Mathematics HL Information for teachers and moderators

Copies of all Internal Assessment portfolio tasks published by the IB

Includes
First edition of previous TSM published 1998 First examinations May 2000
Second edition previous TSM published 2000 First examinations May 2001
Current TSM published 2005 First examinations May 2006
Portfolio tasks for 2009/2010 published 2008

Note
Any tasks contained here CANNOT be submitted for final assessment after the November 2010 examination session

These tasks are NOT acceptable for assessment for examination sessions in 2011 and onward

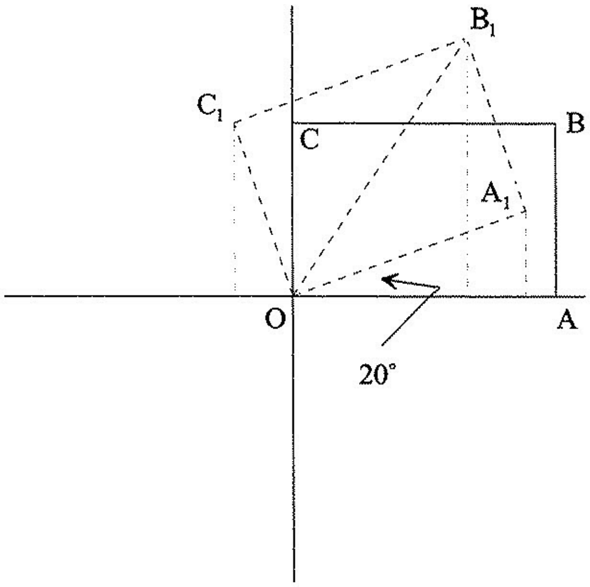
1 Rotating Rectangles

Type I

Work done on this assignment will be assessed against the first five criteria A, B, C, D and E. While the use of a graphic display calculator may be helpful, the use of technology is not specifically required, nor will it be assessed.

A rectangle OABC rotates counter clockwise about the point O as shown in the diagram below, which is not drawn to scale.

Initially O is at $(0, 0)$, A is at $(4, 0)$, B is at $(4, 3)$ and C is at $(0, 3)$.



The diagram shows the rectangle after it has rotated through 20 degrees. The vertices are now labelled O, A_1, B_1, and C_1.

1 Calculate the coordinates of A_1 and C_1.

2 Calculate the coordinates of B_1 from your answers to part 1.

3 By calculating OB and the angle AOB show how the coordinates of B_1 can be calculated directly.

4 Draw the rectangle after it has rotated through larger angles, and through angles measured in a clockwise direction.

5 Using both of the above methods, find expressions for the coordinates of the images of a rectangle of length a and width b for any angle of rotation.

6 Write $a\sin\theta + b\cos\theta$ in the form $r\sin(\theta+\alpha)$, giving r and α in terms of a and b.

7 Write $a\cos\theta - b\sin\theta$ in the form $r\cos(\theta+\alpha)$, giving r and α in terms of a and b.

8 The angle α is called the phase shift and r is called the amplitude. How would you predict the amplitude, phase shift and period of each of the following functions?

(i) $f:\theta \mapsto 3\sin\theta + 4\cos\theta$
(ii) $f:\theta \mapsto 3\sin\theta - 4\cos\theta$
(iii) $f:\theta \mapsto 2\cos\theta - \sin\theta$
(iv) $f:\theta \mapsto \sqrt{5}\sin\theta + \sqrt{11}\cos\theta$

1 Rotating Rectangles (continued)

9 For each of the functions in part 8,

(a) find the derivative and equate it to zero;
(b) find the coordinates of the maximum point that is nearest to $x = 0$.

10 Summarize what you have found out about functions of the form $a\sin\theta + b\cos\theta$. You may find it appropriate to compare the answers you obtained in parts 8 and 9.

2 Estimation of π — Type I

Work done on this assignment will be assessed against criteria A, B, C, D and F. You are therefore expected to use a graphic display calculator and/or computer for this assignment.

The goal of this assignment is to estimate the value of π through finding the area of the unit circle. To find an estimate for this area you need to calculate the areas of n-sided inscribed and circumscribed polygons.

This method was used by the Greek mathematician Archimedes over two thousand years ago.

1 Explain in simple terms why the value of π lies between the areas of the two polygons described above.

2 Develop a general formula for the area of an n-sided equilateral polygon which is inscribed in a circle of radius r.

Hint: Draw a sketch of a typical triangular piece of the n-sided polygon.

3 Develop a general formula for the area of an n-sided equilateral polygon which is circumscribed in a circle of radius r.

4 Using $r = 1$, and n from 10 to 400 (in steps of 10), calculate the values of the areas of the inscribed and circumscribed polygons.

(a) For each polygon how many decimal places of accuracy are there when $n = 400$? Note that $\pi = 3.141592654$ to 9 decimal places.

(b) Calculate the average of the areas of the inscribed and circumscribed polygons for $n = 400$. How many places of accuracy are there?

5 What value of n is necessary to obtain six decimal places of accuracy in both approximations, given that $n \geq 400$.

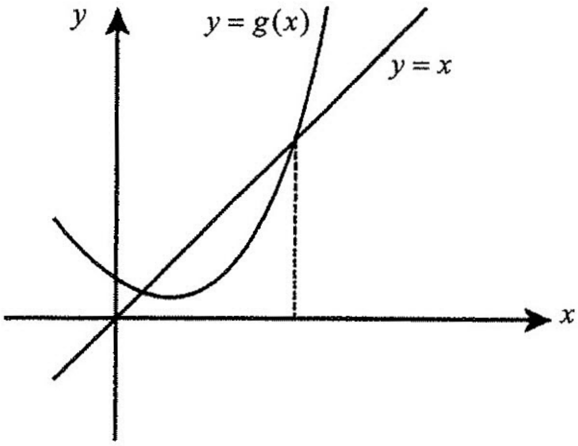
6 Plot the values of the areas you have found against the number of sides. Also label the value of π. Which approximation converges faster? Try to justify your answer.

7 Archimedes also found a value for π using the semi-perimeter of the inscribed and circumscribed polygons. Using this alternative method, follow a similar procedure to the one described previously to find an estimate for the value of π.

3 Finding Zeros of Functions — Type I

Work done on this assignment will be assessed against all six criteria A, B, C, D, E and F. You are therefore expected to use a graphic display calculator and/or computer for this assignment.

There are various numerical methods for obtaining approximate solutions of equations of the form $f(x)=0$. One such method requires the equation to be expressed in the form $x=g(x)$ so that a solution can be interpreted as the value of x where the line $y=x$ intersects the curve $y=g(x)$. This is described as a fixed point of the function $g(x)$.



Part 1

(a) Let $g(x)=\cos x$. Show graphically that $g(x)$ has one fixed point between $x=0$ and $x=2$.

(b) Using the iteration $x_{n+1}=g(x_n)$ and the initial value $x_1=1$, find the first five iterations and explain graphically why the iteration is approaching the fixed point of the function $g(x)$.

(c) Repeat the process, but this time let $g(x)=2\cos x$. Comment on your results and interpret them graphically.

(d) Repeat this process for $g(x)=a\cos x$, using some other values between $a=1$ and $a=2$ to discover which converge to a positive fixed point for $g(x)$. Suggest why this only happens for some values of a.

(e) Determine for which value of a the iteration will converge for $g(x)=a\sin x$ starting with $a=1.2$. Comment on differences in the graphical representation of this iteration compared to those earlier.

3 Finding Zeros of Functions (continued)

Part 2

The only real solution of the equation $x^3 - x - 1 = 0$ is, of course, the zero of the function $f(x) = x^3 - x - 1$.

(a) Show that the equation $f(x) = 0$ can be rearranged in the form $x = x^3 - 1$ and use an iteration to try to find the root of the equation. Describe and explain what happens.

(b) Rearrange the equation in **different** ways in the form $x = g(x)$ and try new iterations, to discover which converge on the root. Explain graphically why they converge.

(c) Repeat steps (a) and (b) for the following equations:

(i) $x^5 - x - 2 = 0$ (ii) $x^3 - 2x - 4 = 0$

Part 3

Summarize what you have found out about fixed point iteration and the conditions for convergence. Justify any conclusions you have reached using graphs as illustrations.

4 Investigating a Sequence of Numbers — Type I

Work done on this assignment will be assessed against the first five criteria A, B, C, D and E. While the use of a graphic display calculator may be helpful, the use of technology is not specifically required, nor will it be assessed.

Part 1

The sequence of numbers $\{a_n\}_{n=1}^{\infty}$ is defined by

$$a_1 = 1\times 1!,\quad a_2 = 2\times 2!,\quad a_3 = 3\times 3!,\quad \ldots.$$

Find the nth term of the sequence.

Part 2

Let $S_n = a_1 + \ldots + a_n$. Investigate S_n for different values of n.

Part 3

Based on your answer to part 2, conjecture an expression for S_n.

Part 4

Prove your conjecture by the use of mathematical induction.

Part 5

Show that a_n can be expressed as $(n+1)!-n!$ and devise an alternative (direct) proof of your conjecture for S_n.

Part 6

Let $c_n = a_n + a_{n+1}$. Write an expression for c_n in factorial notation and simplify it.

Part 7

Let $T_n = c_1 + c_2 + \ldots + c_n$. Investigate T_n for different values of n.

Part 8

Conjecture an expression for T_n.

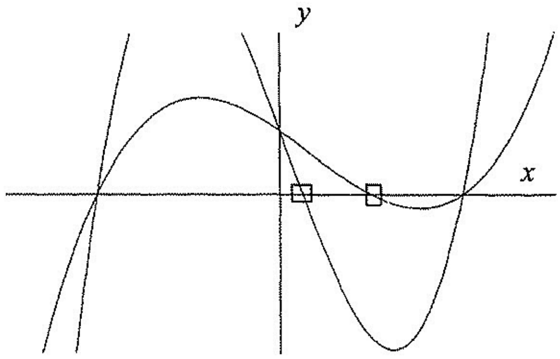
Part 9

Prove your conjecture by any method.

5 The Rational Zeros

Type II

Work done on this assignment will be assessed against the first four criteria A, B, C and D. While the use of a graphic display calculator may be helpful, the use of technology is not specifically required, nor will it be assessed.



1 (a) Graph in turn the following functions (take $x \in [-3, 3]$ and $y \in [-10, 10]$):

(i) $y = x^3 - x^2 - 4x + 4$

(ii) $y = 2x^3 - x^2 - 8x + 4$

(iii) $y = 3x^3 - x^2 - 12x + 4$

(iv) $y = 4x^3 - x^2 - 16x + 4$

(The diagram above shows the graphs of two of these functions.)

In each case find, using any tools available to you, the smallest positive root of $y = 0$, and express it as a fraction. Also find the other roots.

(b) Now graph $y = \frac{2}{3}x^3 - x^2 - \frac{8}{3}x + 4$ and find the smallest positive root of $y = 0$.

(c) Finally graph, with a suitable window, $y = 2x^3 - 3x^2 - 8x + 12$.

What are the roots of $y = 0$?

5 The Rational Zeros (continued)

2 Find the smallest positive root of each of the following functions:

(a) (i) $y = x^3 - x^2 - 9x + 9$

(ii) $y = 3x^3 - x^2 - 27x + 9$

(iii) $y = \frac{3}{2}x^3 - x^2 - \frac{27}{2}x + 9$

(b) (i) $y = 30x^3 - 20x^2 - 270x + 180$

(ii) $y = 2x^3 - 3x^2 - 18x + 27$

(iii) $y = 2x^3 - 5x^2 - 18x + 45$

You may have noticed that there is a pattern relating the rational root $x = \frac{p}{q}$ of the equation $y = ax^3 + \ldots + d = 0$ to the coefficients a and d. The next step is to find this relationship and prove it.

3 Suppose a, b, c and d are integers with $a \neq 0$, and that $f(x)$ is the polynomial defined by $f(x) = ax^3 + bx^2 + cx + d$. Suppose, in addition, that p and q are positive integers with no common factors.

(a) Show that the equation $f\left(\frac{p}{q}\right) = 0$ can be rewritten in the form

$$p(ap^2 + bpq + cq^2) = -dq^3.$$

(b) Hence show that p must be a factor of d.

(c) Show that the equation $f\left(\frac{p}{q}\right) = 0$ can also be rewritten in the form

$$ap^3 = -q(bp^2 + cpq + dq^2).$$

(d) Hence show that q is a factor of a.

(e) Generalize results (b) and (d) to prove the following result:

Rational-zero theorem

Suppose all the coefficients of the polynomial function

$$F(x) = a_n x^n + a_{n-1}x^{n-1} + \ldots + a_2 x^2 + a_1 x + a_0,$$

are integers with $a_n \neq 0$. Let $\frac{p}{q}$ be a rational number in its lowest terms. If $\frac{p}{q}$ is a zero of F, then p is a factor of a_0 and q is a factor of a_n.

5 The Rational Zeros (continued)

4 List all possible candidates for rational zeros of

$$P(x) = 6x^4 - 7x^3 + 8x^2 - 7x + 2 .$$

Determine if P has any rational zeros, if so, find them and all other remaining zeros.

6 The Newton-Raphson Method

Type II

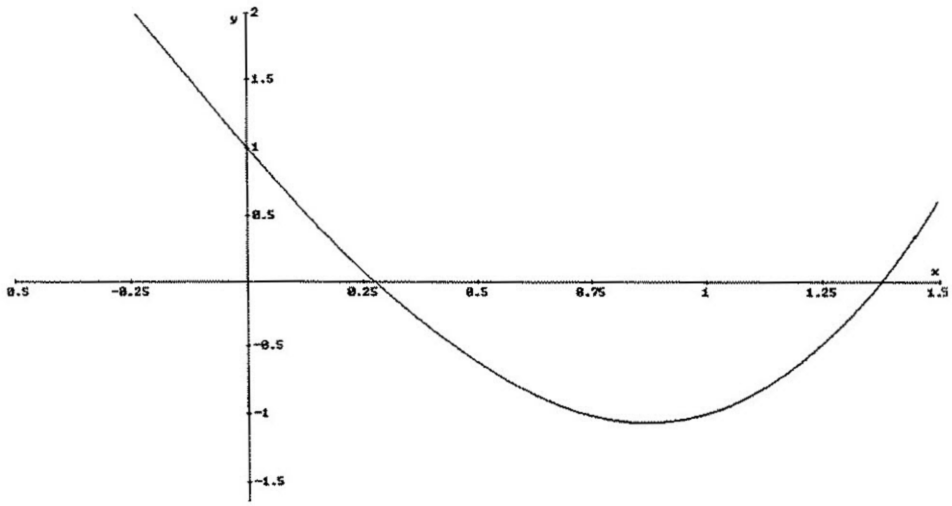
Work done on this assignment will be assessed against the first four criteria A, B, C and D. While the use of a graphic display calculator may be helpful, the use of technology is not specifically required, nor will it be assessed.

If $f(x)$ is a function, then a number r for which $f(r)=0$ is called a **zero** or a **root** of the function $f(x)$, or a **solution** to the equation $f(x)=0$. You are already familiar with ways of solving linear and quadratic equations. However, not all equations admit solutions that can be found exactly. There are methods of finding approximate solutions to equations, one of which is the bisection method, which depends on the intermediate value theorem.

Intermediate value theorem:
Suppose $f(x)$ is continuous on an interval I, and a and b are any two points in I. Then, if y_0 is a number between $f(a)$ and $f(b)$, there exists a number c between a and b such that $f(c)=y_0$. As a consequence of this, if $f(a)<0$ and $f(b)>0$, then there exists a number c between a and b such that $f(c)=0$.

The bisection method uses this fact and successive approximations to locate the zero as shown in the following exercise.

1 Consider the function $f(x)=x^3+x^2-4x+1$, whose graph is given below.

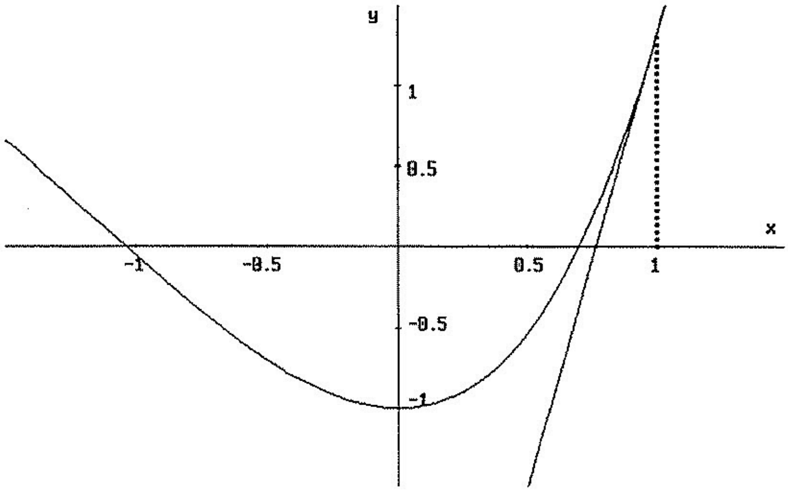


(a) State why the intermediate value theorem guarantees a root between 0 and 1.

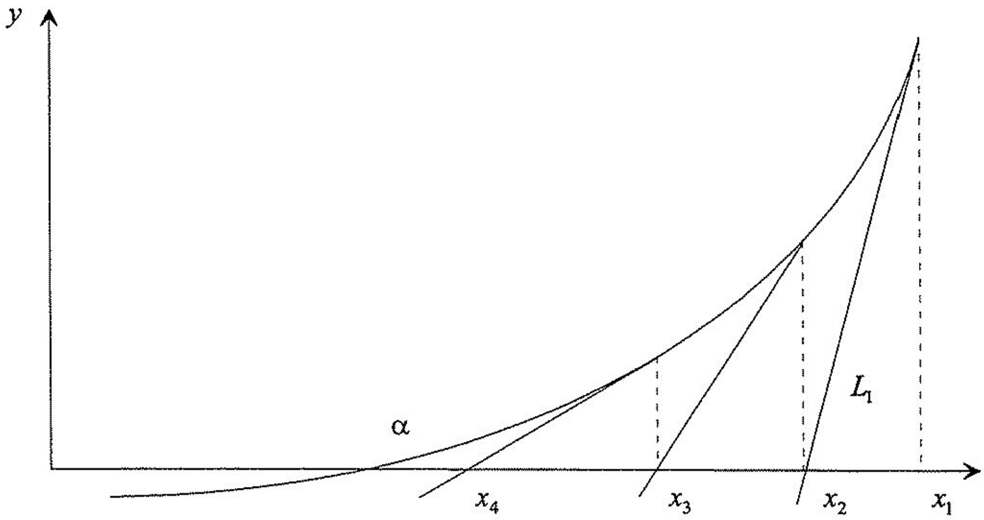
(b) Evaluate $f(0)$ and $f(0.5)$. Taking the midpoint of $[0, 0.5]$ as 0.25, evaluate $f(0.25)$. Decide whether to choose the next midpoint in the interval $[0, 0.25]$ or $[0.25, 0.5]$. Continue the process until you get a root answer correct to two decimal places.

6 The Newton-Raphson Method (continued)

All iterative methods require a first approximation to a root. By sketching the graph we can form an idea of where the function will intersect the x-axis. For example, consider the graph of $f(x)=3e^x-\sin x-2x-4$, which is given below.



Suppose we take the first approximation to the positive root to be $x=1$. By enlarging the region around this root, it can be seen that a better approximation can be found where the tangent at $x=1$ cuts the x-axis.



2 (a) Show that the equation of the tangent line L_1 is approximated by $y=5.61x-4.30$

(b) Show that this line intersects the x-axis at the point where $x\approx 0.77$.

This is an improved estimate of the zero.

(c) This method will generate improved approximations if we repeat it starting with the first estimate of 0.77.

(i) Find the equation of the tangent to the curve at the point with x-coordinate 0.77, then find the x-intercept of this line.

(ii) Repeat the process two more times. What is your improved estimate?

(iii) Substitute this value in the equation of the function. How much error is there?

(d) This method can be generalized to give a formula for improving approximations of solutions of equations of the form $f(x) = 0$.

(i) Show that the equation of L_1 is of the form $y = f(x_1) + f'(x_1)(x - x_1)$.

(ii) Show that $f'(x_1) = \dfrac{f(x_1)}{x_1 - x_2}$.

(iii) Show that L_1 crosses the x-axis at $x_2 = x_1 - \dfrac{f(x_1)}{f'(x_1)}$.

(iv) Find similar expressions for x_3 and x_4.

(v) Generalize this expression for x_{n+1}.

Congratulations, you have discovered the Newton-Raphson method!

3 Apply the Newton-Raphson method to find approximate solutions to each of the following equations, correct to 3 decimal places:

(a) $\sin 2x = x^2$

(b) the positive solution of $e^{2x} = 3\cos x$

(c) $x^3 = 3^x$

(d) $2e^{-x}\cos 4x = 1$, in the interval $[-1, 0]$

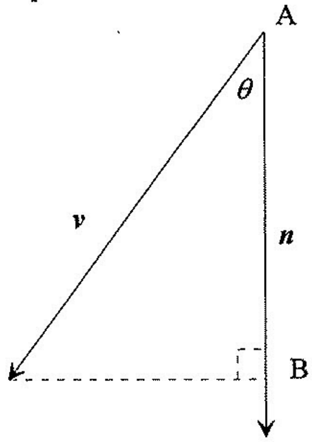
(e) $x^4 = x + 1$

4 Is the choice of the first estimate of the root significant? Explain your answer.

7 Lines and Planes **Type II**

Work done on this assignment will be assessed against the first four criteria A, B, C and D. While the use of a graphic display calculator may be helpful, the use of technology is not specifically required, nor will it be assessed.

Note: To do this activity, you need to refresh your memory concerning the scalar product of two vectors and the properties of vectors and planes.



1 With reference to the diagram above, show that $\mathrm{AB} = \dfrac{\boldsymbol{v} \cdot \boldsymbol{n}}{|\boldsymbol{n}|}$.

2 The point $\mathrm{A}(x_0, y_0, z_0)$ lies outside a plane π with equation $ax + by + cz + d = 0$, as shown in the diagram below. The point $\mathrm{P}(x, y, z)$ lies on the plane π. AB is the distance between the point A and the plane π.

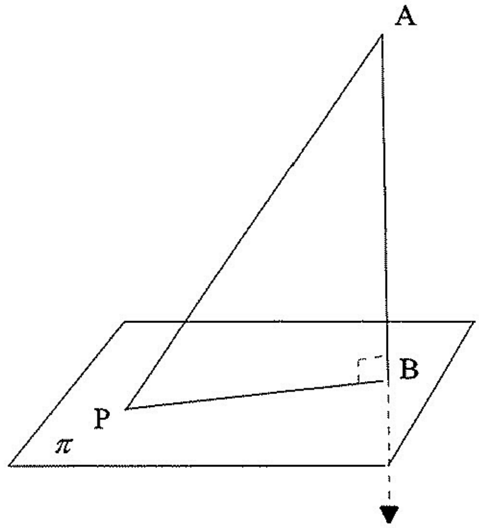
(a) Find the component form of the vector $\overrightarrow{\mathrm{AP}}$.

(b) Use the fact established in part 1 to show that:

$$\mathrm{AB} = \frac{|(x - x_0)a + (y - y_0)b + (z - z_0)c|}{\sqrt{a^2 + b^2 + c^2}}$$

(c) Hence show that:

$$\mathrm{AB} = \frac{|ax_0 + by_0 + cz_0 + d|}{\sqrt{a^2 + b^2 + c^2}}$$



7 Lines and Planes (continued)

3 Consider the two lines:

L_1: $x = 1 + 2t$, $y = 2 - t$, $z = -1 + 3t$

L_2: $x = 2 - 3m$, $y = 2m$, $z = 1 - m$

(a) Show that these two lines do not intersect.

(b) Find the equation of the plane containing L_2 which is parallel to L_1.

(c) Taking any specific point on L_1, find its distance from this plane.

4 Given that $\boldsymbol{l}_1$ and $\boldsymbol{l}_2$ are direction vectors for L_1 and L_2, find the vector product $\boldsymbol{l}_1 \times \boldsymbol{l}_2$.

5 Taking a general point A on L_1 and a general point B on L_2, find the vector $\overrightarrow{\mathrm{AB}}$.

6 Find points A and B such that $\overrightarrow{\mathrm{AB}}$ is parallel to $\boldsymbol{l}_1 \times \boldsymbol{l}_2$.

7 Find $|\overrightarrow{\mathrm{AB}}|$. Compare with your answer to part 3 above. Comment on the result.

8 Absorbing Shocks **Type II**

Work done on this assignment will be assessed against criteria A, B, C, D and F. You are therefore expected to use a graphic display calculator/computer for this assignment.

Springs and shock absorbers in cars are designed to provide comfort to passengers, and stability on the road for the car. Engineers have designed a new type of shock absorber, and we have the task of testing it.

The following test is devised. The front of the car is pushed down 10 cm from its rest position and then released. The displacement of any point on the front of the car from its rest position is measured as a function of time. The car rocks up and down and then returns to rest.

1 Sketch what you think is the graph of the motion.

The maximum distance that any point on the front of the car travels **below** the rest position **after** being pushed downwards is used as a measure of the effectiveness of the shock absorber. Dividing this distance by the original displacement, in this case 10 cm, gives the rebound ratio. A ratio below 1 % is considered acceptable.

For this particular type of shock absorber, the distance y from the rest position is modelled by the function

$$y = -10e^{-10t}[\cos(10t) + \sin(10t)],$$

where t is measured in seconds and y in centimetres.

2 Sketch a graph of this function.

3 Comment on the suitability of this function as a model for the motion.

4 How long does it take the front of the car to reach the rest position for the first time?

5 Find the velocity at that point.

6 How long will it take the front of the car to travel below the rest position for the first time after being pushed?

7 How far does it travel below the rest position?

8 Is this shock absorber acceptable?

9 How long does it take the front of the car to reach the rest position for the second time?

10 Find the velocity at this point.

11 Estimate the time it takes for the front of the car to reach its lowest point after it reaches the rest position for the second time. How far below the rest position is this?

12 Comment on your results.

9 Transformations and Their Matrices **Type II**

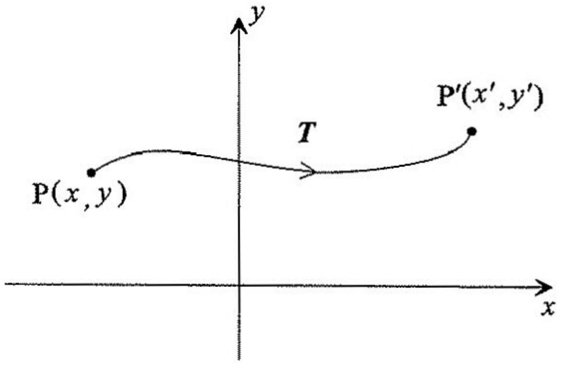
Work done on this assignment will be assessed against the first four criteria A, B, C and D. While the use of a graphic display calculator may be helpful, the use of technology is not specifically required, nor will it be assessed.

Introduction

A linear transformation, T, of the plane which maps the point $P(x, y)$ onto the point $P'(x', y')$ is defined by the equations

$$x' = ax + by$$
$$y' = cx + dy$$

i.e. $T(x, y) = (ax + by, cx + dy)$.



Part 1

For the linear transformation $T_0\ (x, y) = (2x - y, x + 3y)$, find the image of

(a) the point (5, 3);
(b) the unit square OABC, where $O = (0, 0), A = (1, 0), B = (1, 1), C = (0, 1)$.

Part 2

Apply the following transformations

$$T_1\ (x, y) = (-x, y)$$
$$T_2\ (x, y) = (y, -x)$$
$$T_3\ (x, y) = (y, x)$$

to the unit square. In each case draw a diagram showing OABC and O′A′B′C′ and identify the nature of the transformation.

Part 3

Apply the composite transformation $T_2 T_1$ (i.e. T_1 followed by T_2) to the unit square and write the equations for the transformation $T_2 T_1$.

9 Transformations and Their Matrices (continued)

Part 4

Express the equations $x' = ax + by$ and $y' = cx + dy$ in the matrix form $\begin{pmatrix} x' \\ y' \end{pmatrix} = T\begin{pmatrix} x \\ y \end{pmatrix}$ where T is a 2×2 matrix called "the matrix of transformation T".

Use matrix multiplication to find the image of the unit square under T_0 and compare your results with those from part 1 (b).

Part 5

Find the transformation matrices T_1, T_2, T_3, and $T_2\ T_1$ (as defined in part 2 above). Compare your results with those obtained in parts 2 and 3.

Decide whether $T_2\ T_1 = T_1\ T_2$. Comment in terms of the composition of transformations.

Part 6

Find the point (x, y) such that $T_0(x, y) = (-8, -11)$.
Solve $T_0\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} -8 \\ -11 \end{pmatrix}$ using matrix methods.

Part 7

Write a conclusion which explains the relationships between transformations and their matrices. Include the multiplication of matrices and inverse matrices.

10 Population Growth

Type III

Work done on this assignment will be assessed against criteria A, B, C, D and F. You are therefore expected to use a graphic display calculator/computer for this assignment.

Statement of the problem

Given below is the population data for England and Wales between 1801 and 1951. No census was taken in 1941 because of the Second World War.

Year	Population
1801	8 892 536
1811	10 164 256
1821	12 000 236
1831	13 896 797
1841	15 914 148
1851	17 927 609
1861	20 066 224
1871	22 712 266
1881	25 974 439
1891	29 002 525
1901	32 527 843
1911	36 070 492
1921	37 866 699
1931	39 952 377
1951	43 757 888

(a) Using your graphic display calculator make a plot of population against time.

(b) Comment on the population growth during the years 1801-1951.

(c) To estimate how the population is changing the average growth rate over an interval of time can be calculated using the formula:

$$\text{average growth rate} = \frac{\text{change in population}}{\text{time interval}}.$$

For example, between 1801 and 1811 the average growth rate is calculated to three significant figures as 1.27×10^5 people per year.

A more meaningful measure of change of population is the average proportional growth rate defined by:

$$\text{average proportionate growth rate} = \frac{\text{average growth rate}}{\text{population at start of the time interval}}.$$

Construct a table of values of both the average growth rate and the average proportional growth rate for each of the time intervals between the censuses in the given table. Comment on your results.

10 Population Growth (continued)

(d) It is assumed that the proportionate growth rate was constant from the year 1921 onwards. Estimate the populations for the years 1941, 1971, 1991 and 2001.

(e) If the resources of England and Wales could support any population, one model for population growth would assume that the rate of change of population $\frac{\mathrm{d}P}{\mathrm{d}t}$ at time t is proportional to the size of the population P.

Mathematically this becomes $\frac{\mathrm{d}P}{\mathrm{d}t} = kP$, where $k > 0$ is a constant.

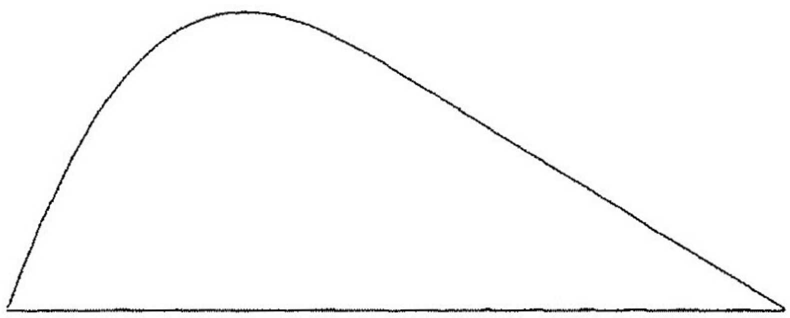
Show that the solution of this differential equation gives an expression of the form $P = A\mathrm{e}^{kt}$. Using the populations of the years 1911 and 1921 find the values of A and k.

(f) Comment on the usefulness of this exponential model. To do so you may find it useful to compare estimates of the populations found in part (d) with estimates obtained from the model used in part (e). The 1991 census found the population of England and Wales to be 49 718 300. Comment on the results you obtained previously and suggest improvements that might be made to improve your predictions.

11 Design of a Ship's Propeller Blade — Type III

Work done on this assignment will be assessed against criteria A, B, C, D and F. You are therefore expected to use a graphic display calculator/computer for this assignment.

A new design for a ship's propeller blade requires that it has a flat shape similar to the one in the diagram below.



For construction purposes, we need to set up the mathematical functions that describe the profile of the propeller blade.

The curved side of the propeller blade is made up of three functions denoted by $P(x), Q(x)$, and $R(x)$, where x is defined as the horizontal distance along the drive shaft to which the propeller blade will be attached. This drive shaft will have a radius of 30 cm. In total 8 blades will be attached to the drive shaft to form the final configuration of the propeller.

The conditions required are as follows:

(i) $P(2) = 3$

(ii) $P(3) = Q(3) = 7$

(iii) $Q(12) = 12$

(iv) $P'(3) = Q'(3)$

(v) $P''(3) = Q''(3)$

(vi) $P''(2) = Q''(12) = 0$

Note that 1 unit represents 10 cm.

$P(x)$ descibes the curved side of the propeller blade between 20 cm and 30 cm, $Q(x)$ describes it for 30 cm to 120 cm, and $R(x)$ describes it between 120 cm and 210 cm.

Through some research work the design engineers have found that the functions $P(x)$ and $Q(x)$ are cubic, while $R(x)$ is linear. Condition (i) ensures that the curved part of the propeller blade starts at the correct location. Condition (ii) will guarantee that the two parts meet, conditions (iv) and (v) ensure smoothness at the joining point of $P(x)$ and $Q(x)$, and condition (vi) ensures smoothness at the ends of each part.

11 Design of a Ship's Propeller Blade (continued)

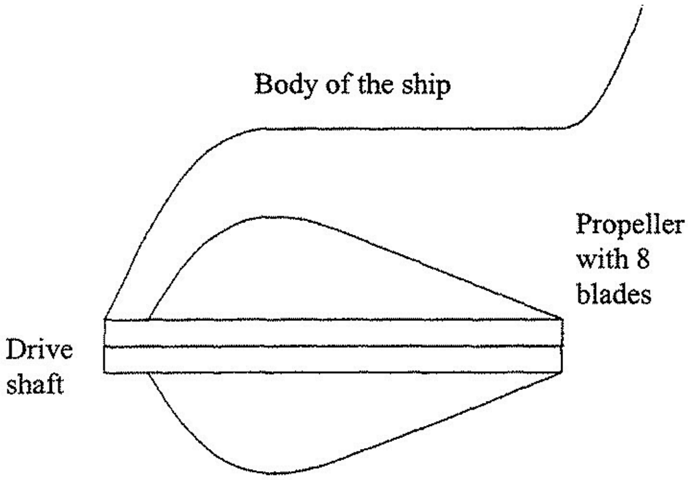
1 Using the conditions given, set up a system of simultaneous equations that will enable you to find the functions $P(x)$ and $Q(x)$.

2 Using suitable technology as a tool, solve this system of equations to find the functions $P(x)$ and $Q(x)$.

3 Given that $R(x)$ must continue the smoothness of $Q(x)$ find its equation.

4 Find the total area of one blade of the propeller.

5 The ship makers must construct the hull to allow sufficient space between the hull and the propeller. For example the starting point of the blade is 20 cm along the drive shaft. Find a function to describe the shape of the hull as shown in the diagram below that is needed to keep an open space between the hull and propeller of at least 20 cm.



12 Some History and Complex Numbers — Type IV

Work done on this assignment will be assessed against the first four criteria A, B, C and D. While the use of a graphic display calculator may be helpful, the use of technology is not specifically required, nor will it be assessed.

Introduction

1 (a) Expand $y = \sin x$, $y = \cos x$, and $y = e^x$ as polynomials in x up to the term in x^7.

(b) Find an expression for e^{ix} and simplify it.

(c) Find expressions for $i \sin x$ and $\cos x + i \sin x$. Simplify the last expression.

(d) Write down the "obvious" relation that emerges.

2 (a) If $y = \cos\theta + i\sin\theta$ find $\frac{dy}{d\theta}$.

(b) Relate $\frac{dy}{d\theta}$ to iy in an equation, and solve that differential equation by the variables-separable method.

(c) Compare your results to part 1(d) above.

3 (a) Substitute some values including 2π, π and $\frac{\pi}{2}$ into the results of parts 1 and 2 above.

(b) Find $(e^{ix})^2$, $(e^{ix})^3$, and $(e^{ix})^n$, and relate these answers to $\cos kx + i \sin kx$ for $k \in \mathbb{Z}$.

(c) State De Moivre's theorem and prove it using mathematical induction. Express De Moivre's theorem in another form.

Research

4 Write a report discussing the following.

(a) The influence of Cardano and Bombelli on the emergence of complex numbers.

(b) De Moivre's contributions to the development of complex numbers.

13 The Use of i in Solving Quadratic Equations

Type IV

Work done on this assignment will be assessed against the first four criteria A, B, C and D.

For the quadratic equation $ax^2 + bx + c = 0$ there is no real solution if $b^2 - 4ac < 0$. A solution can be found if we define the number $\mathrm{i} = \sqrt{-1}$.

(a) Write a research report on the use of i in solving quadratic equations when $b^2 - 4ac < 0$, commenting on significant contributions made to this branch of mathematics by mathematicians such as Euler.

(b) Extend your report to include a detailed description of how solutions of this type of quadratic equation may be represented graphically, paying particular attention to the contributions of Gauss, Wessel and Argand.

1 Binomial Coefficients **Type I**

Work done on this assignment will be assessed against criteria A, B, C, D and E. While the use of a graphic display calculator may be helpful, the use of technology is not specifically required, nor will it be assessed.

The binomial coefficients in the expansion of $(a+b)^n$, $n \in \mathbb{Z}^+$ are defined as

$$\binom{n}{r} = \frac{n!}{r!(n-r)!},\ n \ge r\ (= nCr).$$

1 Write down the first 13 rows of the Pascal Triangle.

2 Prove the symmetrical property for the coefficients, that is $\binom{n}{r} = \binom{n}{n-r}$.

3 Calculate:

(a) $\binom{5}{3} + \binom{5}{4}$, (b) $\binom{8}{2} + 2\binom{8}{3} + \binom{8}{4}$, (c) $\binom{9}{4} + 3\binom{9}{5} + 3\binom{9}{6} + \binom{9}{7}$,

(d) $\binom{7}{2} + 4\binom{7}{3} + 6\binom{7}{4} + 4\binom{7}{5} + \binom{7}{6}$.

4 By comparing your results in part 3 with some other binomial coefficients try to find the following sums:

(a) $\binom{n}{r-1} + \binom{n}{r}$, (b) $\binom{n}{r-2} + 2\binom{n}{r-1} + \binom{n}{r}$,

(c) $\binom{n}{r-3} + 3\binom{n}{r-2} + 3\binom{n}{r-1} + \binom{n}{r}$.

Prove your results.

5 There is a formula connecting any $(k+1)$ successive coefficients in the n^{th} row of the Pascal Triangle with a coefficient in the $(n+k)^{\text{th}}$ row. Find this formula.

6 Show that your formula works for the sum $\displaystyle\sum_{r=0}^{8}\binom{8}{r}\binom{27}{8-r}$.

7 By comparing the corresponding coefficients of the expansions of $(1+x)^n \times (1+x)^k$ and $(1+x)^{n-k}$, prove your formula in part 5.

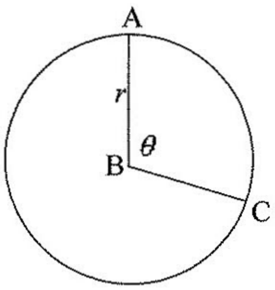
8 Use the results from parts 2 and 5 to find the sum $\displaystyle\sum_{r=0}^{n}\binom{n}{r}^2$.

9 Show that your formula works for the sum $\displaystyle\sum_{r=0}^{13}\binom{13}{r}^2$.

7 Volumes of Cones	Type III

Work done on this assignment will be assessed against criteria A, B, C, D and F. You are therefore expected to use a graphic display calculator/computer for this assignment.

A cone is to be made from the minor sector ABC of a circle of paper of radius r by removing the sector as shown. The angle θ is obtuse.



1 Find an expression for the volume of the cone in terms of r and θ.

2 By using the substitution $x = \dfrac{\theta}{2\pi}$, express the volume as a function of x.

3 Draw the graph of this function using the calculator. Hence find the values of x and θ for which this volume is a maximum. Give your answer for x to four decimal places.

When the circle is cut there are two sectors so it is possible to make two cones.

4 Find the value or values of x for which the sum of the volumes of the two cones is a maximum.

8 Modelling Sunrise

Type III

Work done on this assignment will be assessed against criteria A, B, C, D and F. You are therefore expected to use a graphic display calculator/computer for this assignment.

Answer the following questions.

1 How does the time of year affect the time of sunrise?

2 If you were to draw a graph of sunrise times against time of year, what kind of a graph might you get?

3 How might the location of the place at which you record the sunrise time affect the sunrise time?

4 Consider the relative sunrise times of two places with (i) the same latitude, (ii) the same longitude.

5 Is dawn earlier or later in the North? Is this always true?

6 Go to the following website and use it to obtain the sunrise times in Geneva for the year 1998:

http://aa.usno.navy.mil/AA/data/docs/RS OneYear.html

7 Record sunrise times every two weeks. Record the week numbers and the times in two data lists, L_1 and L_2. (Don't forget to change the times into decimals of an hour. You could write a short program to do this or use the one below. You may need to change the 25).

```
PROGRAM:HOURS
:For(C,1,25)
:Prompt H,M
:H+M/60→L2(C)
:Disp L2(C)
:End
```

8 Find a regression function for the curve that appears.

9 Predict the sunrise time for the 26th week of the year in Geneva.

10 Convert the time to hours and minutes using the "fpart" facility on your calculator.

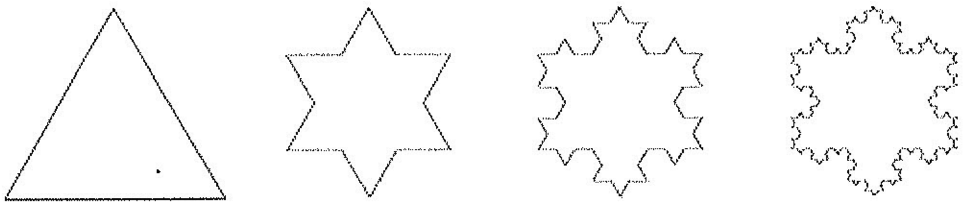
11 Repeat parts 6–10 for four more cities, which are 15, 30, 45 and 60 degrees east of Geneva.

12 For each city find the regression curve. Predict the regression equation for Mandalay and find its sunrise time in the 33rd week.

Type I tasks

1 The Koch snowflake

HL Type I



Stage 0 Stage 1 Stage 2 Stage 3

Description

In 1904 Helge von Koch identified a fractal that appeared to model the snowflake. The fractal is built by starting with an equilateral triangle and removing the inner third of each side, building another equilateral triangle at the location where the side was removed, and then repeating the process indefinitely. The process is illustrated above, showing the original triangle at stage 0 and the resulting figures after one, two and three iterations.

Method

Let N_n= the number of sides, l_n= the length of a single side, P_n= the length of the perimeter, and A_n= the area of the snowflake, at the n^{th} stage.

1. Using an initial side length of 1, create a table that shows the values of N_n , l_n , P_n and A_n for $n = 0$, 1, 2 and 3. Use exact values in your results. Explain the relationship between successive terms in the table for each quantity N_n , l_n , P_n and A_n.
2. Using a GDC or a suitable graphing software package, create graphs of the four sets of values plotted against the value of n. Provide separate printed output for each graph.
3. For each of the graphs above, develop a statement in terms of n that generalizes the behaviour shown in its graph. Explain how you arrived at your generalizations. Verify that your generalizations apply consistently to the sets of values produced in the table.
4. Investigate what happens at $n = 4$. Use your conjectures from step 3 to obtain values for N_4 , l_4 , P_4 and A_4. Now draw a large diagram of one "side" (that is, one side of the original triangle that has been transformed) of the fractal at stage 4 and clearly verify your predictions.
5. Using the relationship between successive terms of N_n , l_n , P_n and A_n, use a spreadsheet or similar software to find the values of each up to a value for n where A_{n+1} equals A_n to six places of decimals. What is the value of n at this point?
6. Explain what happens to the perimeter and area as n gets very large. What conclusion can you make about the area as $n \rightarrow \infty$? Comment on your results.
7. Using your general expression for A_n and the iterative relationship between A_{n+1} and A_n already established in step 1 above, prove the general expression for A_n by induction.

2 Series and induction

HL Type I

Description

The objective of this task is to investigate patterns and formulate conjectures about numerical series. It is expected that students can recall that $1 + 2 + 3 + \ldots + n = \dfrac{n(n+1)}{2}$.

Method

1. Consider the sequence $\{a_n\}_{n=1}^{\infty}$ where

$$a_1 = 1 \cdot 2$$
$$a_2 = 2 \cdot 3$$
$$a_3 = 3 \cdot 4$$
$$a_4 = 4 \cdot 5$$
$$\vdots$$

   Find an expression for a_n, the general term in the sequence.

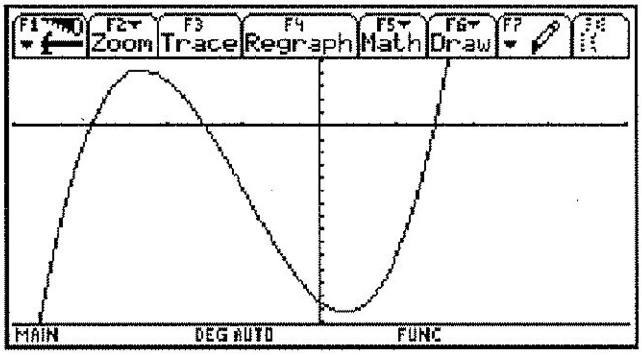
2. Consider the series $S_n = a_1 + a_2 + a_3 + \ldots + a_n$ where a_k is defined as above.
   (a) Determine several values of S_k, including $S_1, S_2, S_3, \ldots, S_6$ and note observations.
   (b) Formulate a conjecture for a general expression for S_n.
   (c) Prove your conjecture by induction.
   (d) Using the above result, calculate $1^2 + 2^2 + 3^2 + 4^2 + \ldots + n^2$.
3. Consider $T_n = 1 \cdot 2 \cdot 3 + 2 \cdot 3 \cdot 4 + 3 \cdot 4 \cdot 5 + \ldots + n(n+1)(n+2)$.
   (a) Determine several values of T_k, including $T_1, T_2, T_3, \ldots, T_6$ and note observations.
   (b) Formulate a conjecture for a general expression for T_n.
   (c) Prove your conjecture by induction.
   (d) Using the above result, calculate $1^3 + 2^3 + 3^3 + 4^3 + \ldots + n^3$.
4. Consider $U_n = 1 \cdot 2 \cdot 3 \cdot 4 + 2 \cdot 3 \cdot 4 \cdot 5 + 3 \cdot 4 \cdot 5 \cdot 6 + \ldots + n(n+1)(n+2)(n+3)$.
   (a) Determine several values of U_k, including $U_1, U_2, U_3, \ldots, U_6$ and note observations.
   (b) Formulate a conjecture for a general expression for U_n.
   (c) Prove your conjecture by induction.
   (d) Using the above result, calculate $1^4 + 2^4 + 3^4 + 4^4 + \ldots + n^4$.
5. With the patterns noted above, can you formulate a conjecture for the series
   $1^k + 2^k + 3^k + 4^k + \ldots + n^k$?

3 Zeros of cubic functions

HL Type I

Method

1. Consider the cubic function: $f(x)=2x^3+6x^2-4.5x-13.5$. Graph this function and find an appropriate window so that the graph looks like this:



Record your window values. State the roots of this cubic and confirm using the Remainder Theorem. Then, taking the roots two at a time, find the equations of the tangent lines to the average of two of the three roots. Find where the tangent lines at the average of the two roots intersect the curve again. Does this observation hold regardless of which two roots you average? State a conjecture concerning the roots of the cubic and the tangent lines at the average value of these roots.

2. Test your conjecture in other similar cubic functions.

3. Prove your conjecture.

4. Investigate the above properties with cubic functions which have:

   (a) one root

   (b) two roots

   (c) one real and two complex roots.

4 Properties of quartics

HL Type I

Description

A quartic function with a "W" shape has two points of inflection, Q and R. In this investigation a line is drawn through Q and R to meet the quartic function again at P and S. The ratio PQ:QR:RS is to be investigated using specific examples to form a conjecture, and then examined formally to prove the findings.

Method

1. Graph the function $f(x) = x^4 - 8x^3 + 18x^2 - 12x + 24$.
2. Find the coordinates of the points of inflection Q and R. Determine the points P and S, where the line QR intersects the quartic function again, and calculate the ratio PQ:QR:RS.
3. Simplify this ratio so that PQ $= 1$ and comment upon your results.
4. Choose another quartic function with a "W" shape and investigate the same ratios.
5. Form a conjecture and formally prove it using a general quartic function.
6. Extend this investigation to other quartic functions that are not strictly of a "W" shape.

5 Orbital radii of planets and moons of Jupiter

HL Type II

Description

The purpose of this task is to construct mathematical models for the orbital radius of the planets of the solar system and for the moons of Jupiter. The models will be used to predict the location of other moons, or in the case of the solar system, the asteroids. A Sicilian monk observed such an asteroid in the winter of 1800–01 using a similar model. Gauss, a mathematical prodigy and famous mathematician, calculated its orbit, and the asteroid Ceres was discovered shortly afterwards.

Method

1. Below is a chart giving the planets, their order from the Sun x and their average orbital radius in millions of kilometres y. An asteroid has been added to the table.

	Mercury	Venus	Earth	Mars	Asteroid	Jupiter	Saturn	Uranus	Neptune	Pluto
x	1	2	3	4	5	6	7	8	9	10
y	57.9	108.2	149.6	227.9	T	778.3	1249	2871	4504	5914

   (a) Use the information in the chart to find an equation relating x and y, and represent this graphically.

   (b) Use the mathematical model to predict the orbital radius of an asteroid that may lie in position 5 on the chart. Comment on the applicability of your model in predicting the orbital radius of an asteroid.

2. Galileo discovered the four Galilean moons of Jupiter in 1610, shortly after he had invented the telescope. They are named Io, Europa, Ganymede and Callisto. It has since been discovered that Jupiter has many moons and that Ganymede, the largest, is in fact moon number 7. A chart of their order x and orbital radius y from the mother planet Jupiter in thousands of kilometres is given below.

	Io	Europa	Ganymede	Callisto
x	5	6	7	8
y	422	670.9	1070	1883

   (a) Use the information in the chart to find an equation relating the average orbital radius (y) with the moon number (x) counting from Jupiter.

   (b) Use the mathematical model to predict the orbital radii of moons that may lie in positions 3, 4, 9 and 10. Use any astronomical data you have (for example, from web sites such as www.solarviews.com/eng/jupiter.htm) to comment on your model's ability to predict orbital radii of Jupiter's moons.

   (c) Do all the planets or moons fit your mathematical models? Evaluate the usefulness of your models.

3. Finally, develop mathematical models for only the four innermost planets and the four Galilean moons by reducing all the orbital radii by a factor that makes the orbital radius of the first planet or moon equal to one. Give possible reasons for the similarities and/or differences in the two models.

6 Flow rate

HL Type II

Description

The following data describes the flow rate of the Nolichucky River in Tennessee between 27 October 2002 and 2 November 2002, taken from the web site water.usgs.gov/pubs/wri/wri934076/stations/03465500.html.

Time	0	6	12	18	24	30	36	42	48	54	60	66	72
Flow	440	450	480	570	680	800	980	1090	1520	1920	1670	1440	1380

Time	78	84	90	96	102	108	114	120	126	132	138	144
Flow	1300	1150	1060	970	900	850	800	780	740	710	680	660

The time is measured in hours past midnight, starting at 00:00 on 27 October. The flow is measured in cubic feet per second (cfs).

Method

1. Find the lines of best fit and describe the features of:

   (a) the original data

   (b) the rate of change of the data.

   Consider the rate of flow between 00:00 on 27 October and 00:00 on 2 November.

2. When did the river have its greatest rate of flow?
3. Estimate this flow rate.
4. What weather pattern could account for the shape of the graph?
5. How much water flowed past the measuring station between 00:00 on 28 October and 00:00 on 29 October?
6. Approximately when was the flow rate at 1000 cfs?
7. The river is controlled by a series of dams. During which days did the amount of flowing water increase?
8. What was the average flow rate from 00:00 on 28 October to 00:00 on 2 November?
9. At what times did the river flow at that rate?

7 Radiometric dating

HL Type II

Part 1

Description

Rolling 216 dice and taking out the sixes each time the dice are rolled has developed the following data. In this way the number of dice left to roll decreases. The chart shows the roll number and the number of dice to be rolled. Students are encouraged to perform this experiment and to record their own data.

Roll	0	1	2	3	4	5	6	7	8
Number	216	180	150	125	104	87	73	61	51

Roll	9	10	11	12	13	14	15	16	17
Number	43	36	25	21	18	15	13	11	9

Roll	18	19	20	21	22	23	24	25	26
Number	7	7	6	6	5	5	4	4	4

Roll	27	28	29	30	31	32	33	34	35
Number	3	3	3	3	3	2	2	2	2

Roll	36	37	38	39	40
Number	2	2	1	1	0

Method

Plot this data and conjecture one or more equations that will model the behaviour. Test your models against the data and illustrate their closeness of fit on a graph.

1. Use your model to predict on which roll the number of dice will be exactly half the original number.
2. Check the theoretical result in step 1 above against the experimental data.
3. Repeat steps 1 and 2 to predict and check at which roll the number of dice will be one-quarter that of the original number.
4. Conjecture a model for 12-sided dice where each die showing a 12 is removed after each roll and where the original number of dice is 100. Predict the number of rolls after which the number of dice will be half the original number.
5. Use a random number function to generate random numbers from 1 to 12 to simulate a 12-sided die and develop data similar to that in the table above. Use this data to test the model you produced in step 4. Show how you did this by downloading software printouts for the first three steps.

Part 2

Description

Many radioactive elements can be used as geological clocks. Each radioactive element decays at its own nearly constant rate. Once this rate is known, geologists can estimate the length of time over which decay has been occurring by measuring the amount of radioactive parent element and the amount of stable daughter elements. Each radioactive element has its own unique half-life. A half-life is the time it takes for half the parent radioactive element to decay to a daughter product. Radioactive carbon-14 is present in the atmosphere and all living things, which absorb it. Upon death a plant or animal no longer absorbs carbon-14 and its amount starts to decrease through radioactive decay so that after 5730 years there is only half the amount of carbon-14 that would be expected in a living plant or animal.

Method

1. Use the information above to conjecture a model for the number of carbon-14 atoms that have not decayed from an initial number N_0.
2. In a sample of bone discovered in a Neolithic grave, the amount of radioactive carbon-14 was found to be only 1% of that in living things. How old was the bone?
3. Explain why carbon-14 is not used for dating dinosaur bones.
4. What fundamental assumption has been made in the use of carbon-14 for dating purposes? Explain how scientists might try to validate the scientific use of carbon-14 in dating.

8 Creating a logistic model

HL Type II

Description

A geometric population growth model takes the form $u_{n+1} = r \cdot u_n$ where r is the growth factor and u_n is the population at year n. For example, if the population were to increase annually by 20%, the growth factor is $r = 1.2$, and this would lead to an exponential growth. If $r = 1$ the population is stable. A logistic model takes a similar form to the geometric, but the growth factor depends on the size of the population and is variable. The growth factor is often estimated as a linear function by taking estimates of the projected initial growth rate and the eventual population limit.

Method

1. A hydroelectric project is expected to create a large lake into which some fish are to be placed. A biologist estimates that if 10,000 fish were introduced into the lake, the population of fish would increase by 50% in the first year, but the long-term sustainable limit would be about 60,000. From the information above, write two ordered pairs in the form (u_0, r_0), (u_n, r_n) where $U_n = 60,000$. Hence, determine the slope and equation of the linear growth factor in terms of U_n.
2. Find the logistic function model for u_{n+1}.
3. Using the model, determine the fish population over the next 20 years and show these values using a line graph.
4. The biologist speculates that the initial growth rate may vary considerably. Following the process above, find new logistic function models for u_{n+1} using initial growth rates $r = 2,\ 2.3$ and 2.5. Describe any new developments.
5. A peculiar outcome is observed for higher values of the initial growth rate. Show this with an initial growth rate of $r = 2.9$. Explain the phenomenon.
6. Once the fish population stabilizes, the biologist and the regional managers see the commercial possibility of an annual controlled harvest. The difficulty would be to manage a sustainable harvest without depleting the stock. Using the first model encountered in this task with $r = 1.5$, determine whether it would be feasible to initiate an annual harvest of 5,000 fish after a stable population is reached. What would be the new, stable fish population with an annual harvest of this size?
7. Investigate other harvest sizes. Some annual harvests will be sustainable, and others will cause the population to die out. Illustrate your findings graphically.
8. Find the maximum annual sustainable harvest.
9. Politicians in the area are anxious to show economic benefits from this project and wish to begin the harvest before the fish population reaches its projected steady state. The biologist is called upon to determine how soon fish may be harvested after the initial introduction of 10,000 fish. Again using the first model in this task, investigate different initial population sizes from which a harvest of 8,000 fish is sustainable.

INVESTIGATING DIVISIBILITY **HL TYPE I**

Background

In order to satisfactorily complete this assignment, the following areas of the core curriculum should have been covered:

- Factorization of polynomials
- Mathematical induction
- Pascal's triangle
- Binomial theorem and notation $\begin{pmatrix} n \\ r \end{pmatrix}$

1. Factorize the expression $P(n) = n^x - n$ for $x \in \{2, 3, 4, 5\}$. Determine if the expression is always divisible by the corresponding *x*. If divisible use mathematical induction to **prove** your results by showing whether $P(k+1) - P(k)$ is always divisible by *x*. Using appropriate technology, explore more cases, summarize your results and make a conjecture for when $n^x - n$ is divisible by *x*.

2. Explain how to obtain the entries in Pascal's Triangle, and using appropriate technology, generate the first 15 rows. State the relationship between the expression $P(k+1) - P(k)$ and Pascal's Triangle. Reconsider your conjecture and revise if necessary.

Write an expression for the x^{th} row of Pascal's Triangle. You will have noticed that $\begin{pmatrix} x \\ r \end{pmatrix} = k$, $k \in \mathbb{N}$. Determine when *k* is a multiple of *x*.

3. Make conclusions regarding the last result in part 2 and the form of proof by induction used in this assignment. Refine your conjecture if necessary, and prove it.

4. State the converse of your conjecture. Describe how you would prove whether or not the converse holds.

INVESTIGATING RATIOS OF AREAS AND VOLUMES **HL TYPE I**

Background

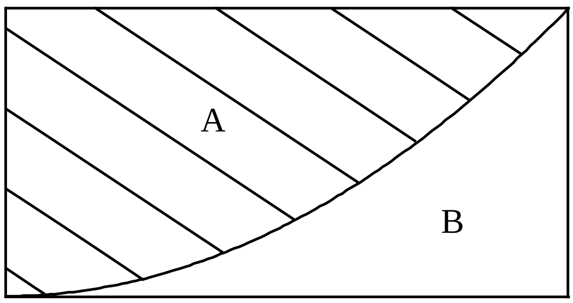
The following areas of the core curriculum should have been covered in order to successfully complete this assignment:

- area under a curve
- volumes of revolution
- integration using the power rule
- functions of the form $y = x^n$, $n \in \mathbb{Z}$ and $n \in \mathbb{Q}$.

Introduction

In this portfolio assignment you will investigate the ratio of the areas formed when $y = x^n$ is graphed between two arbitrary parameters $x = a$ and $x = b$ such that $a < b$.

1. Given the function $y = x^2$, consider the region formed by this function from $x = 0$ to $x = 1$ and the x-axis. Label this area B. Label the region from $y = 0$ to $y = 1$ and the y-axis area A.

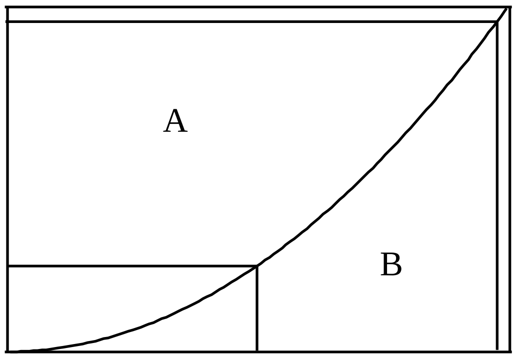


Find the ratio of area A: area B.

Calculate the ratio of the areas for other functions of the type $y = x^n$, $n \in \mathbb{Z}^+$ between $x = 0$ and $x = 1$. Make a conjecture and test your conjecture for other subsets of the real numbers.

(This task continues on the following page)

2. Does your conjecture hold only for areas between $x = 0$ and $x = 1$? Examine for $x = 0$ and $x = 2$, $x = 1$ and $x = 2$, *etc*. (See diagram)



3. Is your conjecture true for the general case $y = x^n$ from $x = a$ to $x = b$ such that $a < b$ and for the regions defined below? If so prove it; if not explain why not.

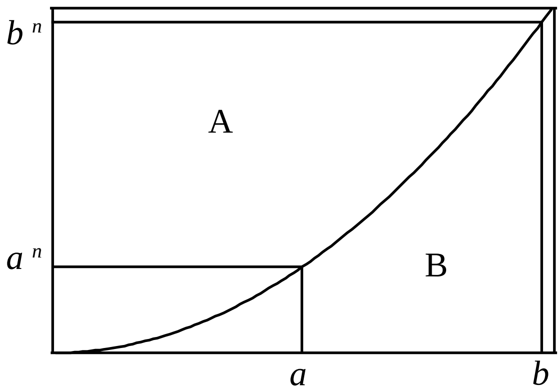
Area A: $y = x^n$, $y = a^n$, $y = b^n$ and the *y*-axis
Area B: $y = x^n$, $x = a$, $x = b$ and the *x*-axis

4. Are there general formulae for the ratios of the volumes of revolution generated by the regions A and B when they are each rotated about

(a) the *x*-axis?

(b) the *y*-axis?

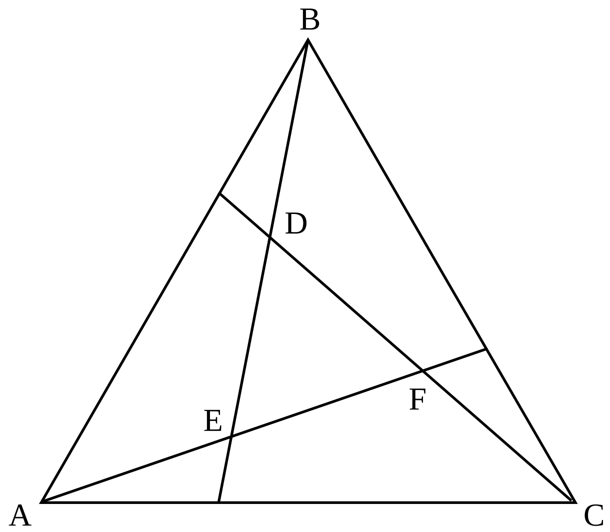
State and prove your conjecture.



THE SEGMENTS OF A POLYGON **HL TYPE I**

To undertake this task, the student should use a geometry tool such as *Geometer's Sketchpad*, *Autograph*, *Cabri Jr* or *TI-Nspire.*

1. In an equilateral triangle ABC, a line segment is drawn from each vertex to a point on the opposite side so that the segment divides the side in the ratio 1:2, creating another equilateral triangle DEF (see below).



(a) What is the ratio of the areas of the two equilateral triangles? To answer this question,

(i) create the above diagram with your geometry package.

(ii) measure the lengths of the sides of the two equilateral triangles.

(iii) find the areas of the two equilateral triangles and the ratio between them.

(b) Repeat the procedure above for at least two other side ratios, 1:n.

(c) By analyzing the results above, conjecture a relationship between the ratios of the sides and the ratio of the areas of the triangles.

(d) Prove this conjecture analytically.

2. Does this conjecture hold for non-equilateral triangles? Explain.

(This task continues on the following page)

3. (a) Do a similar construction in a square where each side is divided into the ratio of 1:2. Compare the area of the inner square to the area of the original square.

(b) How do the areas compare if each side is divided in the ratio 1:n? Record your observations, describe any patterns noted, and formulate a conjecture.

(c) Prove the conjecture.

4. If segments were constructed in a similar manner in other regular polygons (*e.g.* pentagons, hexagons, *etc*.), would a similar relationship exist? Investigate the relationship in another regular polygon.

PARABOLA INVESTIGATION **HL TYPE I**

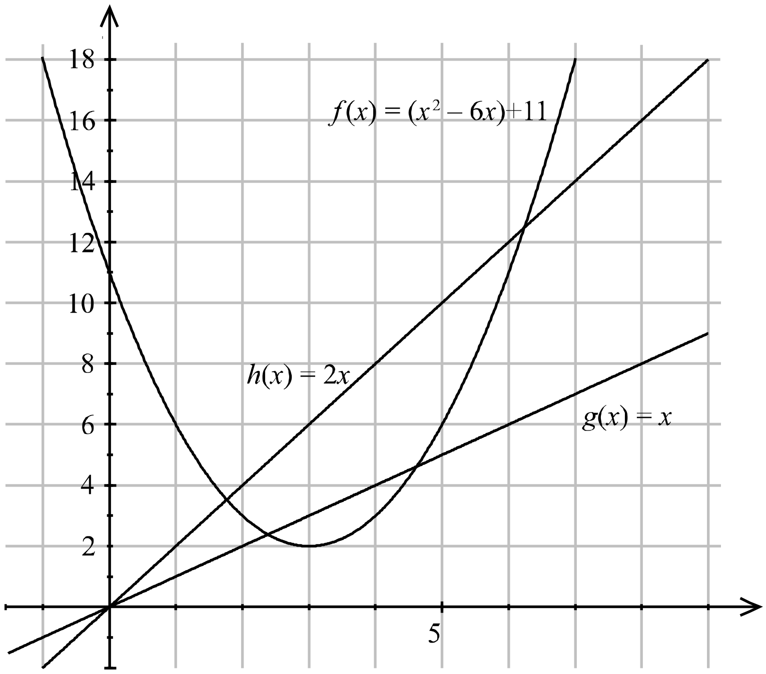
Description

In this task, you will investigate the patterns in the intersections of parabolas and the lines $y = x$ and $y = 2x$. Then you will be asked to prove your conjectures and to broaden the scope of the investigation to include other lines and other types of polynomials.

Method

1. Consider the parabola $y = (x-3)^2 + 2 = x^2 - 6x + 11$ and the lines $y = x$ and $y = 2x$.

- Using technology find the four intersections illustrated on the right.
- Label the x-values of these intersections as they appear from left to right on the x-axis as x_1, x_2, x_3, and x_4.
- Find the values of $x_2 - x_1$ and $x_4 - x_3$ and name them respectively S_L and S_R.
- Finally, calculate $D = |S_L - S_R|$.



2. Find values of D for other parabolas of the form $y = ax^2 + bx + c$, $a > 0$, with vertices in quadrant 1, intersected by the lines $y = x$ and $y = 2x$. Consider various values of a, beginning with $a = 1$. Make a conjecture about the value of D for these parabolas.

3. Investigate your conjecture for any real value of a and any placement of the vertex. Refine your conjecture as necessary, and prove it. Maintain the labeling convention used in parts 1 and 2 by having the intersections of the first line to be x_2 and x_3 and the intersections with the second line to be x_1 and x_4.

4. Does your conjecture hold if the intersecting lines are changed? Modify your conjecture, if necessary, and prove it.

5. Determine whether a similar conjecture can be made for cubic polynomials.

6. Consider whether the conjecture might be modified to include higher order polynomials.

DESIGNING A FREIGHT ELEVATOR **HL TYPE II**

In this activity, you will first explore a possible model for the motion of a heavy-duty freight elevator used to raise and lower equipment and minerals in a mineshaft. You will evaluate the model for its strengths and weaknesses and then create a set of specifications to develop a model of your own.

Analyzing a possible model

The formula $y = 2.5t^3 - 15t^2$ represents the position of the elevator, *y*, measured in meters ($y = 0$ represents ground level) and *t* represents time measured in minutes ($t = 0$ is the starting time). We know that the trip up and down the shaft, ignoring time spent at the foot of the shaft, is approximately six minutes and that the depth of the shaft is no more than 100 metres.

Use suitable computer software, or your calculator in parametric mode, to visualize the motion of the elevator in the shaft. Create a table of values for the position of the elevator for $t = 0, 1, 2, 3, 4, 5, 6$ minutes.

If using your calculator in parametric mode, you may want to view the motion along the line $x = -1$. Changing the increment value of *x* may be helpful. Continuing in parametric or changing to function mode, enter the displacement, velocity and acceleration functions.

1. Use these functions to:

(a) interpret the original vertical line motion simulation,

(b) explain the meaning of the negative, positive and zero values of the velocity graph,

(c) explain the relationship between velocity and acceleration in the intervals when the elevator speeds up, slows down, and is at rest,

(d) evaluate the usefulness and identify the problems of the model in the given situation.

Creating your own model

2. List specifications for a redesign of the freight elevator model.

3. Hence, create a model of your own. You may use a single function or you may combine functions in a piecewise function. In using piecewise functions, make sure that the joining points (except perhaps at the bottom of the shaft) are smoothly connected. Interpret what this means mathematically.

4. Explain how your model addresses the problems of the given model and satisfies the specifications of a well-functioning elevator for the mining company.

Applying your model

Explain how your model may be modified to be useful in other situations.

MODELLING THE HEIGHTS OF SAPLINGS — HL TYPE II

Description

The following data come from a random sample of 1000 saplings collected from a representative 0.25 km^2 area in a forest. Your job is to find a way to use this random sample to develop a *probability density function* (*pdf*) to model the population of all such saplings in the area in which the sample was taken.

Heights (m)	**Frequency**
0.00–0.25	**61**
0.25–0.50	**160**
0.50–0.75	**209**
0.75–1.00	**202**
1.00–1.25	**158**
1.25–1.50	**105**
1.50–1.75	**58**
1.75–2.00	**29**
2.00–2.25	**12**
2.25–2.50	**4**
2.50–2.75	**1**
2.75–3.00	**1**

Checking available models

1. Find a graphical way to show the data.

2. Add two other columns to the chart, entering the expected frequencies for the 1000 saplings for both a normal and Poisson distribution. Test each as a possible model for the data.

3. Could either of these be developed as a probability density function for the data? Explain.

Creating your own model

4. List the requirements for a probability density function.

5. A useful method for finding a *pdf* for a given data set is to begin with a function that best describes the data's associated *cumulative relative frequency curve*. What traits must this function possess in order to be defined as a *cumulative density function* (*cdf*) for the *pdf* you are creating?

6. Find a suitable function for this *cdf*, check it to see if it has the necessary traits, and refine it if necessary.

7. What is the relationship between your *cdf* and the required *pdf* ? Explain. Find the *pdf* for this *cdf* and evaluate it. If it does not satisfy all conditions, might it still be acceptable?

8. Continue the search for a good *cdf* and *pdf*. Explain why your final result is the best and indicate its limitations.

Applying your model

9. If you were asked to develop a *pdf* for a different forest region, comment on the applicability of the current model.

MODELLING PROBABILITIES IN GAMES OF TENNIS **HL TYPE II**

Description

The purpose of this task is to analyze the relationship between certain probability distributions and the process of match play in tennis. We will construct a probability model first for two players of predictable abilities. We will also investigate two models of resolving ties.

Method

Part 1: Club practice.

1. Suppose two players have played against each other often enough to know that Adam wins about twice as many points as Ben does. The two decide to play 10 points of practice at their club.

(a) What would be an appropriate model for the distribution of X, the number of points won by Adam? Do you have any concerns about its validity? Are there limitations to its value?

(b) Suppose a point that Adam wins is designated as A and one that Ben wins as B. Using the distribution you have chosen, calculate all possible values of the random variable X, *the number of points won by Adam*, and draw a histogram from your chart. Document any technology that you are using to help with the calculations and/or graphing.

(c) Find the expected value and standard deviation of this distribution. In this context, what can you say about what usually happens in these 10-point practices?

Part 2: Non-extended play games.

2. When Adam and Ben play against each other in club events, their probabilities of winning points are approximately the same as above. In club play, the tennis rules are generally followed (win with at least four points and by at least two points in each game), but to save court time, no game is allowed to go beyond 7 points. This means that if *deuce* is called (each player has 3 points), the next point determines the winner. Show that there are 70 possible ways that such a game might be played. To assist with this let Y be the number of points played. What values can Y take? For each possible value of Y find the number of possible ways that such a game could be played, and show the probability model for such a game. Be sure to define a random variable for the distribution.

3. Using the model you developed in 2, what is the **probability** that Adam wins the game? What are the **odds** that he wins?

4. Generalise this to find the probability that Player C wins in terms of *c* and *d*, where *c* represents the probability that Player C wins a point and *d* represents the probability that Player D wins a point.

(This task continues on the following page)

Part 3: Extended play games.

5. When Adam and Ben play against each other in tournaments outside the club, their point-winning probabilities remain the same (2/3 and 1/3, respectively), but the rules now require that players win by 2 points and therefore, that games may in theory be infinitely long.

Show that although Adam's point odds against Ben are 2:1, his game odds are almost 6:1. Be sure to consider separately the cases of non-deuce and deuce games.

6. Suppose that, more generally, Player C's probability of winning a point is c and Player D's probability of winning a point is d. Write formulas in terms of c and d for these probabilities: probability that Player C wins without deuce being called, probability that deuce is called, and probability that Player C wins given that deuce is called. Using these formulas to aid calculation, find the odds that Player C wins for $c = 0.5,\ 0.55,\ 0.6,\ 0.7,\ 0.9$ and any other values you would like to test. A spreadsheet approach is encouraged.

7. What expression represents the **odds** in such situations? What happens when the winning probabilities are close together or when one player is almost certain to win each point?

8. Evaluate the usefulness and limitations of such probability models.

MODELLING THE COURSE OF A VIRAL ILLNESS AND ITS TREATMENT HL TYPE II

Description

When viral particles of a certain virus enter the human body, they replicate rapidly. In about four hours, the number of viral particles has doubled. The immune system does not respond until there are about 1 million viral particles in the body.

The first response of the immune system is fever. The rise in temperature lowers the rate at which the viral particles replicate to 160 % every four hours, but the immune system can only eliminate these particular viral particles at the rate of about 50 000 viral particles per hour. Often people do not seek medical attention immediately as they think they have a common cold. If the number of viral particles, however, reaches 10^{12}, the person dies.

Modelling infection

1. Model the initial phase of the illness for a person infected with 10 000 viral particles to determine how long it will take for the body's immune response to begin.

2. Using a spreadsheet, or otherwise, develop a model for the next phase of the illness, when the immune response has begun but no medications have yet been administered. Use the model to determine how long it will be before the patient dies if the infection is left untreated.

Modelling recovery

An antiviral medication can be administered as soon as a person seeks medical attention. The medication does not affect the growth rate of the viruses but together with the immune response can eliminate 1.2 million viral particles per hour.

3. If the person is to make a full recovery, explain why effective medication must be administered before the number of viral particles reaches 9 to 10 million.

The antiviral medication is difficult for the body to adapt to, so it must *initially* be carefully introduced to the body over a four-hour time period of **continuous** intravenous dosing. This means the same amount of medication is entering the body at any given time during the first 4 hours. At the same time, however, the kidneys eliminate about 2.5 % of this medication per hour. The doctor has calculated that the patient needs at least 90 micrograms of medication to begin and maintain the rate of elimination of 1.2 million viral particles.

4. Create a mathematical model for this four-hour period so that by the end of the four hour period the patient has 90 micrograms of medication in their body. Find the solution to your model analytically, or estimate its solution with the help of technology.

Once the level of medication has reached 90 micrograms the patient is taken off the intravenous phase and given injections every four hours. The kidneys will still be working to eliminate the medication, so the doctor must calculate the additional dosage, D accordingly. Dosage D should allow for maintenance of a minimum of 90 micrograms within the patient's bloodstream throughout the treatment regimen.

(This task continues on the following page)

5. What dosage, *D*, administered every four hours from the end of the first continual intravenous phase, would allow for the patient to maintain at least 90 micrograms of the medication in his system? Make sure you take into account the kidneys' rate of elimination. Explain carefully how you came to this number.

6. Determine the last possible time from the onset of infection to start the regimen of medication. How long it will take to clear the viral particles from the patient's system? Show on a graph the entire treatment regimen from the time treatment begins until the viral particles are eliminated.

Analyzing your models

Analyze all your models discussing any assumptions you have made, the strengths and weaknesses of the models, and the reliability of your results.

Applying your model

7. Explain how your models could be modified for use if the patient were not an adult, but a young child.