \frac{x^3}{3!} + \frac{x^5}{5!} - \dots\right)$$

$$= \log\left[2\left(1 + \frac{x}{2} - \frac{x^3}{12} + \frac{x^5}{240} - \dots\right)\right]$$

$$= \log 2 + \log\left(1 + \frac{x}{2} - \frac{x^3}{12} + \frac{x^5}{240} - \dots\right)$$

Letting $z = \frac{x}{2} - \frac{x^3}{12} + \frac{x^5}{240} - \dots$ we have that $\log(2+\sin x) = \log 2 + \log(1+z)$

$$= \log 2 + z - \frac{z^2}{2} + \frac{z^3}{3} - \dots$$

Therefore, we have that the expansion of $\log(2+\sin x)$ is given by

$$\log 2 + \left(\frac{x}{2} - \frac{x^3}{12} + \frac{x^5}{240} - \dots\right) - \frac{1}{2}\left(\frac{x}{2} - \frac{x^3}{12} + \frac{x^5}{240} - \dots\right)^2 + \frac{1}{3}\left(\frac{x}{2} - \frac{x^3}{12} + \frac{x^5}{240} - \dots\right)^3 - \dots$$

That is, $\log(2+\sin x) = \log 2 + \frac{x}{2} - \frac{x^2}{8} - \frac{x^3}{24} + \frac{x^4}{24} - \frac{x^5}{60} + \dots$

4.2.2 TAYLOR EXPANSIONS INVOLVING MULTIPLICATION

We had previously anticipated this section in subsection 4.1.5 when, having calculated Maclaurin's series for $\sin x$, $\cos x$ and $\tan x$, we showed that the relationship $\tan x = \frac{\sin x}{\cos x}$ is also valid when we substitute in all three expansions of $\sin x$, $\cos x$ and $\tan x$.

EXAMPLE 4.9 Find the first three terms of the expansion of $e^{x^2}\arctan x$.

Solution

We separate it into the expansion of e^{x^2} and that of $\arctan x$.

The expansion of e^{x^2}:

Let $f(x) = e^{x^2} \Rightarrow f'(x) = 2xe^{x^2} \therefore f'(0) = 0$ and $f(0) = 1$.

Next, $f''(x) = 2e^{x^2} + 4x^2e^{x^2} \Rightarrow f''(0) = 2$.

Then, $f'''(x) = 4xe^{x^2} + 8xe^{x^2} + 8x^3e^{x^2} \Rightarrow f'''(0) = 0$.

Next, $f^{(iv)}(x) = 12e^{x^2} + 48x^2e^{x^2} + 16x^4e^{x^2} \Rightarrow f^{(iv)}(0) = 12$.

Therefore, $e^{x^2} = 1 + 2\cdot\frac{x^2}{2!} + 12\cdot\frac{x^4}{4!} + \dots = 1 + x^2 + \frac{x^4}{2} + \dots$.

Now we determine the expansion for $\arctan x$:

Let $g(x) = \arctan x \therefore g(0) = 0$.

Next, $g'(x) = \frac{1}{1+x^2} \Rightarrow g'(0) = 1$ and $g''(x) = -\frac{2x}{(1+x^2)^2} \Rightarrow g''(0) = 0$.

Then, $g'''(x) = \frac{6x^2-2}{(1+x^2)^3} \Rightarrow g'''(0) = -2$.

Therefore, $\arctan x = x - 2 \cdot \frac{x^3}{3!} + \ldots = \arctan x = x - \frac{x^3}{3} + \ldots$.

We now bring the two together:

$$e^{x^2}\arctan x = \left(1 + x^2 + \frac{x^4}{2} + \ldots\right)\left(x - \frac{x^3}{3} + \ldots\right)$$

$$= x + \frac{2}{3}x^3 + \frac{1}{6}x^5 + \ldots$$

EXAMPLE 4.10

Expand the product $4\cos^2 x\ln(x+1)$ to the term of the expansion in x^5.

Solution

This example has a slight extra complication: there are two products. The first is $\cos^2 x = \cos x \cos x$, but we already know the expansion for $\cos x$.

If, we deal with $\cos^2 x$ directly, we would need to calculate all the derivatives. It is easier to recall the earlier expressions:

$$\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \ldots \text{ and } \ln(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \ldots$$

First we find the expansion for $\cos^2 x$:

$$\cos^2 x = \left(1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \ldots\right)\left(1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \ldots\right)$$

$$= 1 - x^2 + \frac{5}{12}x^4 + \ldots$$

Therefore, putting the two expansions together we have

$$4\cos^2 x\ln(x+1) = 4\left(1 - x^2 + \frac{5}{12}x^4 + \ldots\right)\left(x - \frac{x^2}{2} + \frac{x^3}{3} - \ldots\right)$$

Therefore, to the 5th term of approximation we finally have

$$4\cos^2 x\ln(x+1) = 4 - 2x^2 + \frac{11}{3}x^4 - \frac{32}{15}x^5 + \ldots$$

EXAMPLE 4.11

Expand the expression $\frac{\tan x}{(1+x)^4}$ as far as the 5th term.

Solution

We already know that $\tan x = x + \frac{x^3}{3} + \frac{2x^5}{15} + \dots$.

Now we have to expand the other term as $(1+x)^{-4}$:

Let $f(x) = (1+x)^{-4} \therefore f'(x) = -4(1+x)^{-5}$ so that $f(0) = 1$ and $f'(0) = -4$.

Next, $f''(x) = 20(1+x)^{-6} \therefore f''(0) = 20$.

$f'''(x) = -120(1+x)^{-7} \therefore f'''(0) = -120$.

$f^{(iv)}(x) = 840(1+x)^{-8} \therefore f^{(iv)}(0) = 840$.

$f^{(v)}(x) = -6720(1+x)^{-9} \therefore f^{(iv)}(0) = -6720$.

Therefore, $(1+x)^{-4} = 1 - 4x + 10x^2 - 20x^3 + 35x^4 - 56x^5 + \dots$.

The product, to the 5th term in approximation, then becomes

$$\frac{\tan x}{(1+x)^4} = \left(x + \frac{x^3}{3} + \frac{2x^5}{15} + \dots\right)(1 - 4x + 10x^2 - 20x^3 + 35x^4 - 56x^5 + \dots)$$

$$= x - 4x^2 + 10x^3 - 20x^4 + 35x^5 + \frac{x^3}{3} - \frac{4x^4}{3} + \frac{10x^5}{3} + \frac{2x^5}{15} + \dots$$

$$= x - 4x^2 + \frac{31}{3}x^3 - \frac{64}{3}x^4 + \frac{527}{15}x^5 + \dots$$

EXERCISES 4.2

1. Find the Maclaurin expansion of the following:

(a) e^{2x} (b) $e^{-\frac{1}{2}x}$ (c) e^{x^2} (d) $e^{\sin x}$

(e) $\sin(2x)$ (f) $\cos\left(\frac{1}{2}x\right)$ (g) $\sin(x^2)$ (h) $\arctan(2x)$

2. Find the Maclaurin expansion of the following:

(a) $\ln\left(1 + \frac{1}{2}x\right)$ (b) $\ln(1 + x^3)$ (c) $\ln(1 + \cos x)$

3. (a) Expand the expression $\frac{\sin x}{(1+x)}$ as far as the 5th term.

(b) Expand the expression $e^{-x}\cos x$ as far as the 4th term.

(c) Expand the expression $(\sin x)^2 \cdot \ln(1+x)$ as far as the 4th term.

(d) Expand the expression $\frac{\cos(x^2)}{\sqrt{1+x}}$ as far as the 5th term.

4.3 APPLICATIONS - REVISITED

4.3.1 Two definite integrals

First let us see the usefulness of Maclaurin's expansion in calculating the integral of a function which is not an elementary function, i.e. it cannot be integrated directly.

EXAMPLE 4.12

Evaluate the definite integral $\int_0^a e^{-x^2}dx$.

Solution

We can substitute 'x' by '$-x^2$' in the Maclaurin series for the function e^x and get directly the new series expansion for the above elementary function:

$$\begin{aligned} e^{-x^2} &= 1 + (-x^2) + \frac{(-x^2)^2}{2!} + \frac{(-x^2)^3}{3!} + \ldots + \frac{(-x^2)^n}{n!} + \ldots \\ &= 1 - x^2 + \frac{x^4}{2!} - \frac{x^6}{3!} + \ldots + (-1)^n \cdot \frac{x^{2n}}{n!} + \ldots \end{aligned}$$

Therefore integrating both sides between 0 and a we obtain

$$\begin{aligned} \int_0^a e^{-x^2}dx &= \int_0^a \left(1 - x^2 + \frac{x^4}{2!} - \frac{x^6}{3!} + \ldots\right)dx \\ &= \left[x - \frac{x^3}{3} + \frac{x^5}{10} - \frac{x^7}{42} + \ldots\right]_0^a \\ &= a - \frac{a^3}{3} + \frac{a^5}{10} - \frac{a^7}{42} + \ldots \end{aligned}$$

This allows us to calculate the integral, for any value of a, with the approximation desired.

EXAMPLE 4.13

Evaluate the definite integral $\int_0^a \frac{\sin x}{x}dx$.

Solution

We already know the Maclaurin series for $\sin x$, i.e., (4.11a). Thus dividing by x we have

$$\begin{aligned} \frac{\sin x}{x} &= \frac{1}{x} \cdot \left(x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \ldots\right) \\ &= 1 - \frac{x^2}{3!} + \frac{x^4}{5!} - \frac{x^6}{7!} + \ldots \end{aligned}$$

Integrating both sides between 0 and a we then get

$$\int_0^a \frac{\sin x}{x}dx = \int_0^a\left(1 - \frac{x^2}{3!} + \frac{x^4}{5!} - \frac{x^6}{7!} + \ldots\right)dx$$

$$= \left[x - \frac{x^3}{3 \cdot 3!} + \frac{x^5}{5 \cdot 5!} - \frac{x^7}{7 \cdot 7!} + \ldots\right]_0^a$$

$$= a - \frac{a^3}{3 \cdot 3!} + \frac{a^5}{5 \cdot 5!} - \frac{a^7}{7 \cdot 7!} + \ldots$$

EXERCISES 4.3

1. Using the first three terms of the Taylor polynomial, approximate $\int_{-1}^{1} \cos(x^2)dx$.

2. Using the first three terms of the Taylor polynomial, approximate $\int_{-1}^{1} \frac{\sin x}{x}dx$.

3. Using the first four terms of the Taylor polynomial, approximate $\int_0^{0.1} \ln(1 + \sin x)dx$.

4. Using the first four terms of the Taylor polynomial, approximate $\int_{-0.2}^{0.2} e^{-\frac{1}{2}x^2} dx$.

5. Using the first five terms of the Taylor polynomial, approximate $\int_1^2 \frac{1}{1 + \ln x}dx$.

6. Using the first four terms of the Taylor polynomial, approximate $\int_{-1}^{2} \sin(x^2)dx$.

7. Using the first four terms of the Taylor polynomial, approximate $\int_0^1 \arctan(\sqrt{x})dx$.

5.1 INTRODUCTION

CHAPTER 5

5.1.1 REVIEWING DEFINITIONS

In this section we briefly revisit definitions and work covered in the core section of the course.

A differential equation, often abbreviated to d.e., is an expression that contains an initially unknown function y, its independent variable x and some of its successive derivative(s) $y', y'', y''', \ldots$. In other words, it is an expression that can be written in the form:

$$F(x, y, y', y'', y''', \ldots) = 0$$

This equation may contain combinations of all or some of its implicit components, i.e., product, quotients, powers and so on.

For example, we have:

1. $2y' + y^2 = x - 1$.
   Rewriting the expression as $2y' + y^2 - x + 1 = 0$,
   we have that $F(x, y, y') \equiv 2y' + y^2 - x + 1$

2. $y'' - 4y' - 5y = \frac{1}{2}x$
   Rewriting the expression as $y'' - 4y' - 5y - \frac{1}{2}x = 0$,
   we have that $F(x, y, y', y'') \equiv y'' - 4y' - 5y - \frac{1}{2}x$

3. $\left(\frac{dy}{dx}\right)^2 - \frac{d^2y}{dx^2} = 9x^3$
   Rewriting the expression as $\left(\frac{dy}{dx}\right)^2 - \frac{d^2y}{dx^2} - 9x^3 = 0$,
   we have that $F\left(x, \frac{dy}{dx}, \frac{d^2y}{dx^2}\right) \equiv \left(\frac{dy}{dx}\right)^2 - \frac{d^2y}{dx^2} - 9x^3$

This may make it more or less difficult for us to find a **solution** to the d.e., that is, to find an expression for y that satisfies the given d.e.. Generally we shall seek an explicit solution of y in terms of x, but we shall see that this task may prove to be sometimes laborious or even impossible. In this chapter we shall only concern ourselves with differential equations that contain terms involving x, y and y' . That is, we shall only be solving equations of the form $F(x, y, y') = 0$.

Equations such as these are called **first-order differential equations**. The order of a differential equation is determined by the order of the highest derivative of the function y.

We shall only treat differential equations of the **second–order** (i.e., those containing y'') in one specific simple case: when we are given the solution to the particular d.e.. That is we will be given an expression in y and we will then have to verify that this solution satisfies the particular given d.e.. We will not be required to solve the d.e., but rather verify that the given solution satistfies the d.e.

For example, if asked to verify that $y = e^{2x}$ is a solution to the d.e., $y'' - 3y' + 2y = 0$ we proceed as follows:

If $y = e^{2x} \Rightarrow y' = 2e^{2x}$ and $y'' = 4e^{2x}$.

Substituting into the L.H.S of the d.e., we have:

$$\begin{aligned} \text{L.H.S} &= 4e^{2x} - 3 \times (2e^{2x}) + 2 \times (e^{2x}) \\ &= 4e^{2x} - 6e^{2x} + 2e^{2x} \\ &= 0 \\ &= \text{R.H.S} \end{aligned}$$

Therefore, the equation $y = e^{2x}$ satisfies the d.e., and so is a solution to the given d.e..

We shall further restrict ourselves to differential equations in which the term in y' appears as a **linear term**, i.e., in the first power of y', and not as a term of some other power of y' such as $(y')^2$, $(y')^3$, $(y')^{1/2}$ etc. These involve more complicated methods of solving the d.e., and are outside the I.B syllabus.

Therefore we shall only concern ourselves with the restricted class of **linear first-order differential equations**.

EXERCISES 5.1

1. Show that the differential equations are satisfied by the given equations.

(a) $\dfrac{dy}{dx} = \dfrac{4}{\sqrt{4-x^2}}$; $\quad y = 4\arcsin\left(\dfrac{x}{2}\right)$.

(b) $y'' + 9y = 10e^x$; $\quad y = \sin 3x + e^x$.

(c) $\dfrac{dy}{dx} = \dfrac{1}{2\sqrt{y}}$; $\quad y = \sqrt{x}$.

(d) $y'' = -y$; $\quad y = \sin x$.

(e) $\dfrac{dy}{dx} = x + y$; $\quad y = e^x - x - 1$.

(f) $\dfrac{dy}{dx} - y = e^x$; $\quad y = xe^x$.

(g) $\dfrac{d^2y}{dx^2} + 2e^{-y} = 0$; $\quad y = 2\ln x$.

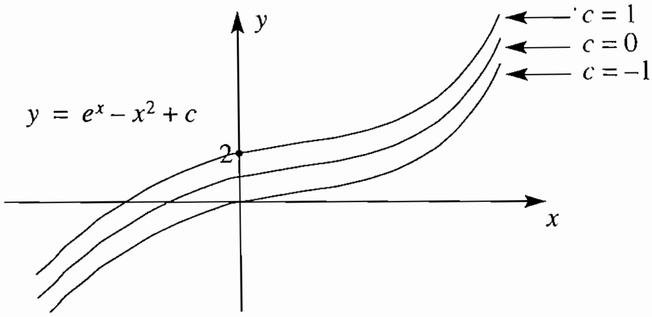
5.2 GEOMETRICAL METHOD

5.2.1 THE SLOPE FIELD

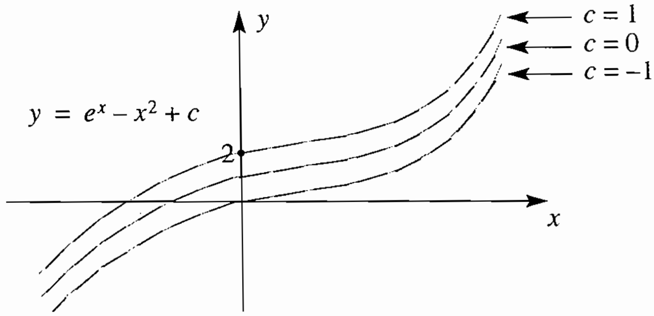
In the core section of the syllabus differential equations were solved analytically through the use of specific methods. In this section we shall solve differential equations geometrically. We do this by considering the **slope field** generated by the differential equation.

So, what is a slope field? Before we answer this question we quickly review what solving a differential equation involves.

Consider the d.e. $\frac{dy}{dx} = e^x - 2x$. The solution to this equation is given by $y = e^x - x^2 + c$, where c is a real constant – a constant that is uniquely determined by making use of an initial condition. This provides us with a family of curves – i.e., for different values of c we end up with different solution curves (as shown below):



Using these graphs, we can 'display' the gradient (or slope) at points on the curve. We do this by drawing line segments along the curves, which basically 'point the way' along the solution curve.



In short, the slope field displays the general direction (or flow) of the curve.

Alright then, having described a slope field – why are they important? Well, as the slope fields represent the values of the derivative along the curve there is no need to solve the d.e. to obtain a solution curve. That is, we can 'visualize' the solution curve by simply looking at the slope field [or at the very least, observe the behaviour of the solution curve]. And all of this is done without the need to obtain the solution curve – hence the strength and usefulness of this method – that is, obtaining a 'solution' by making use of the information that is based only the values of y', that is of $\frac{dy}{dx}$.

Using the TI–89, we can easily obtain many more slope lines of the d.e. $\frac{dy}{dx} = e^x - 2x$, with Figure A showing only the slope field and Figure B showing both the slope field and the three cases of the solution curve passing through the initial points $(0, 0)$, $(0, 1)$ and $(0, 2)$:

Figure A

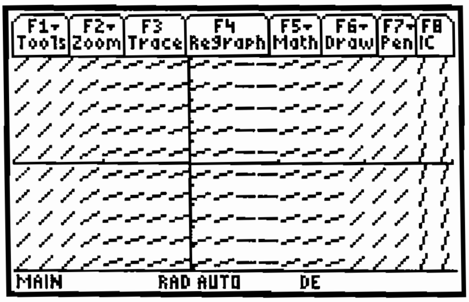
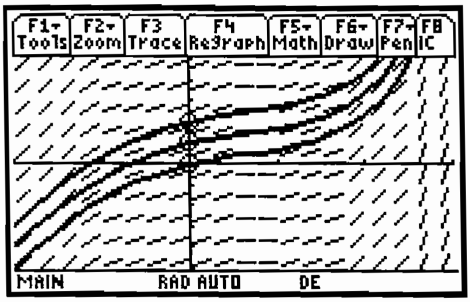


Figure B



Next, we consider an example and produce a solution showing you a step-by-step approach for how to generate the slope field.

EXAMPLE 5.1

Sketch the slope field for the d.e. $\frac{dy}{dx} = x + y$ and in particular, sketch the graph of the solution curve passing through the point $(0, 1)$.

Solution

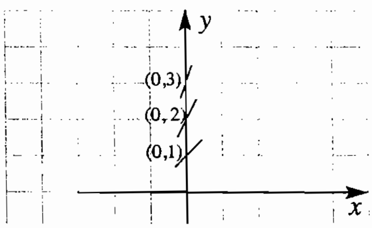
The d.e. $\frac{dy}{dx} = x + y$ informs us that the gradient of the solution curve is given by the sum of the coordinates, x and y.

So, for example, at the coordinates $(0, 1)$, $\frac{dy}{dx} = 0 + 1 = 1$.

Similarly, at the point with coordinates $(0, 2)$, $\frac{dy}{dx} = 0 + 2 = 2$.

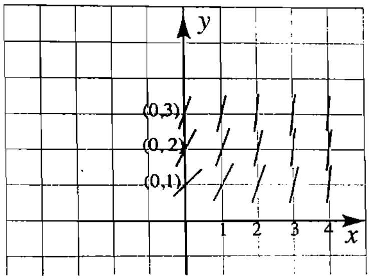
Then, at the point with coordinates $(0, 3)$, $\frac{dy}{dx} = 1 + 2 = 3$.

We start by drawing in the three slope lines we have found so far:



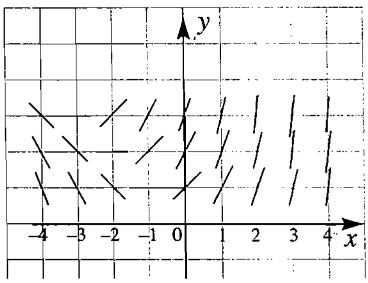
Then we simply continue with this process. Obviously a table of values will help.

	1	2	3	4	1	2	3	4	1	2	3	4
	1	1	1	1	2	2	2	2	3	3	3	3
	2	3	4	5	3	4	5	6	4	5	6	7

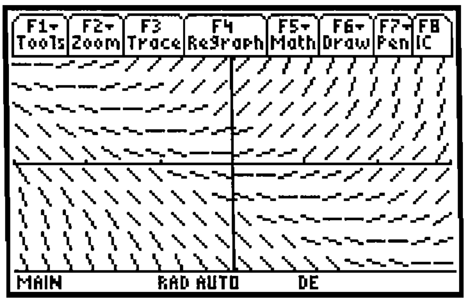


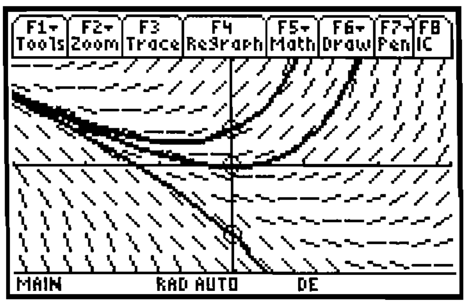
We obviously need more slope lines to have a better idea of the behaviour of the solution curve.

[redacted]	–1	–2	–3	–4	–1	–2	–3	–4	–1	–2	–3	–4
	1	1	1	1	2	2	2	2	3	3	3	3
	0	–1	–2	–3	1	0	–1	–2	2	1	0	–1



Of course, this does take some time, but the more coordinates we use the clearer the behaviour of the solution curve becomes. As with all repetitive processes, the use of a graphics calculator or spreadsheet goes a long way to making life easier. We plot the slope lines using the TI–89 and include a second display that includes the solution curve that passes through the point (0,0), (0, 1) and (0, –2):





Note that it is important that the slope lines are joined correctly! By this, we mean that the 'natural flow' of the solution curve is used to identify the different curves. This means that the line segments should be short and that as many coordinates (as practicable) are used – otherwise it is possible to accidently join the slope lines from two different solution curves!

EXAMPLE 5.2

Sketch the slope field for the d.e. $\frac{dy}{dx} = xy + 1$ and in particular, sketch the graph of the solution curves passing through the points (0, 0), (0, 1) and (0, 2).

Solution

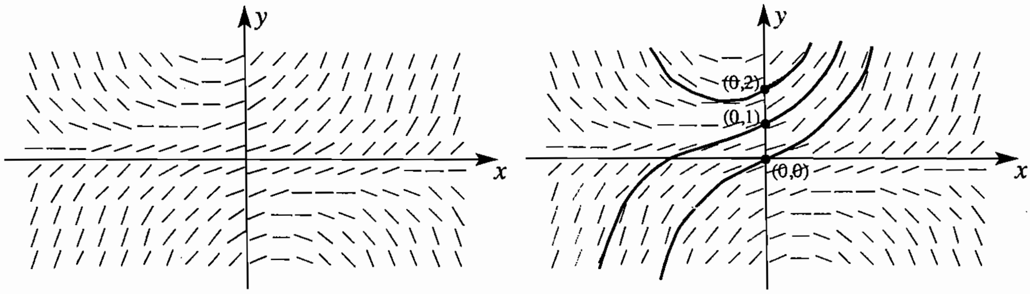
We start by setting up a table of values for x, y and $\frac{dy}{dx} = xy + 1$:

For example, at the point (–3, 1), $\frac{dy}{dx} = -3 \times 1 + 1 = -2$.

	–3	–2	–1	0	1	2	3	–3	–2	–1	0	1	2	3
	1	1	1	1	1	1	1	2	2	2	2	2	2	2
	–2	–1	0	1	2	3	4	–5	–3	–1	1	3	5	7

Again, it is clear that many more gradient values need to be determined so that the behaviour of the solution curves can be accurately observed.

Slope field:



EXERCISES 5.2

1. Construct the slope field for each of the following differential equations:

(a) $\frac{dy}{dx} = x + 1$ (b) $\frac{dy}{dx} = y + 1$ (c) $\frac{dy}{dx} = y - x$

2. On the slope fields of Q.1., draw the solution curve that passes through the point (1, 1).

3. (a) Sketch the slope field of the d.e., $\frac{dy}{dx} = \frac{x}{y}$.

(b) Draw the solution curve on the slope field such that $x = 2$ when $y = 1$.

4. (a) Sketch the slope field for the d.e., $\frac{dy}{dx} = x^2 + y^2$.

(b) If the solution curve passes through the point (0, 1), find an approximate value of y when $x = 1$.

5. Consider the differential equation $\frac{dy}{dx} = \frac{y}{x^2 - x}$.

(a) Determine the values of x for which the d.e., is undefined.
(b) Sketch the slope field for this d.e.
(c) Draw the curve that passes through the point $(2, 1), x > 0$.

6. Sketch the slope field for the following differential equations, identifying the solution curve that passes through the point (1, 1).

(a) $\frac{dy}{dx} = 1 - xy$ (b) $\frac{dy}{dx} = 1 - \frac{y}{x}$

7. Sketch the slope field for the following differential equations, identifying the solution curve that passes through the point (1, 1):

(a) $\frac{dy}{dx} = \frac{y-1}{x-2}$ (b) $\frac{dy}{dx} = \frac{x-y}{x+y}$

8. Construct the slope field for the differential equation $\frac{dy}{dx} = xe^{-y}$, identifying the solution curve that passes through the point (0, 1).

9. Construct the slope field for the differential equation $\frac{dy}{dx} = 2x - y^2$, identifying the solution curve that passes through the point

(a) (0, 1).
(b) (0, –1).

10. Consider the differential equation $\frac{dy}{dx} = \frac{ax - y}{ax + y}$. The gradient of the solution curve at the point (2, 1) is 0.5.

(a) Determine the value of a.
(b) Construct the slope field for the differential equation using the value of a in (a).
(c) Draw the solution curve passing through the point
  i. (0, 1).
  ii. (0, –1).

5.3 NUMERICAL METHOD

5.3.1 EULER'S METHOD

Euler's method is a procedure used to construct a numerical approximation for the solution curve of a first-order d.e., $\frac{dy}{dx} = f(x, y)$ such that the solution curve must satisfy the initial condition $y = y_0$ when $x = x_0$.

Euler's method is based on the use of the linear approximation method derived from differential calculus. That is, while we know that $f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$, replacing '$\lim_{h \to 0}$' with '$h$ is small', we have that

$$f'(x) \approx \frac{f(x+h) - f(x)}{h} \text{ for small } h.$$

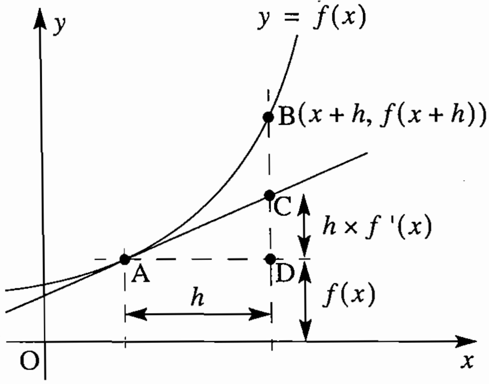
That is,

For small h, $f(x+h) \approx f(x) + h \times f'(x)$

So, how does this help and what does it mean?

From the diagram, the y–coordinate of point B is given by $f(x+h)$ and the y–coordinate of point C is given by $f(x) + h \times f'(x)$, where point C lies on the tangent line to the curve at point $A(x, f(x))$.

This means that the y–coordinate of B can be approximated by the y–coordinate of point C and so, we have that $f(x+h) \approx f(x) + h \times f'(x)$.



Of course, the smaller the value of h the closer point C will be to point B and so the better the approximation.

Let's see how this works by making use of only the derivative function and an initial point that is known to belong on the solution curve. Let's say that the derivative function, $f'(x) = \frac{1}{2}x + 1$, that the solution curve passes through the point (2,3) and that we want to determine an approximate value for $f(x)$ at $x = 2.1$.

While it is true that we could easily determine the equation of the function $y = f(x)$ from the information given, the point here is to only use the derivative function (or differential equation) and the initial point.

Using our linear approximation we have, $f(2.1) = f(2 + 0.1) \approx f(2) + 0.1 \times f'(2)$.

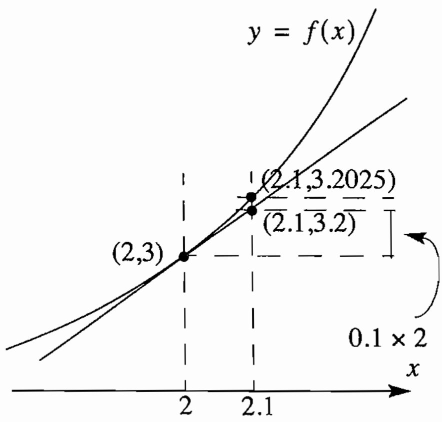
[i.e., $h = 0.1$, $f(2) = 3$ and $f'(2) = \frac{1}{2} \times 2 + 1 = 2$].

This gives, $f(2.1) \approx 3 + 0.1 \times 2 = 3.2$.

For the sake of making a comparison, we state what the original function, passing through (2, 3), was.

$$f(x) = \frac{1}{4}x^2 + x.$$

Now, $f(2.1) = \frac{1}{4} \times (2.1)^2 + 2.1 = 3.2025$.



Our approximation of 3.2 turns out to be pretty good!

The key here is that we were able to determine an approximate value of the function given only information about its derivative function (i.e., a differential equation) and an initial value.

We also note that if we wished to determine the approximate value of $f(3)$ given the same information, i.e., that $f'(x) = \frac{1}{2}x + 1$ and that the curve passes through the point (2, 3), we would have that

$$\begin{aligned} f(3) &= f(2+1) \approx f(2) + 1 \times f'(2) \\ &= 3 + 1 \times 2 \\ &= 5. \end{aligned}$$

The actual value is in fact, $f(3) = \frac{1}{4} \times 3^2 + 3 = 5.25$ – which is not as good an approximation as the previous approximation. This highlights the importance when determining approximations, of making use of *small values of h* to improve the approximation.

The need to use small values of h would mean, for the example we've just looked at, that rather than using $h = 1$ (in getting from $x = 2$ to $x = 3$) we could make use of 10 smaller steps, getting us from $x = 2$, to $x = 2.1$ to $x = 2.2$, ..., to $x = 2.9$ to $x = 3.0$. We consider how the first couple of steps would work:

We've already worked out the result for $x = 2$, to $x = 2.1$, where we obtained $f(2.1) \approx 3.2$.

Next, to determine $f(2.2)$ we have:

$$\begin{aligned} f(2.2) &= f(2.1 + 0.1) \approx f(2.1) + 0.1 \times f'(2.1) \\ &\approx 3.2 + 0.1 \times \left(\frac{1}{2} \times 2.1 + 1\right) \\ &= 3.405. \end{aligned}$$

Similarly, to determine $f(2.3)$ we would have:

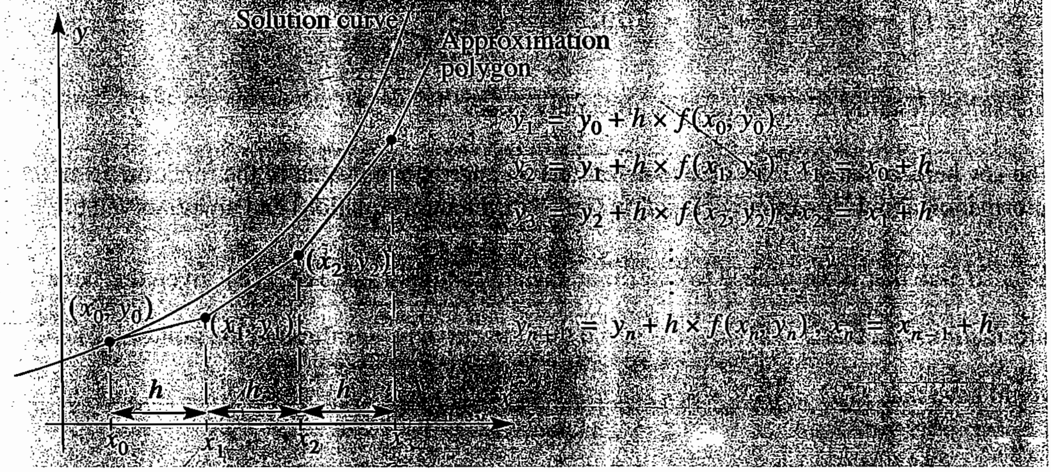
$$\begin{aligned} f(2.3) &= f(2.2 + 0.1) \approx f(2.2) + 0.1 \times f'(2.2) \\ &\approx 3.405 + 0.1 \times \left(\frac{1}{2} \times 2.2 + 1\right) \\ &= 3.615. \end{aligned}$$

In the same way we can continue applying this process repeatedly – i.e., we can generate an *iterative process*.

Before proceeding with a formal look at the derivation of Euler's approximation, we provide a summary of what is needed to obtain an approximation.

Given a d.e., $\frac{dy}{dx} = f(x, y)$ (i.e., the derivative function) with an initial value (x_0, y_0) on the solution curve (i.e., on the function), the successive approximations to the solution curve are generated by the *recurrence equation* $y_{n+1} = y_n + h \times f(x_n, y_n)$ where $x_{n+1} = x_n + h$ and where h is small.

This process generates a set of approximations $y_1, y_2, \ldots, y_{n+1}$ given by:



We have used the expression $\frac{dy}{dx} = f(x, y)$ to highlight the fact that the d.e., can have both x and y terms in its equation. For example, we could have a d.e., defined by $\frac{dy}{dx} = 2x + y$ rather than by an expression in terms of x only.

It is important to recognise this fact because, while it might be that if the d.e. is entirely in terms of x it will then be easily solved, with the additional y–terms, an algebraic solution might not be possible. If this is so, our only recourse is to obtain an approximate solution using a numerical method.

EXAMPLE 5.3

Find an approximate value of y at $x = 5$ given that $\frac{dy}{dx} = \sqrt{xy+1}$ and that the curve passes through the point (4.5, 1). Give the answer to 3 decimal places.

Solution

Starting with $x_0 = 4.5$, $y_0 = 1$ and using a step of 0.1, i.e., $h = 0.1$ in the recursive equation $y_{n+1} = y_n + h \times f(x_n, y_n)$, where $f(x, y) = \sqrt{xy+1}$, we have:

$y_1 = y_0 + h \times f(x_0, y_0) = 1 + 0.1 \times \sqrt{4.5 \times 1 + 1} = 1.2345$.

Next, $x_1 = x_0 + 0.1 = 4.5 + 0.1 = 4.6$.

$y_2 = y_1 + h \times f(x_1, y_1) \approx 1.2345 + 0.1 \times \sqrt{4.6 \times 1.2345 + 1} = 1.4929$.

Next, $x_2 = x_1 + 0.1 = 4.6 + 0.1 = 4.7$.

$$y_3 = y_2 + h \times f(x_2, y_2) \approx 1.4929 + 0.1 \times \sqrt{4.7 \times 1.4929 + 1} = 1.7760 .$$

Next, $x_3 = x_2 + 0.1 = 4.7 + 0.1 = 4.8$.

$$y_4 = y_3 + h \times f(x_3, y_3) \approx 1.7760 + 0.1 \times \sqrt{4.8 \times 1.7760 + 1} = 2.0846 .$$

Next, $x_4 = x_3 + 0.1 = 4.8 + 0.1 = 4.9$.

$$y_5 = y_4 + h \times f(x_4, y_4) \approx 2.0846 + 0.1 \times \sqrt{4.9 \times 2.0846 + 1} = 2.4195 .$$

Next, $x_5 = x_4 + 0.1 = 4.9 + 0.1 = 5.0$.

$$y_6 = y_5 + h \times f(x_5, y_5) \approx 2.4195 + 0.1 \times \sqrt{4.9 \times 2.4195 + 1} = 2.7780 .$$

That is, an approximate solution at $x = 5$ is $y = 2.778$ (to 3 d.p.).

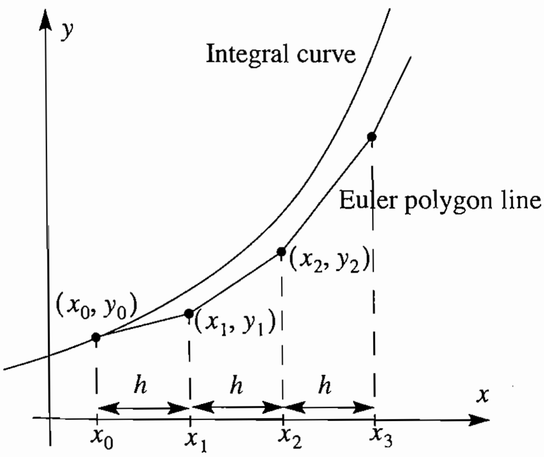
In the example above, it can be useful to write the recursive equation as

$$y_{n+1} = y_n + h \times f(x_n, y_n) = y_n + h \times \sqrt{x_n y_n + 1}$$

In this form it is then easier to make use of the computational powers of your graphics calculator or indeed, of a spreadsheet.

Having demonstrated Euler's method informally, we now look at how it is constructed more formally.

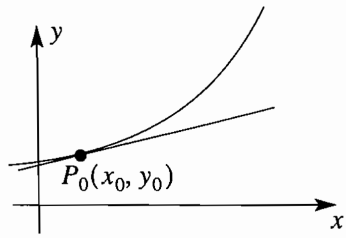
Geometrically we want to find the integral curve (or solution curve) passing through the point $P_0(x_0, y_0)$. Euler's method contructs, from the starting point P_0, successive approximations for the solution at equally spaced x-values, $x_1, x_2, x_3, \ldots$, such that $x_{n+1} = x_n + h$ where h is the constant **step size**. Once the corresponding values of $y_1, y_2, y_3, \ldots$ are calculated by the method detailed below, the coordinates of the points $P_1, P_2, P_3, \ldots, P_n$ will be generated. Once we have the points we join these points by straight line segments (called the **Euler polygon line**) to obtain an approximation to the integral curve. We shall see that as the step size $h \to 0$, we will improve the accuracy of the polygonal line and thus get a better approximation to the integral curve.



Let us describe Euler's method in detail:

From the starting point $P_0(x_0, y_0)$ we draw the line with initial slope $\frac{dy}{dx} = f(x_0, y_0)$. This has the straight line equation

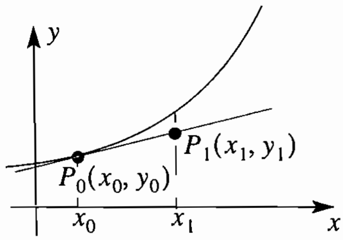
$$(y - y_0) = f(x_0, y_0)(x - x_0)$$



It follows that this line passing through $P_0(x_0, y_0)$ is tangent to the integral curve.

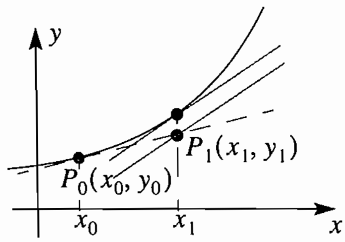
On this line we take the next point $P_1(x_1, y_1)$, where $x_1 = x_0 + h$ (or $h = x_1 - x_0$) and the corresponding value of y will be given by

$$y_1 = y_0 + f(x_0, y_0)(x_1 - x_0)$$
$$= y_0 + h \times f(x_0, y_0)$$



This value will then be the approximate value of the solution at $x_1 = x_0 + h$. Through this point $P_1(x_1, y_1)$ we next draw a line whose slope is $\frac{dy}{dx} = f(x_1, y_1)$ with equation

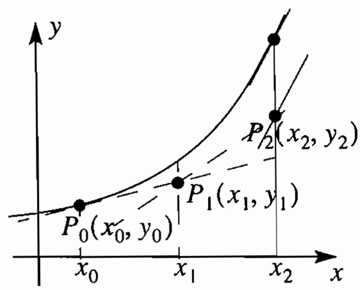
$$(y - y_1) = f(x_1, y_1)(x - x_1)$$



Then, with $x_2 = x_1 + h$, (or $h = x_2 - x_1$) the corresponding value of y_2 will accordingly be

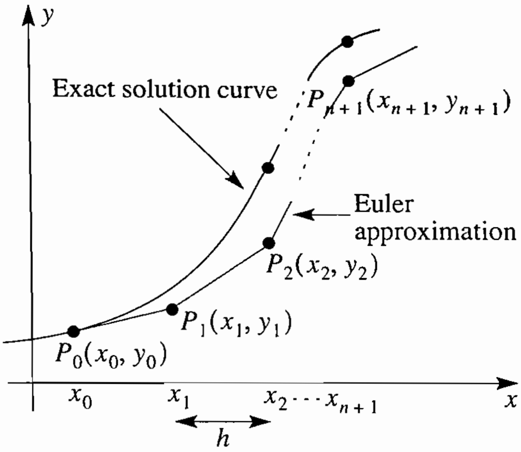
$$y_2 = y_1 + f(x_1, y_1)(x_2 - x_1)$$
$$= y_1 + h \times f(x_1, y_1)$$

which will be an approximate solution at $x = x_2$.



The next approximation will be the line drawn through the point $P_2(x_2, y_2)$, which will lead to generating the point P_3 and so on.

The method can then be generalised as constructing a sequence of coordinates (x_n, y_n) such that $x_{n+1} = x_n + h$ is used to generate the values $y_{n+1} = y_n + h \times f(x_n, y_n)$



The smaller the step size h, the better out approximation will be. In order to give a better and more precise idea of the method, let us now see an example to see how it all works out.

EXAMPLE 5.4

Consider the d.e. $\frac{dy}{dx} = \frac{xy}{2}$. Find an approximation to the value of y_9 when $x_9 = 0.9$, given that the curve passes through the point (0, 1).

Solution

The initial point is (0, 1) and we need to determine the value of y_9. To get from x_0 to x_9 take 10 steps, so that our step size is $h = 0.1$. This gives:

$$x_0 = 0, x_1 = 0.1, x_2 = 0.2, x_3 = 0.3, \ldots x_9 = 0.9$$

Therefore the corresponding values of y are given by

$$y_1 = y_0 + h \times f(x_0, y_0) = y_0 + 0.1 \times \frac{x_0 y_0}{2} = 1 + 0.1 \times \frac{0 \times 1}{2} = 1.$$

Next, $$y_2 = y_1 + h \times f(x_1, y_1) = y_1 + 0.1 \times \frac{x_1 y_1}{2} = 1 + 0.1 \times \frac{0.1 \times 1}{2} = 1.005.$$

Similarly, $$y_3 = y_2 + h \times f(x_2, y_2) = y_2 + 0.1 \times \frac{x_2 y_2}{2} = 1.005 + 0.1 \times \frac{0.2 \times 1.005}{2}$$
$$= 1.0151.$$

Next, $$y_4 = y_3 + h \times f(x_3, y_3) = y_3 + 0.1 \times \frac{x_3 y_3}{2} = 1.0151 + 0.1 \times \frac{0.3 \times 1.0151}{2}$$
$$= 1.0303.$$

Next, $$y_5 = y_4 + h \times f(x_4, y_4) = y_4 + 0.1 \times \frac{x_4 y_4}{2} = 1.0303 + 0.1 \times \frac{0.4 \times 1.0303}{2}$$
$$= 1.0509.$$

The process continues in the same way using the recursive equation

$$y_{n+1} = y_n + h \times \frac{x_n y_n}{2}, x_{n+1} = x_n + h.$$

This produces the following results:

0	0	1
1	0.1	1
2	0.2	1.005
3	0.3	1.0151
4	0.4	1.0303
5	0.5	1.0509
6	0.6	1.0772
7	0.7	1.1095
8	0.8	1.1483
9	0.9	1.1942

Therefore, the approximate value when $x = 0.9$ is $y = 1.1942$.

In order to appreciate the accuracy of Euler's method we could try to solve the original d.e., and make a comparison. Namely, we need to solve for y when $x = 0.9$ given that $\frac{dy}{dx} = \frac{xy}{2}$ and that the curve passes through the point with coordinates (0, 1):

From $\frac{dy}{dx} = \frac{xy}{2}$ we have $\frac{1}{y}\frac{dy}{dx} = \frac{x}{2} \Rightarrow \int \frac{1}{y}\frac{dy}{dx}dx = \frac{1}{2}\int x dx$

$$\ln y = \frac{1}{4}x^2 + c.$$

Solving for c we use (0, 1): $\ln 1 = \frac{1}{4}(0)^2 + c \Leftrightarrow c = 0$.

Therefore we have that $\ln y = \frac{1}{4}x^2 \Leftrightarrow y = e^{\frac{1}{4}x^2}$.

We can now calculate the value of this function for $x = 0.9$ which is $y = 1.2245$. Upon comparison with Euler's approximation, the percentage error is just 2.5% – not bad at all.

EXAMPLE 5.5

Consider the d.e., $\frac{dy}{dx} = -0.2(xy - 50)$. Find the approximation to the value of y when $x = 10$, given that the curve passes through the point (0, 100).

Solution

Starting with $x = 0$ and using steps of 1 (i.e., $h = 1$), we obtain the sequence

$$x_0 = 0, x_1 = 1, x_2 = 2, \ldots, x_9 = 9, x_{10} = 10$$

With $\frac{dy}{dx} = -0.2(xy - 50)$, we have the recursive equation is $y_{n+1} = y_n + h \times f(x_n, y_n)$.

Then, with $f(x_n, y_n) = -0.2(x_n y_n - 50)$, $x_0 = 0, y_0 = 100$ and $h = 1$ we have:

$$y_{n+1} = y_n + 1 \times [-0.2(x_n y_n - 50)]$$
$$= y_n - 0.2(x_n y_n - 50).$$

$$
\begin{aligned}
n=0, \qquad y_1 &= y_0 - 0.2(x_0y_0 - 50) = 100 - 0.2(0 \times 100 - 50) \\
&= 110\,. \\
n=1, \qquad y_2 &= y_1 - 0.2(x_1y_1 - 50) = 110 - 0.2(1 \times 110 - 50) \\
&= 98\,. \\
n=2, \qquad y_3 &= y_2 - 0.2(x_2y_2 - 50) = 98 - 0.2(2 \times 98 - 50) \\
&= 68.8\,.
\end{aligned}
$$

Obviously, a spreadsheet is very useful at this stage. Two such outputs are shown below:

Table for step size $h = 1$:

n	x(n)	y(n)	
0	0	100	
1	1	110	110
2	2	98	98
3	3	68.8	68.8
4	4	37.52	37.52
5	5	17.504	17.504
6	6	10	10
7	7	8	8
8	8	6.8	6.8
9	9	5.92	5.92
10	10	5.264	5.264

Table for step size $h = 0.1$:

n	x(n)	y(n)	
0	0	100	
1	0.1	110	110
2	0.2	117.8	117.8
3	0.3	123.088	123.088
4	0.4	125.70272	125.70272
5	0.5	125.646502	125.6465024
6	0.6	123.081852	123.0818522
7	0.7	118.31203	118.3120299
8	0.8	111.748346	111.7483457
9	0.9	103.86861	103.8686104
10	1	95.1722605	95.17226053
11	1.1	86.1378084	86.13780842
12	1.2	77.1874906	77.18749057

89	8.9	5.65371362	5.653713622
90	9	5.59010337	5.590103374
91	9.1	5.5279173	5.5279173
92	9.2	5.46710781	5.467107814
93	9.3	5.40762944	5.407629437
94	9.4	5.34943868	5.349438685
95	9.5	5.29249396	5.292493958
96	9.6	5.23675544	5.236755438
97	9.7	5.182185	5.182184997
98	9.8	5.1287461	5.128746103
99	9.9	5.07640374	5.076403741
100	10	5.02512433	5.025124334

y(n-1) – 0.2(x(n-1)y(n-1) – 50)

This gives us the (approximate) result that when $x = 10, y = 5.264$ (using step size of $h = 1$) and $y = 5.025$ (using step size of $h = 0.1$).

EXERCISES 5.3

1. (a) i. Solve the differential equation $\frac{dy}{dx} = 2x + 1$ given that the curve passes through the point $(0, 0)$.

ii. Determine the value of y when $x = 2$.

(b) i. Using step sizes of $h = 0.2$, use Euler's method to approximate the value of y when $x = 2$.

ii. Calculate the percentage error involved in using this approximation.

2. (a) i. Solve the differential equation $\frac{dy}{dx} = 3x^2, y(1) = 2$.

ii. Determine the value of y when $x = 3$.

(b) i. Using step sizes of $h = 0.1$, use Euler's method to approximate the value of y when $x = 3$.

ii. Calculate the percentage error involved in using this approximation.

3. (a) For the d.e., $\frac{dy}{dx} = y + 1$, $y(2) = 0$, use Euler's method with step size 0.5 to find an approximate value for y when $x = 4$.

(b) Next, using step sizes of 0.2, calculate an approximate value of y when $x = 4$.

(c) Compare the error involved in using step sizes of 0.5 and 0.2 when determining an approximate value of y when $x = 4$.

4. (a) Consider the initial-value problem, $\frac{dy}{dx} = 2xy, y(1) = 2$.

Use Euler's method to calculate an approximate value of $y(2)$ using

i. step sizes of 0.1.

ii. step sizes of 0.2.

(b) Compare the results of (a) with the exact value of $y(2)$.

5. Using Euler's method with a step size of 0.2 calculate an approximate value of y when $x = 1$ for the d.e., $\frac{dy}{dx} = \frac{x-y}{x+y}$ given that the curve passes through the point (0, 1).

6. Use Euler's method, with a step size of 0.5, to calculate an approximate value for $y(4)$ for the solution of $\frac{dy}{dx} = (x + y - 1)^2$ given that the curve passes through the point (0, 2).

7. Use Euler's method, with a step size of 0.1, to calculate an approximate value for $y(1)$ for the solution of $\frac{dy}{dx} = x^2 + y^2$ given that the curve passes through the point (0, 1).

8. Given the initial-value problem, $\frac{dy}{dx} = y - y^2$, $y(0) = 0.5$, find an approximate value of $y(1)$ using Euler's method with step size $h = 0.2$.

9. Given the initial-value problem, $\frac{dy}{dx} = y + y^2$, $y(0) = 0.5$, find an approximate value of $y(1)$ using Euler's method with step size $h = 0.2$.

10. Use Euler's method, with a step size of 0.1, to calculate an approximate value for $y(2.6)$ given that $\frac{dy}{dx} = 3x - 2y + 1$ and that the curve passes through the point (2, 5).

11. (a) Sketch the slope field along with the solution curve to the initial-value problem, $\frac{dy}{dx} = x + y^2$, $y(0) = 1$.

(b) Using Euler's method with step size $h = 0.2$, calculate an approximate value of $y(1)$.

12. Given the initial-value problem, $\frac{dy}{dx} = \sin(x^2)$, $y(0) = 2$, find an approximate value of $y(2)$ using Euler's method with step size $h = 0.5$.

13. (a) i. Solve the initial-value problem $\frac{dy}{dx} = 3x^2 - 2x$, $y(1) = 0$.

ii. Calculate $y(3)$.

(b) Using Euler's method find an approximate value of $y(3)$ using step sizes of

i. $h = 0.5$.

ii. $h = 0.1$.

(c) Determine the percentage error in using Euler's method in calculating $y(3)$ with

i. $h = 0.5$.

ii. $h = 0.1$.

14. (a) i. Show that the equation $\sqrt{xy} - x = 1$ satisfies the initial-value problem

$$\frac{dy}{dx} = \frac{2\sqrt{xy} - y}{x},\ y(1) = 4.$$

ii. Calculate $y(2)$.

(b) Use Euler's method to find an approximate value of $y(2)$ using step sizes of

i. $h = 0.2$.

ii. $h = 0.1$.

(c) Determine the percentage error in using Euler's method in calculating $y(3)$ with

i. $h = 0.2$.

ii. $h = 0.1$.

5.4 TWO NEW FIRST ORDER D.E.S

5.4.1 HOMOGENEOUS FIRST ORDER DIFFERENTIAL EQUATION

Sometimes we can use a substitution that will enable us to reduce a difficult differential equation to one that can be solved either by the method of separation of variables or some other method. Such equations are known as **Homogeneous first-order differential equations**.

If a function F of two variables x and y (say) is such that $F(x, y) = F(tx, ty)$ for all values of x, y and t, we say that F is homogeneous (of degree zero).

For example, the function $F(x, y) = \frac{x^2 - y^2}{xy}$ is a homogeneous function, because

$$F(tx, ty) = \frac{(tx)^2 - (ty)^2}{(tx)(ty)} = \frac{t^2(x^2 - y^2)}{t^2xy} = \frac{x^2 - y^2}{xy} = F(x, y).$$

This result shows that $F(x, y)$ only depends on the ratio $\frac{y}{x}$, and so, F can be considered as a function of only one variable, $u = \frac{y}{x}$.

For the function above, we have $F(x, y) = \frac{x^2 - y^2}{xy} = \frac{\frac{x^2 - y^2}{x^2}}{\frac{xy}{x^2}} = \frac{1 - \left(\frac{y}{x}\right)^2}{\left(\frac{y}{x}\right)} = \frac{1 - u^2}{u}$.

That is, we have that $F(x, y) = f(u)$, where $u = \frac{y}{x}$.

We are now in a position to consider this new type of differential equation:

A homogeneous first-order differential equation is an equation of the form

$$\frac{dy}{dx} = F(x, y) \text{ or } \frac{dy}{dx} = f\left(\frac{y}{x}\right)$$

where F is a homogeneous function.

From the discussion above, it seems appropriate to use the substitution, $y = ux$.

That is, with $y = ux$, we have $\frac{dy}{dx} = u\frac{dx}{dx} + x\frac{du}{dx} = u + x\frac{du}{dx}$.

As $\frac{dy}{dx} = F(x, y)$, and $F(x, y) = f(u)$, we have $u + x\frac{du}{dx} = f(u)$

Therefore, we have $$x\frac{du}{dx} = f(u) - u$$

which is a **variable separable** type d.e. We can now solve this d.e. using the techniques used in the core section of the course.

EXAMPLE 5.6

Solve the differential equation $\frac{dy}{dx} = \frac{x^2 - y^2}{xy}$.

Solution

The d.e., $\frac{dy}{dx} = \frac{x^2 - y^2}{xy}$ is a homogeneous first order differential equation and so we use the substitution $y = ux \Rightarrow \frac{dy}{dx} = u + x\frac{du}{dx}$. This transforms our d.e. to

$$u + x\frac{du}{dx} = \frac{x^2 - (ux)^2}{x(ux)}$$

$$= \frac{x^2(1 - u^2)}{x^2 u}$$

That is,
$$u + x\frac{du}{dx} = \frac{1 - u^2}{u}$$

Therefore, we have
$$x\frac{du}{dx} = \frac{1 - u^2}{u} - u$$

$$\Rightarrow x\frac{du}{dx} = \frac{1 - 2u^2}{u}$$

$$\Rightarrow \frac{u}{1 - 2u^2}\frac{du}{dx} = \frac{1}{x}$$

Integrating b.s.w.r.t. x:
$$\int \frac{u}{1 - 2u^2}\frac{du}{dx}dx = \int \frac{1}{x}dx$$

Therefore,
$$-\frac{1}{4}\ln(1 - 2u^2) = \ln x + c$$

Substituting back for $y = ux$, i.e., $u = \frac{y}{x}$, we have;

$$-\frac{1}{4}\ln\left(1 - 2\left(\frac{y}{x}\right)^2\right) = \ln x + c$$

That is, $x^4 - 2x^2y^2 = k$, where $k > 0$.

EXAMPLE 5.7

Solve the initial value problem $x\frac{dy}{dx} = y + xe^{y/x}$, $y(1) = 2$.

Solution

We observe that $x\frac{dy}{dx} = y + xe^{y/x}$ (after dividing through by x) can be written as

$$\frac{dy}{dx} = \frac{y}{x} + e^{y/x}.$$

Next we observe that the ratio $\frac{y}{x}$ appears prominently in the d.e., and so we use the substitution $u = \frac{y}{x}$ from where we have $y = ux \Rightarrow \frac{dy}{dx} = u + x\frac{du}{dx}$.

Therefore, the d.e., is transformed to $u + x\frac{du}{dx} = u + e^u$.

Hence, $x\frac{du}{dx} = e^u$

$\Leftrightarrow e^{-u}\frac{du}{dx} = \frac{1}{x}$

Antidifferentiating b.s.w.r.t x: $\therefore \int e^{-u}\frac{du}{dx}dx = \int \frac{1}{x}dx$

$-e^{-u} = \ln x + c$

Substituting back for u, we have: $-e^{-y/x} = \ln x + c$

When $x = 1, y = 2$ so that $-e^{-2} = \ln 1 + c \Leftrightarrow c = -e^{-2}$.

Therefore, $-e^{-y/x} = \ln x - e^{-2}$ or, $\ln x = e^{-2} - e^{-y/x}$.

The key to solving such differential equations is to recognise the need to use the substitution $u = \frac{y}{x}$ so that the original d.e., can be reduced to a more manageable d.e., which can be solved by some method that we are familiar with.

EXERCISES 5.4.1

1. Find the general solution to the homogeneous first order differential equations:

(a) $\frac{dy}{dx} = \frac{x+y}{x}$ (b) $\frac{dy}{dx} = \frac{y^2+xy}{x^2}$ (c) $2xy\frac{dy}{dx} = 3y^2 - x^2$

(d) $\frac{dy}{dx} - \frac{y}{x} = e^{\frac{y}{x}}$ (e) $\frac{dy}{dx} = \frac{x-y}{x}$ (f) $\frac{dy}{dx} = \frac{x-y}{x+y}$

2. Find the particular solution to the initial value problem, $\frac{dy}{dx} = \frac{3y^2+x^2}{2xy}$, $y(1) = 2$.

3. Find the particular solution for the differential equation $(x^2+y^2)\frac{dy}{dx} = xy$, given that when $x = 1, y = 1$.

4. By using the substitution $u = x + y$, show that the differential equation $\frac{dy}{dx} = x + y$ can be reduced to the d.e. $\frac{du}{dx} = u + 1$.

Hence show that the general solution is given by $x + y + 1 = ke^x$.

5. Use the substitution $u = x + y + 1$, to show that the differential equation $\frac{dy}{dx} = \frac{1}{x+y+1}$ can be reduced to the d.e., $\frac{du}{dx} - 1 = \frac{1}{u}$.

Hence show that the general solution is given by $x + y + 2 = ke^y$.

6. Use the substitution $y = ux$, to show that the differential equation $x\frac{dy}{dx} = y^2 + x^2 + y$ can be reduced to the d.e., $\frac{du}{dx} = u^2 + 1$.

Hence show that the general solution is given by $y = x\tan(x + c)$.

7. Use the substitution $y = xu$, to show that the d.e., $\frac{dy}{dx} = \frac{y + \sqrt{x^2 + y^2}}{x}, x > 0$ can be reduced to the d.e., $x\frac{du}{dx} = \sqrt{u^2 + 1}$.

Hence show that if the curve passes through the point $(1, 0)$, the particular solution is given by $y = \frac{1}{2}(x^2 - 1)$.

8. Solve the initial-value problem $x\frac{dy}{dx} = y + x^2\sin x$, $\left(\frac{\pi}{2}, \frac{\pi}{2}\right)$.

9. Solve the initial-value problem $xy\frac{dy}{dx} = y^2 - x^2$, $(1, 0)$.

10. Solve the initial-value problem $x\frac{dy}{dx} - 2y = x$, $(1, 1)$.

5.4.2 SOLVING DIFFERENTIAL EQUATIONS USING THE INTEGRATING FACTOR

We begin by reviewing the form of a first-order linear equation, namely;

A first-order linear differential equation, has the form

$$\frac{dy}{dx} + P(x)y = Q(x),$$

where $P(x)$ and $Q(x)$ are continuous functions defined on some interval $a < x < b$.

The reason that this type of d.e. is referred to as linear and of the first order is that the dependent variable y and $\frac{dy}{dx}$ both occur in the first degree.

The simplest case of first order d.e.s. occurs when $P(x) = 0$, giving us the d.e., $\frac{dy}{dx} = Q(x)$. If $Q(x)$ is 'nice', this can be easily solved. However, when $P(x)$ is given as some function of x or even a constant, we need to use a 'trick' (or rather method) similar to what we needed to do when we used the substitution $y = ux$ in section 5.4.1 – the aim of which was to reduce a difficult d.e., into one that is more manageable.

General solution process for this type of d.e.:

We start by multiplying the d.e., $\frac{dy}{dx} + P(x)y = Q(x)$ by an 'arbitrary' function of x, say, $I(x)$.

Giving the result: $$I(x)\frac{dy}{dx} + P(x)I(x)y = I(x)Q(x)$$

We take time to recall the derivative using the product rule $I(x) \times y$:

$$\frac{d}{dx}(I(x)y) = I(x)\frac{dy}{dx} + \frac{dI}{dx}y \Rightarrow I(x)\frac{dy}{dx} = \frac{d}{dx}(I(x)y) - \frac{dI}{dx}y.$$

Therefore, our original d.e. $I(x)\frac{dy}{dx} + P(x)I(x)y = I(x)Q(x)$ can be rewritten as:

$$\left[\frac{d}{dx}(I(x)y) - \frac{dI}{dx}y\right] + P(x)I(x)y = I(x)Q(x)$$

That is, $$\frac{d}{dx}(I(x)y) + \left(P(x)I(x) - \frac{dI}{dx}\right)y = I(x)Q(x).$$

So far, all that we have done is made a substitution, the 'trick' is to now choose our $I(x)$, so that $P(x)I(x) - \frac{dI}{dx} = 0$, i.e., $\frac{dI}{dx} = P(x)I(x)$, reducing the d.e. to a much more manageable one.

However, if $\frac{dI}{dx} = P(x)I(x)$, then $\frac{1}{I(x)}\frac{dI}{dx} = P(x) \Rightarrow \int\frac{1}{I(x)}\frac{dI}{dx}dx = \int P(x)dx$.

$$\Rightarrow \ln I(x) = \int P(x)dx + c$$

$$\therefore I(x) = Ae^{\int P(x)dx}.$$

However, as the constant is arbitrary and because we are not looking for a family of solutions, we can simply set $A = 1$. So we have [redacted], which is known as the **integrating factor**.

Multiplying our original d.e., by the integrating factor produces a 'nice' result, which enables us to rewrite the d.e as:

$$\frac{d}{dx}\left(ye^{\int P(x)dx}\right) = Q(x)e^{\int P(x)dx} \Rightarrow ye^{\int P(x)dx} = \int Q(x)e^{\int P(x)dx}dx,$$

from where we can then find an equation for y.

Notice that the equation is of the first order and as such, the solution will contain only one arbitrary constant. Therefore, there is no need to include an arbitrary constant when finding the integrating factor, $I(x) = e^{\int P(x)dx}$.

Even though this seems like a somewhat daunting expression, the process is straightforward, as the following example shows:

EXAMPLE 5.8

Find the general solution to the differential equation $\frac{dy}{dx} - \frac{2}{x}y = x^3$.

Solution

We recognise this as a d.e. of the form $\frac{dy}{dx} + P(x)y = Q(x)$ with $P(x) = -\frac{2}{x}$ and $Q(x) = x^3$, meaning that we can use the integrating factor:

$$I(x) = e^{\int\left(-\frac{2}{x}\right)dx} = e^{-2\ln x} = x^{-2}.$$

Therefore, using $\frac{d}{dx}\left(ye^{\int P(x)dx}\right) = Q(x)e^{\int P(x)dx}$ we have:

$$\frac{d}{dx}(yx^{-2}) = x^3 \times x^{-2}$$

$$= x.$$

Next, antidifferentiating b.s.w.r.t. x: $\int\frac{d}{dx}(yx^{-2})dx = \int x dx$

$$\therefore yx^{-2} = \frac{1}{2}x^2 + c$$

And so, $y = \frac{1}{2}x^4 + cx^2$.

EXAMPLE 5.9

Find the general solution to the differential equation $y' + 2y = 2e^x$.

Solution

In this instance we have that $P(x) = 2$ and $I(x) = e^{\int 2dx} = e^{2x}$.
Multiplying the original d.e., by e^{2x} throughout gives;

$$e^{2x}y' + 2e^{2x}y = 2e^x \times e^{2x}$$

That is, $\frac{d}{dx}(e^{2x}y) = 2e^{3x}$

Therefore, $\int\frac{d}{dx}(e^{2x}y)dx = \int 2e^{3x}dx$

So that, $e^{2x}y = \frac{2}{3}e^{3x} + c$

Therefore, $y = \frac{2}{3}e^x + ce^{-2x}$.

EXAMPLE 5.10

Find the general solution to the differential equation $\frac{dy}{dx}+\frac{y}{x+1}=x^2$

Solution

Again, we recognise this as a d.e. of the form $\frac{dy}{dx}+P(x)y = Q(x)$ with $P(x) = \frac{1}{x+1}$ and $Q(x) = x^2$, meaning that we can use the integrating factor:

$$\begin{aligned} I(x) &= e^{\int\left(\frac{1}{x+1}\right)dx} \\ &= e^{\log_e(x+1)} \\ &= x+1 \end{aligned}$$

Therefore from $\quad \frac{d}{dx}(I(x)y) = Q(x)\times I(x)$

we have $\quad \frac{d}{dx}(y(x+1)) = x^2(x+1)$

Integrating b.s.w.r.t. x we have; $\int\frac{d}{dx}(y(x+1))dx = \int x^2(x+1)dx$

$$\therefore y(x+1) = \frac{1}{4}x^4+\frac{1}{2}x^3+c$$

$$\Rightarrow y = \frac{x^4}{4(x+1)}+\frac{x^3}{2(x+1)}+\frac{c}{x+1}.$$

EXAMPLE 5.11

(a) Show that $\frac{d}{dx}(\ln(x+\sqrt{x^2+4})) = \frac{1}{\sqrt{x^2+4}}$

(b) Find the solution of the differential equation $(x^2+4)\frac{dy}{dx}+xy = 1$, if $y(0) = 1$.

Solution

(a) Using the chain rule we have;

$$\frac{d}{dx}(\ln(x+\sqrt{x^2+4})) = \frac{1+\frac{x}{\sqrt{x^2+4}}}{x+\sqrt{x^2+4}} = \frac{\frac{\sqrt{x^2+4}+x}{\sqrt{x^2+4}}}{x+\sqrt{x^2+4}} = \frac{1}{\sqrt{x^2+4}} \text{ as required.}$$

(b) We start by rearranging the d.e.,:

$$(x^2+4)\frac{dy}{dx}+xy = 1$$

$$\Rightarrow \frac{dy}{dx}+\frac{x}{(x^2+4)}y = \frac{1}{(x^2+4)}$$

This is a d.e., of the form $\frac{dy}{dx} + P(x)y = Q(x)$ with $P(x) = \frac{x}{(x^2+4)}$ and $Q(x) = \frac{1}{(x^2+4)}$, meaning that we can use the integrating factor:

$$I(x) = e^{\int\left(\frac{x}{x^2+4}\right)dx} = e^{\frac{1}{2}\ln(x^2+4)}$$
$$= e^{\ln\sqrt{(x^2+4)}}$$
$$= \sqrt{x^2+4}$$

So that
$$\frac{d}{dx}(y\sqrt{x^2+4}) = \frac{1}{(x^2+4)} \times \sqrt{x^2+4}$$
$$\frac{d}{dx}(y\sqrt{x^2+4}) = \frac{1}{\sqrt{x^2+4}}$$

Therefore, from (a), we have
$$y\sqrt{x^2+4} = \ln(x+\sqrt{x^2+4}) + c$$
$$\Rightarrow y = \frac{1}{\sqrt{x^2+4}}\ln(x+\sqrt{x^2+4}) + \frac{c}{\sqrt{x^2+4}}.$$

Using the fact that $y(0) = 1$, we have, $1 = \frac{1}{2}\ln 2 + \frac{c}{2} \Rightarrow c = 2 - \ln 2$.

Therefore, the solution is $y = \frac{1}{\sqrt{x^2+4}}\ln(x+\sqrt{x^2+4}) + \frac{2-\ln 2}{\sqrt{x^2+4}}$.

We conclude this section by looking at an application that involves the use of the integrating factor to solve the d.e. obtained.

EXAMPLE 5.12 A container is initially filled with 100 litres of a salt solution containing 50 kg of salt. Brine containing 2 kg of salt per litre runs into a tank at a rate of 6 litres/min, and runs out at a rate of 4 litres/min.

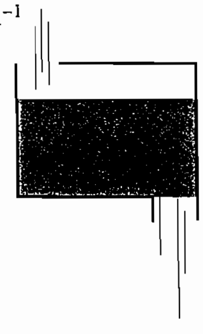
(a) Assuming that the mixture is kept uniform by stirring, set up a differential equation describing the relationship for the amount of salt in the container at any time t minutes.

(b) How much salt will there be in the tank after 10 minutes?

Solution

(a) Let $s(t)$ kg be the amount of salt in the container at any time t minutes.

Amount in = $2 \text{ kg l}^{-1} \times 6 \text{ l min}^{-1}$
$= 12 \text{ kg min}^{-1}$



Amount out = $\left(\frac{s}{100+2t} \text{ kg l}^{-1}\right) \times (4 \text{ l min}^{-1})$
$= \frac{4s}{100+2t} \text{ kg min}^{-1}$

Then, we have that the rate of change in the amount of salt in the container must be given by

$$\frac{ds}{dt} = (\text{Amount in} - \text{Amount out}).(\text{per minute})$$

The 'Amount in' (per minute) is 12 kg min^{-1}, while the

'Amount out' per minute is $= \frac{4s}{100+2t}\text{kg min}^{-1}$

The last term was derived as follows:

First, we need to determine the concentration, C, of salt in the container at any time t, and, given that

$$C = \frac{\text{Amount of salt in container at any time } t}{\text{Volume of solution in container at any time } t}$$

we have,

$$C = \frac{s}{100+(6-4)t},$$

this is because

i. by definition, we have that s kg is the amount of salt in the tank at any time t,

ii. initially there are 100 litres of solution in the container and every minute this increases by $(6-4) = 2$ litres per minute, so that after t minutes, there will be an extra $2t$ litres in the container.

Therefore, we have the differential equation, $\frac{ds}{dt} = 12 - \frac{4s}{100+2t}, t \geq 0$.

This can also be written as, $\frac{ds}{dt} + \frac{4s}{100+2t} = 12$.

(b) This d.e. is of the form $\frac{dy}{dx} + P(x)y = Q(x)$, and so we can use the integrating factor method to solve it.

The integrating factor is given by $e^{\int \frac{4}{100+2t}dt} = e^{2\ln(100+2t)} = (100+2t)^2$.

Therefore, we have

$$\frac{d}{dt}((100+2t)^2 s) = 12(100+2t)^2, t \geq 0$$

$$\Rightarrow (100+2t)^2 s = 2(100+2t)^3 + c, t \geq 0$$

$$\Rightarrow s = 2(100+2t) + \frac{c}{(100+2t)^2}$$

Now, when $t = 0, s = 50$, $\therefore c = -1500000$, so that $s = 2(100+2t) - \frac{1500000}{(100+2t)^2}, t \geq 0$.

Therefore, when $t = 10$, $s = 135.83$, i.e., approximately 135.83 kg of salt remains.

EXERCISES 5.4.2

1. Find the general solution to the following first order linear differential equations:

(a) $\frac{dy}{dx} - \frac{2}{x}y = x^4, x > 0$

(b) $\frac{dy}{dx} + \frac{2}{x}y = x^4, x > 0$

(c) $\frac{dy}{dx} + 2y = e^{-x}$

(d) $\frac{dy}{dx} + 2y = x$

(e) $\frac{dy}{dx} + \frac{1}{x}y = \frac{\ln x}{x^2}, x > 0$

(f) $\frac{dy}{dx} + \frac{y}{x-1} = x$

2. Solve the following differential equations:

(a) $\frac{dy}{dx} = e^{-x} - y, \; y(0) = 1$

(b) $(x^2+1)\frac{dy}{dx} - xy = 0, \; y(0) = 3$

(c) $5\frac{dx}{dt} + \frac{15x}{50-t} = 1, \; x(0) = -45$

(d) $x\frac{dy}{dx} + y = 4x^2, \; y(1) = 0$

(e) $\frac{dy}{dx} = \frac{1}{x+y^2}, \; y(-1) = 0$

(f) $\frac{dy}{dx} + \frac{2}{x}y = \frac{\sin x}{x^2}, y(\pi) = 1, x > 0$

3. Solve the differential equation $y'\cos x = y\sin x + \sin 2x$.

4. Solve the initial-value problem $(1+x)\frac{dy}{dx} + y = 1 + x, \; y(1) = \frac{1}{4}$.

5. Solve the differential equation $\frac{dy}{dx} + y = p(x)$, where $p(x) = \begin{cases} 1, & 0 \le x \le 1 \\ 0, & x > 1 \end{cases}$.

6. Solve the initial-value problem $\frac{dy}{dx} + \frac{xy}{4-x^2} = 1, y(0) = 1, |x| < 2$.

7. By using the substitution $u = x^2$, show that the differential equation $\frac{dy}{dx} = \frac{x}{x^2y+y^3}$ can be transformed to the differential equation $\frac{du}{dy} - 2yu = 2y^3$.
Hence, find an implicit equation relating x and y.

8. Solve the initial-value problem $\tan x\frac{dy}{dx} + y = \sin x, y(\pi) = 0$

9. A large tank holds 100 litres of brine which is made up with 80 kg of salt. A second solution is run into the tank at a rate of 10 litres/min. The mixture is kept uniform by constantly stirring and is allowed to flow out at the same rate.
Set up and solve, the differential equation for the amount of salt present in the tank at any time t, $t \geq 0$, for each of the following cases:

(a) i. If the second solution is fresh water.
ii. If the second solution has a salt concentration of 1kg/litre.

Now consider the case where the solution is flowing out at rate of 8 litres/min.
Set up and solve, the differential equation for the amount of salt present in the tank at any time t, $t \geq 0$, for each of the following cases:

(b) i. If the second solution is fresh water.
ii. If the second solution has a salt concentration of 1kg/litre.

10. The charge Q on a charging capacitor in an RC circuit with constant voltage V is given by the differential equation

$$R\frac{dQ}{dt} + \frac{Q}{C} = V,$$

where $Q(0) = 0$. Find $Q(t)$.

11. A 10-volt battery is connected to a simple series circuit and satisfies the differential equation $\frac{1}{2}\frac{di}{dt} + 10i = 10$, $i(0) = 0$, where i is the current flowing in the circuit at any time t seconds.
(a) Solve the initial-value problem.
(b) Sketch the solution curve to the initial-value problem.
(c) i. Calculate the current flowing in the circuit after 0.05 sec.
ii. Determine the 'maximum' current that can flow in this circuit.

12. A differential equation used to model the height, x m, of a species of tree at age t years is given by $\frac{dx}{dt} + \alpha x = \beta(t)$ where α is a constant and $\beta(t)$ is a function of t.

For the case that $\beta(t) = \beta$ (a real constant), solve the d.e., and show that this species of tree grows to a height that will remain under $\frac{\beta}{\alpha}$ m.

ANSWERS

CHAPTER 1

Exercises 1.1.1

1. (a) $\frac{1}{2}, \frac{1}{6}, \frac{1}{12}, \frac{1}{20}$ (b) $\frac{10}{3}, \frac{11}{3}, 4, \frac{13}{3}$ (c) $6, \frac{11}{2}, \frac{16}{3}, \frac{21}{4}$ (d) $3, 6, 11, 20$

2. (a) $4, \frac{8}{3}, \frac{12}{5}, \frac{16}{7}$ (b) $1, \frac{\sqrt{2}}{2}, \frac{\sqrt{3}}{3}, \frac{1}{2}$ (c) $\frac{1}{4}, \frac{2}{9}, \frac{3}{16}, \frac{4}{25}$

3. (a) $u_n = \frac{1}{3^{n-1}}, n \in \mathbb{Z}^+$ (b) $u_n = \frac{1}{(-3)^{n-1}}, n \in \mathbb{Z}^+$

(c) $u_n = \frac{1}{n^2+1}, n \in \mathbb{Z}^+$

4. (a) $u_1 = 3.4142, u_5 = 5.1623, u_{10} = 6.4721 ; u_{50} = 12$

(b) $u_1 = 1, u_5 = 2.2361, u_{10} = 3.1623 ; u_{50} = 7.0711$

(c) $u_1 = 0.7854, u_5 = 1.3734, u_{10} = 1.4711 ; u_{50} = 1.5508$

5. (a) $1, \frac{4}{\sqrt{2}}, 9\sin\left(\frac{\pi}{8}\right), 16\sin\left(\frac{\pi}{16}\right)$ (b) $\sqrt{3}-1, 2-\sqrt{2}, \sqrt{5}-\sqrt{3}, \sqrt{6}-1$

(c) $\frac{1}{2}, \frac{1}{3}, \frac{1}{8}, \frac{1}{30}$

6. (a) $\frac{3}{2}, 2, \frac{5}{2}, 3, \frac{7}{2}$ (b) $5, \frac{5}{4}, \frac{5}{9}, \frac{5}{16}, \frac{1}{5}$ (c) $\frac{1}{2}, \frac{3}{4}, \frac{7}{8}, \frac{15}{16}, \frac{31}{32}$

(c) $0, 1, \frac{\sqrt{3}}{2}, \frac{1}{\sqrt{2}}, \sin\left(\frac{\pi}{5}\right)$

7. (a) $1.25, 2.4414, 2.5216, 2.5658, 2.5937$

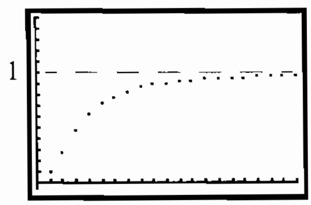
(b) $0, 0.3466, 0.3662, 0.3466, 0.3219$ (c) $0, 2, 2.5981, 2.8284, 2.9389$

8. $1, 4, 5, 2(1+\sqrt{6})$ **9.** $2, 4, 2^{3/2}, 2^{7/4}, 2^{13/8}$ **10.** $1, 1, 2, 3, 5, 8$

Exercises 1.1.2

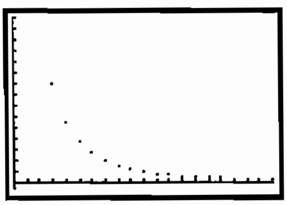
1.

(a)



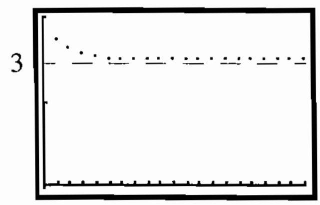
As n increases indefinitely, u_n increases, approaching the limit 1 from below.

(b)



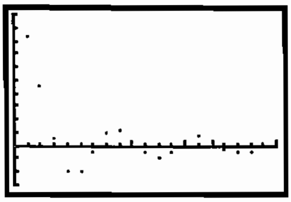
As n increases indefinitely, u_n decreases, approaching the limit 0 from above.

(c)



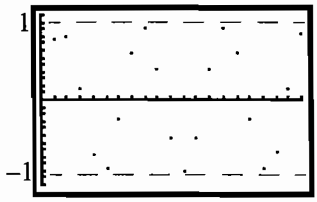
As n increases indefinitely, u_n decreases, approaching the limit 3 from above.

(c)



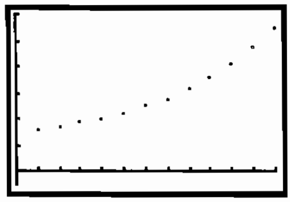
As n increases indefinitely, u_n oscillates, approaching the limit 0 from above and below.

(d)



As n increases indefinitely, u_n oscillates between the values –1 and +1.

(e)



As n increases indefinitely, u_n increases indefinitely.

Exercises 1.2.1

1. (a) $\frac{1}{2}$ (b) $\frac{1}{2}$ (c) $-\frac{1}{3}$ (d) 0 (e) doesn't exist (f) 0

2. (a) 0 (b) 2

3. (a) 1 (b) $\frac{1}{2}$ (c) 0 (d) ∞ (e) 0 (f) $-\infty$

4. (a) $\frac{b}{d}$ (b) $\frac{1}{4}$ (c) 0

5. (a) $\frac{2}{3}$ (b) $\frac{3}{4}$

6. (a) $u_2 = \frac{1}{2}, u_3 = \frac{1}{3}, u_4 = \frac{1}{4}$ (b) 0

7. 0

8. (a) $5, \frac{11}{3}, \frac{17}{5}, \frac{23}{7}$ (b) i. $\frac{an+b}{cn+d}$ ii. $\frac{a}{c}$

Nb: The answer to part (b) depends on the expressions you are working with - however, the answer must be the ratio of the leading coefficients of the linear functions

Exercises 1.2.2

1. See soln. manual.

2. (a) 0; converges (b) π; converges (c) $\frac{3}{2}$; converges

(d) oscillates between –1 and 1; diverges (e) 0; converges

(f) 3; converges (g) oscillates; diverges (h) 1; converges

(i) ∞; diverges (j) 0; converges (k) oscillates; diverges

(l) 1; converges (m) ∞; diverges (n) ∞; diverges

(o) 0; converges

3. (a) e^3 (b) e^{-2} (c) $e^{1/2}$ (d) e^{-2} (e) $e^{-3/4}$ (f) e^{-2}

4. (a) converges; 0 (b) doesn't converge (c) converges; 2

5. (a) i. see soln. manual ii. $L = \frac{1}{3}$ iii. see soln. manual

(b) i. see soln. manual ii. $L = 2$ iii. see soln. manual

(c) i. see soln. manual ii. $L = -\frac{1}{2}$ iii. see soln. manual

6. (a) i. see soln. manual ii. $L = \frac{2}{3}$ (b) see soln. manual

Exercises 1.2.3

1. i. (a); (d); (e) ii. (c); (d) iii. (a); (d); (e)
iv. (a); (d) v. (c); (d) vi. (a); (d); (e)
vii. (b); (d); (e) viii. (c); (d) ix. (c)
x. (b); (d) xi. (b); (d) xii. (a); (d)

2. (a) see soln. manual (b) $\frac{2}{3}$ (c) i. converges ii. 0

3. (a) see soln. manual (b) $\frac{1}{2}$ (c) i. converges ii. 0

4. converges to 0 **5.** see soln. manual

6. (a) see soln. manual (b) 1

Exercises 1.2.4

1. (a) 4 (b) 8 (c) 0 (d) $-\frac{1}{e}$ (e) $\frac{1}{3}$ (f) 0

(g) $e^{1/2}$ (h) 1 (i) 6 (j) 3 (k) $\frac{1}{2}$ (l) $\frac{1}{2}$

Exercises 1.2.5

1. (a) oscillating between $-1+\frac{1}{n}$ and $1+\frac{1}{n}$. (b) min u.b = 1; max l.b = −1

2. (a) see graphs (b) u.b = $\frac{1}{6}$; l.b = $-\frac{35}{81}$ (c) converges to 0.

5. (a) see graphs (b) i. $n \le 9$ ii. $n \ge 9$ (c) $9(0.9)^9$ (e) 0

6. (a) $\frac{81}{256}$ (b) 0 (c) 0 (d) diverges **10.** 3

11. (a) 0 (b) $\frac{1}{2}$, if $a = 1$; $-\frac{1}{2}$, if $a = -1$ and n is odd; ∞, if $a = -1$ and n is even

(c) $\frac{1}{a}$ **12.** (b) −1 **13.** (c) convergent **14.** (c) 2

Exercises 1.3

1. (a) −3 (b) 1 (c) 2 **2.** (a) 0 (b) 0 (c) 2

3. (a) 1 (b) $-\frac{1}{2}$ (c) $\frac{1}{6}$ **4.** (a) 0 (b) 0 (c) 0

5. (a) doesn't exist. (b) $-\frac{1}{2}$ (c) 1 **6.** see soln. manual

7. (a) undefined (b) 0 (c) 0

8. (a) 0 (b) $\frac{1}{2}$ (c) $-\frac{1}{7}$ (d) 0 (e) 0 (f) doesn't exist

(g) 2 (h) 5 (i) $-\frac{3}{2}$

9. (a) i. $-\frac{12}{23}$ ii. e (b) -2 **10.** 1 **11.** e **12.** 1

13. (a) e (b) 1 (c) 1 **14.** (a) 1 (b) 1

15. (a) $2e$ (b) e **16.** (a) $\frac{\alpha}{\beta}$ (b) 1

17. (a) $\frac{1}{2}(\beta^2 - \alpha^2)$ (b) $\frac{1}{2\beta}\sin(2\beta)$

Exercises 1.4

1. (a) doesn't exist (b) $\frac{1}{2}$ (c) $\frac{1}{8}$ (d) doesn't exist

(e) 1 (f) $\frac{1}{2}$

2. (a) see soln. manual (b) $3 + 3\sqrt[3]{2}$

3. (a) 2 (b) doesn't exist (c) $-\frac{1}{2}$

4. (a) 3 (b) 0 (c) doesn't exist

5. (a) $-xe^{-x} - e^{-x} + c$ (b) 1 **6.** (a) $xe^x - e^x + c$ (b) -1

7. (a) see soln. manual (b) 6 sq. units

8. (a) $(a+1)e^{-a}$ (b) $\left(\frac{a+1}{a}\right)e^{-a^2}$ **9.** (a) $\frac{\pi}{2} - \arctan(a)$ (b) $\frac{\pi}{4}$

10. (a) 0 (b) doesn't exist **11.** (a) 1 (b) doesn't exist

12. $p \geq 1$ **13.** $p \leq -1$ **14.** $\varnothing$ **15.** $p \in \mathbb{R}\backslash\{-1\}$

16. (a) doesn't exist (b) π cubic units

17. (a) $2\sqrt{3}$ (b) doesn't exist (c) doesn't exist

18. (a) see soln. manual (b) e^{-1} (c) see soln. manual

19. (a) see soln. manual (b) i. 1 ii. convergent

20. converges **21.** see soln. manual

22. (a) $(1-x)(1+x)(1+x^2)$ (b) see soln. manual (c) see soln. manual

23. see soln. manual

24. (a) see soln. manual (b) $\frac{1}{\sqrt{2x}} \leq \frac{x}{\sqrt{x^3+1}} \leq \frac{1}{\sqrt{x}}, x \geq 1$

(c) i. doesn't exist ii. doesn't exist

(d) i. diverges ii. diverges (e) doesn't converge

Exercises 2.1

1. $\frac{7}{6}$ **2.** 1; converges **3.** $\frac{3}{2}$; diverges **4.** diverges

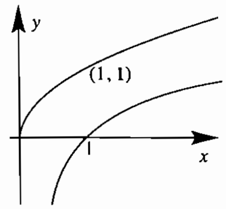
5. $\frac{1}{2}$;converges

6. (a) converges (b) converges (c) diverges (d) converges
(e) converges (f) diverges (g) diverges (h) diverges

7. (a) diverges (b) converges (c) diverges (d) converges

8. converges to 1 **9.** converges to 1

10. (a) i.



ii. 0 (b) converges

11. (c) i. $\frac{1}{n+1} - \frac{1}{n+2}$ **13.** diverges

14. (a) diverges (b) converges (c) diverges **15.** (b) diverges

16. (a) converges (b) converges

Exercises 2.2

1. see soln. manual **2.** see soln. manual

3. (a) i. $\frac{1467}{754}$ ii. $\frac{1539035}{679354}$ (b) i. $\pi - \arctan\left(\frac{n}{2}\right)$

ii. Using 5 terms, error ~ 0.7610; using 10 terms, error ~ 0.3948

4. (a) i. ~ 0.89486 ii. ~ 0.91991

(b) i. $\frac{1}{e^n}(n+1)$ ii. 5 terms, error ~ 0.0404; 9 terms, error ~ 0.001234

5. (a) i. ~ 0.6418 ii. ~ 0.7939

(b) i. $\frac{2}{\sqrt{n+2}}$ ii. 8 terms, error ~ 0.6325; 9 terms, error ~ 0.001234

6. (a) $0 \le R_{100} < \int_{100}^{\infty} \frac{3}{x^4} dx = \frac{1}{10^6}$ (b) $0 \le R_{50} < \int_{50}^{\infty} \frac{2}{x(\ln(x))^4} dx = \frac{1}{(\ln 52)^2} \approx 0.0641$

(c) ~0

7. (a) ≥ 485,165,196 (b) ≥ 20,001 (c) 16 terms

8. (a) i. ~ 1.1975 ii. $\frac{1}{2n^2}$ (b) ~ 0.005 (c) 101 terms

9. (a) i. ~ 0.4049 ii. $\frac{1}{2}e^{-n^2}$ (b) ~ 1.860×10^{-44} (c) 4 terms

10. (a) i. ~0.9765 ii. $\frac{\pi^2}{8} - \frac{1}{2}(\arctan(n))^2$ (b) ~ 0.1876 (c) 3142 terms

11. (b) using $\int_0^1 \frac{1}{\sqrt{1+x}} dx = 2\sqrt{2} - 2 \approx 0.8284$

Exercises 3.1

1. (b) ≥ 1000 terms **2.** (b) ≥ 46 terms **3.** ≥ 14 terms

4. (b) ≥ 10 terms **5.** not convergent **7.** 1×10^{-6}

Exercises 3.2

1. yes **4.** (a) converges absolutely (b) divergent (c) divergent
5. i. conditionally convergent ii. divergent iii. divergent
6. (a) ~ 0.6321 (b) $\sim 2.5052 \times 10^{-8}$
7. (a) i. ~ 0.94703 ii. $\sim 5.1419 \times 10^{-6}$ (b) ≥ 100 terms
8. (a) ~ 3.6108 (7 terms) (b) ~ -5.9 (5 terms) **9.** 10 terms
10. (a) converges absolutely (b) conditionally convergent
11. (a) converges absolutely (b) not convergent

Exercises 3.3

2. $x = 0$ **3.** (a) $|x| < 3$ (b) $x \in [-1, 1[$ (c) $x \in \mathbb{R}$
4. (a) i. 5 ii. $]-5, 5[$ (b) i. 0.25 ii. $]-0.25, 0.25[$
(c) i. 2 ii. $-1 < x < 3$ (d) i. 1 ii. $]-1, 1[$
(e) i. 1 ii. $-1 < x < 1$ (f) i. 1.5 ii. $]-1, 2[$
5. converges for $-7 \leq x < -3$
6. (a) $R = \frac{1}{3}; \left[-\frac{1}{3}, \frac{1}{3}\right[$ (b) $R = \infty;]-\infty, \infty[$ (c) $R = 1; [-2, 0[$
7. (a) 2 (b) $[1, 5[$ **8.** (a) $-3 < x < 3$ (b) $x \in \mathbb{R}$ (c) $[-4, 4[$
9. $\left]-\frac{b+c}{a}, \frac{b+c}{a}\right[$ **10.** (a) 2 (b) 1 **11.** $\frac{1}{3}$ **12.** 3
13. (a) 5 (b) 14 **14.** $a = 1, b = 4$ **15.** (a) $[0, 4]$ (b) $[1, 5]$
16. $R = \frac{1}{2}$; interval of convergence is $\left[\frac{1}{2}, \frac{3}{2}\right[$

Exercises 3.4

1. (a) $\sum_{n=0}^{\infty} \frac{1}{2^{n+1}} x^n, |x| < 2$ (b) $\sum_{n=0}^{\infty} (-1)^n 2^{2n+1} x^n, |x| < \frac{1}{4}$ (c) $\sum_{n=0}^{\infty} \left(-\frac{4}{3}x\right)^n, |x| < \frac{3}{4}$

2. (a) $\sum_{n=0}^{\infty} x^{2n}, |x| < 1$ (b) $\sum_{n=0}^{\infty} \frac{(-1)^n}{4^{n+1}} x^{2n}, |x| < 2$

(c) $\ln(5) - \sum_{n=0}^{\infty} \frac{x^{n+1}}{(n+1)5^{n+1}}, |x| < 5$

3. (a) $\sum_{n=0}^{\infty} x^{2n+1}, |x| < 1$ (b) $\sum_{n=0}^{\infty} \frac{(-1)^n}{2^{n+1}} x^{n+2}, |x| < 2$

(c) $\sum_{n=0}^{\infty} (-1)^n x^{2(n+1)}, |x| < 1$

4. (a) i. $-\frac{1}{(2+x)^2}$ ii. $\sum_{n=0}^{\infty} \frac{(-1)^n(n+1)}{2^{n+2}} x^n, |x| < 2$ iii. $R = 2$

(b) i. $\sum_{n=0}^{\infty} \frac{(-1)^n(n+1)(n+2)}{2^{n+4}} x^n, |x| < 2$ ii. $\sum_{n=0}^{\infty} \frac{(-1)^n(n+1)(n+2)}{2^{n+4}} x^{n+2}, |x| < 2$

5. $\sum_{n=0}^{\infty} \frac{(-1)^n}{7^{n+1}} x^{7n+1} + c$ **6.** (a) **ii.** $a = 3, b = 3$ **iii.** $]-3, 3[$ (b) see soln. manual

7. (a) $\sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{(2n+1)^2} + c, |x| < 1$ (b) $\sum_{n=0}^{\infty} \frac{(-1)^n x^{4n+2}}{2(2n+1)^2} + c, |x| < 1$

(c) $\sum_{n=0}^{\infty} \frac{(-1)^n x^{8n+7}}{(2n+1)(8n+7)} + c, |x| < 1$

8. $\sum_{n=0}^{\infty} \frac{(-1)^n}{n+1} x^{n+1}$; 0.405532. need 9 terms in the series to estimate ln(1.5) to within 0.0001.

9. (a) $x^2 - \frac{x^6}{3!} + \frac{x^{10}}{5!} - \frac{x^{14}}{7!} + \ldots$ (b) ~ 0.3579

10. (a) $[-1, 1]$ (b) $[-1, 1[$ (c) $]-1, 1[$ **11.** (b) 0.000000303

12. (a) $\sum_{n=0}^{\infty} x^{2n}, |x| < 1$ (b) $\sum_{n=0}^{\infty} nx^{2n-1}, |x| < 1$ **13.** $\sum_{n=0}^{\infty} \frac{2x^{2n+1}}{2n+1}, |x| < 1$

14. (a) $[-1, 1]$ (b) $]-1, 1[$ (c) $]-1, 1[$

15. (b) $\sum_{n=0}^{\infty} \frac{(-1)^n}{(2n)!} x^n$ (c) ~ 0.7635

16. (b) $1 - x + \frac{x^3}{3} - \frac{x^4}{6} + \frac{x^5}{24} - \frac{x^6}{720} + \ldots, x \in]-\infty, \infty[$ (c) 0.00437786

Exercises 4.1

1. (b) $s_2(x) = 1 + x + \frac{x^2}{2}$; $s_3(x) = 1 + x + \frac{x^2}{2} + \frac{x^3}{6}$; $s_4(x) = 1 + x + \frac{x^2}{2} + \frac{x^3}{6} + \frac{x^4}{24}$

see sol. manual for graphs.

(c) As n increases, the graph of $y = s_n(x)$ is a better fit to the graph of $y = f(x)$ in the vicinity of $x = 0$.

2. (b) $s_2(x) = x - \frac{x^3}{3!} + \frac{x^5}{5!}$; $s_6(x) = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \frac{x^9}{9!} - \frac{x^{11}}{11!} + \frac{x^{13}}{13!}$;

$s_8(x) = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \frac{x^9}{9!} - \frac{x^{11}}{11!} + \frac{x^{13}}{13!} - \frac{x^{15}}{15!} + \frac{x^{17}}{17!}$

see sol. manual for graphs.

(c) As n increases, the graph of $y = s_n(x)$ is a better fit to the graph of $y = f(x)$ in the vicinity of $x = 0$. Also, as n increases, the graph of $y = s_n(x)$ approximates $\sin(x)$ more closely over an increasingly wider domain.

3. (b) $s_2(x) = 1 - \frac{x^2}{2!} + \frac{x^4}{4!}$; $s_6(x) = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \frac{x^8}{8!} - \frac{x^{10}}{10!} + \frac{x^{12}}{12!}$;

$s_8(x) = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \frac{x^8}{8!} - \frac{x^{10}}{10!} + \frac{x^{12}}{12!} - \frac{x^{14}}{14!} + \frac{x^{16}}{16!}$

see sol. manual for graphs.

(c) As n increases, the graph of $y = s_n(x)$ is a better fit to the graph of $y = f(x)$ in the vicinity of $x = 0$. Also, as n increases, the graph of $y = s_n(x)$ approximates $\cos(x)$ more closely over an increasingly wider domain.

4. (a) $1+2x+2x^2+\frac{4}{3}x^3$ (b) $x-\frac{x^2}{2}+\frac{x^3}{3}-\frac{x^4}{4}$ (c) $2x-\frac{4}{3}x^3+\frac{4}{15}x^5-\frac{8}{315}x^7$

(d) $x+\frac{x^3}{3}+\frac{2}{15}x^5+\frac{17}{315}x^7$ (e) $1-\frac{x^2}{2}-\frac{x^4}{12}-\frac{x^6}{45}$

(f) $1+(\ln(2))x+(\ln(2))^2\cdot\frac{x^2}{2!}+(\ln(2))^3\cdot\frac{x^3}{3!}$

5. (a) $1+x-\frac{2}{3}x^3$ (b) $1-x^2+\frac{x^4}{2}$ (c) $x-x^2+\frac{x^3}{3}$

(d) $x+\frac{x^3}{3!}+\frac{3x^5}{5!}$ (e) $1+\frac{x^2}{2!}+\frac{5x^4}{4!}$ (f) $x\ln(x)+\frac{x^2}{2}-\frac{x^3}{8}$

6. (a) $1+(\ln(a))x+\frac{(\ln(a))^2}{2!}x^2+\frac{(\ln(a))^3}{3!}x^3+\ldots$ (b) $\frac{1}{\ln(a)}\left[x-\frac{x^2}{2}+\frac{x^3}{3}-\frac{x^4}{4}+\ldots\right]$

7. (a) $\sum_{n=0}^{\infty}\frac{e(x-1)^n}{n!}$ (b) $\ln(2)+\frac{x-2}{2}-\frac{(x-2)^2}{8}+\frac{(x-2)^3}{24}-\frac{(x-2)^4}{64}+\ldots$

(c) $\sum_{n=0}^{\infty}(-1)^n\frac{e^2(x+2)^n}{n!}$ (d) $1+\frac{x-2}{2}-\frac{(x-1)^2}{8}+\frac{(x-1)^3}{16}-$

8. (a) $\sum_{k=0}^{n}(-1)^k\frac{x^{4k+2}}{(2k+1)!}$

(b) $x^2-\frac{x^6}{3!}+\frac{x^{10}}{5!}-\ldots+(-1)^n\cdot\frac{x^{4n+2}}{(2n+1)!}+(-1)^{n+1}\cdot\frac{x^{4n+6}}{(2n+3)!}+\ldots$

9. $\sim 0.182266\ldots$; error ≤ 0.000064

10. (a) $32+20(x-4)+\frac{15}{4}(x-4)^2+\frac{5}{32}(x-4)^3$ (b) 88.25 (c) ~ 0.078125

11. $\frac{\pi}{2}\left(1-\frac{k^2}{8}-\frac{3k^4}{2^7}-\frac{5k^6}{2^9}\right)$ + Error, $0\le k\le 1$; Error $\le \sim 0.01678$ (setting $k=1$).

Exercises 4.2

1. (a) $\sum_{n=0}^{\infty}\frac{2^n x^n}{n!}$ (b) $\sum_{n=0}^{\infty}(-1)^n\frac{x^n}{2^n\cdot n!}$ (c) $\sum_{n=0}^{\infty}\frac{x^{2n}}{n!}$

(d) $1+x+\frac{x^2}{2}-\frac{3x^4}{4!}+\ldots$ (e) $\sum_{n=0}^{\infty}(-1)^n\frac{2^{2n+1}x^{2n+1}}{(2n+1)!}$

(f) $\sum_{n=0}^{\infty}(-1)^n\frac{x^{2n}}{2^{2n}\cdot(2n)!}$ (g) $\sum_{n=0}^{\infty}(-1)^n\frac{x^{4n+2}}{(2n+1)!}$

(h) $\sum_{n=0}^{\infty}(-1)^n\frac{2^{2n+1}x^{2n+1}}{2n+1}$

2. (a) $\sum_{n=0}^{\infty}(-1)^n\frac{x^{n+1}}{2^{n+1}\cdot(n+1)}$ (b) $\sum_{n=0}^{\infty}(-1)^n\frac{x^{3n+3}}{n+1}$

(c) $\ln(2)-\frac{x^2}{4}-\frac{x^4}{96}-\frac{x^6}{1440}+\ldots$

3. (a) $x - x^2 + \frac{5}{6}x^3 - \frac{5}{6}x^4 + \frac{101}{120}x^5 + \ldots$ (b) $1 - x - \frac{2}{3}x^3 - \frac{1}{6}x^4 + \ldots$

(c) $x^3 - \frac{x^4}{2} - \frac{x^6}{12} - \frac{4}{45}x^7 + \ldots$ (e) $1 - \frac{x}{2} - \frac{3}{8}x^2 - \frac{5}{16}x^3 - \frac{29}{128}x^4 + \ldots$

Exercises 4.3

1.	1.8092	**2.**	$1.89\dot{2}$	**3.**	$0.40\dot{6}$	**4.**	0.1987
5.	0.95	**6.**	1.04739	**7.**	0.55873		

Exercises 5.1

1. see soln. manual

Exercises 5.2

See sol. manual

Exercises 5.3

1. (a) i. $y = x^2 + x$ ii. 6 (b) i. ~ 5.6 ii. 6.67%

2. (a) i. $y = x^3 + 1$ ii. 28 (b) i. ~ 26.81 ii. 4.25%

3. (a) ~ 4.0625 (b) ~ 5.1917
(c) $h = 0.5$; error = 36.41%. $h = 0.2$; error = 18.74%

4. (a) i. ~ 25.27 ii. ~ 18.235
(b) $h = 0.1$; error = 37.09%. $h = 0.2$; error = 54.61%

5.	~ 0.648	**6.**	~2.2991×10^{29}	**7.**	~ 7.190	**8.**	~ 0.736
9.	~ 2.629	**10.**	~4.240	**11.**	~ 5.084	**12.**	~ 2.9335

13. (a) i. $y = x^3 - x^2$ ii. 18 (b) i. ~ 13.25 ii. ~ 17.01
(c) i. 26.39% ii. 5.5%

14. (a) i. see soln. manual ii. 4.5 (b) i. ~4.4302 ii. ~4.4667
(c) i. 1.55% ii. 0.74%

Exercises 5.4.1

1. (a) $kx = e^{y/x}$ (b) $kx = e^{-x/y}$ (c) $y^2 = kx^3 + x^2$
(d) $\ln(kx) = -e^{-y/x}$ (e) $x = kx^2(x - 2y)$ (f) $x^2 - 2xy - y^2 = k$

2.	$y^2 + x^2 = 5x^3$	**3.**	$x^2 = y^2(1 + 2\ln(y))$	**4.**	$x + y + 1 = ke^x$
5.	$x + y + 2 = ke^y$	**6.**	$y = x\tan(x + c)$	**7.**	$y = \frac{1}{2}(x^2 - 1)$
8.	$y = x - x\cos(x)$	**9.**	$y^2 = -2x^2\ln(x)$	**10.**	$y = x(2x - 1)$