Monotone Skew-Product Linear Dynamical Systems

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Perron Theorem

Perron Theorem

If A is a $d \times d$ matrix with positive entries, then, denoting its spectral radius by $\rho(A)$, the following holds:

- $\rho(A) > 0$ *is a simple eigenvalue (called the* principal eigenvalue*) of A,*
- *an eigenvector corresponding to $\rho(A)$ can be chosen to have its coordinates positive (then it is called a* principal eigenvector*),*
- *the remaining eigenvalues of A have their moduli less than $\rho(A)$,*
- *any nonzero vector belonging to the invariant subspace corresponding to the remaining eigenvalues has coordinates of opposite signs.*

Perron Theorem: Interpretation in Terms of Dynamical Systems

The (nonnegative) iterates of A form a linear dynamical system, $(A^k)_{k=0}^{\infty}$.

Denote by E the invariant linear subspace corresponding to $\rho(A)$, and by F the complementary invariant subspace.

Then the directions of vectors not belonging to F are exponentially attracted towards the direction of E: there exist $C \geq 1$ and $\nu \in [0, 1)$ such that

$$\frac{\|A^k v\|}{\|A^k u\|} \leq C\nu^k \frac{\|v\|}{\|u\|} \quad v \in F, \ u \in E \setminus \{0\}, \ k = 1, 2, 3, \dots. \tag{1}$$

Basic Concepts: Generalizations

Instead of a singleton we take either

- a compact metrizable space Y on which a (two-sided) continuous flow $\phi = (\phi_t)_{t \in \mathbb{R}}$ acts **(topological case)**,

or

- a measurable space $(\Omega, \mathfrak{F}, \mathbb{P})$ on which a (two-sided) $\mathbb{P}$-preserving ergodic measurable flow $\theta = (\theta_t)_{t \in \mathbb{R}}$ acts **(measurable case)**.

Basic Concepts: Some Motivation

Motivation:

- (in the topological case) a family of systems of linear ordinary differential equations (ODEs)

$$\dot{u} = A(t)\,u, \qquad u \in \mathbb{R}^d,\ t \in \mathbb{R},\ A(t) \in \mathbb{R}^{d\times d},$$

parameterized by $A(\cdot) \in Y$, where Y is the closure, in an appropriate topology, of the set of all time translates of a nonautonomous linear ODEs system,

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  parameterized by $A(\cdot) \in Y$, where Y is the closure, in an appropriate topology, of the set of all time translates of a nonautonomous linear ODEs system,

- (in the measurable case) a family of systems of ODEs

$$\dot{u} = A(\theta_t \omega)\, u, \qquad u \in \mathbb{R}^d,\ t \in \mathbb{R},$$

  parameterized by $\omega \in \Omega$, where $[\,\Omega \ni \omega \mapsto A(\omega) \in \mathbb{R}^{d \times d}\,]$ is (at least) measurable.

Basic Concepts: Some Motivation, cont'd

Another source of examples in the topological case:
Assume that for an autonomous (nonlinear) C^1 system of ODEs

$$\dot{v} = F(v),$$

Y is a compact invariant set. Then for ϕ on Y we take the flow generated by the original ODEs system, and, to fit into the former example, we just assign to $y \in Y$ the matrix function

$$A(t) = [(\mathrm{D}_i F_j)(\phi_t(y))]_{i,j=1}^{d}.$$

Definition of Topological Linear Skew-Product DS with Finite-Dimensional Fibers

A *topological linear skew-product dynamical system* on $Y \times V$, where V is a Euclidean space (a *fiber*), *covering* a continuous flow ϕ on the *base* Y, is given as

$$((\phi_t)_{t \in \mathbb{R}}, (U_y(t))_{y \in Y,\ t \in \mathbb{R}}),$$

where $U_y(t)$ is, for each $y \in Y$ and each $t \in \mathbb{R}$, a linear automorphism of V satisfying the following

- $U_y(0) = \mathrm{Id}_V$ for all $y \in Y$,
- **(cocycle equation)** $U_y(t+s) = U_{\phi_s(y)}(t) \circ U_y(s)$ for any $y \in Y$ and any $s, t \in \mathbb{R}$,
- the mapping $[\, Y \times \mathbb{R} \ni (y,t) \mapsto U_y(t) \in \mathcal{L}(V,V) \,]$ is continuous.

Definition of Measurable Linear Skew-Product DS with Finite-Dimensional Fibers

A *measurable linear skew-product dynamical system* (DS) on $\Omega \times V$, where V is a Euclidean space (a *fiber*), *covering* a measurable flow θ on the *base* Ω, is given as

$$\big((\theta_t)_{t\in\mathbb{R}}, (U_\omega(t))_{\omega\in\Omega,\, t\in\mathbb{R}}\big),$$

where $U_\omega(t)$ is, for each $\omega \in \Omega$ and each $t \in \mathbb{R}$, a linear automorphism of V, satisfying the following

- $U_\omega(0) = \mathrm{Id}_V$ for all $\omega \in \Omega$,
- **(cocycle equation)** $U_\omega(t+s) = U_{\theta_s(\omega)}(t) \circ U_\omega(s)$ for any $\omega \in \Omega$ and any $s, t \in \mathbb{R}$,
- the mapping $[\,\Omega \times \mathbb{R} \ni (\omega, t) \mapsto U_\omega(t) \in \mathcal{L}(V, V)\,]$ is measurable.

Definition of Measurable Linear Skew-Product DS with Finite-Dimensional Fibers, cont'd

The following *Carathéodory conditions* are sufficient for the measurability of $[\,(\omega, t) \mapsto U_\omega(t)\,]$:

- for each $\omega \in \Omega$ the mapping $[\,\mathbb{R} \ni t \mapsto U_\omega(t)\,]$ is continuous,
- for each $t \in \mathbb{R}$ the mapping $[\,\Omega \ni \omega \mapsto U_\omega(t)\,]$ is measurable.

Generalizations and Extensions

- A fiber V is an infinite-dimensional Banach space (appears, e.g., when the skew-product system is generated by a system of parabolic partial differential equations (PDEs) or a system of delay ODEs).

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  - the assumption that $U_y(t)$ (or $U_\omega(t)$) is a linear isomorphism is much too strong; it is replaced by stipulating that $U_y(t)$ or $U_\omega(t)$ are bounded linear operators; as a consequence, we have a linear skew-product **semi**dynamical system $((\phi_t)_{t\in\mathbb{R}}, (U_y(t))_{y\in Y,\, t\geq 0})$ covering a (two-sided) dynamical system ϕ;

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  - continuous/measurable dependence of $U_y(t)$ (or $U_\omega(t)$) in the norm topology is too strong an assumption; the strong operator topology is natural here;
  - even in the strong operator topology, joint continuity (from the right) at $t = 0$ can be a problem.

Generalizations and Extensions, cont'd

- The dynamical system ϕ (or θ) on the base is defined only for $t \geq 0$ (in such a case we say that ϕ or θ is a *one-sided semiflow*).

Standing Assumption

In the measurable case (one-sided semiflow on the base space is permitted) we make an additional assumption.

(A) The functions

$$\left[\Omega \ni \omega \mapsto \sup_{0 \le s \le 1} \ln^{+} \| U_{\omega}(s) \| \right]$$

and

$$\left[\Omega \ni \omega \mapsto \sup_{0 \le s \le 1} \ln^{+} \| U_{\theta_s(\omega)}(1 - s) \| \right]$$

belong to $L_1((\Omega, \mathfrak{F}, \mathbb{P}))$.

Standing Assumption, cont'd

Under (A), Kingman's subadditive ergodic theorem states the existence of $\lambda_{\mathrm{top}} \in [-\infty, \infty)$ (the *top Lyapunov exponent*) such that

$$\lim_{t\to\infty} \frac{\ln \|U_\omega(t)\|}{t} = \lambda_{\mathrm{top}}$$

for $\mathbb{P}$-a.e. $\omega \in \Omega$.

Standing Assumption, cont'd

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for $\mathbb{P}$-a.e. $\omega \in \Omega$.

Also, we assume that Ω is a Lebesgue space and that the measure $\mathbb{P}$ is complete.

Oseledets Filtration

The following is a (tiny) part (the first stage, indeed) of the Oseledets theorem.

Oseledets Filtration

Let V be separable. Assume that $U_\omega(1)$ is a compact operator for all $\omega \in \Omega$. Let $\lambda_{\mathrm{top}} > -\infty$. Then there exist

- *a $\mathbb{P}$-measurable subset $\widetilde{\Omega} \subset \Omega$ with $\theta_t(\widetilde{\Omega}) = \widetilde{\Omega}$ for all $t \in \mathbb{R}$, $\mathbb{P}(\widetilde{\Omega}) = 1$,*
- *a family $\{\widetilde{F}_1(\omega)\}_{\omega \in \widetilde{\Omega}}$ of subspaces of V of constant finite codimension, depending measurably* (as elements of the Grassmannian) *on ω,*
- $\lambda_2 \in [-\infty, \lambda_{\mathrm{top}})$

such that for any $\omega \in \widetilde{\Omega}$ there holds:

Oseledets Filtration, cont'd

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- $U_\omega(t)\,\widetilde{F}_1(\omega) \subset \widetilde{F}_1(\theta_t(\omega))$ *for all* $t \ge 0$,
- $$\lim_{t\to\infty} \frac{\ln \left\| U_\omega(t)\restriction_{\widetilde{F}_1(\omega)} \right\|}{t} = \lambda_2,$$
- $$\lim_{t\to\infty} \frac{\ln \| U_\omega(t)\, u \|}{t} = \lambda_{\mathrm{top}}$$

  for all $u \in V \setminus \widetilde{F}_1(\omega)$.

Oseledets Decomposition

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Let the dynamical system θ on $(\Omega, \mathfrak{F}, \mathbb{P})$ be two-sided, and let V be separable. Assume that $U_\omega(1)$ is a compact operator for all $\omega \in \Omega$. Let $\lambda_{\mathrm{top}} > -\infty$. Then there exists a family $\{\widetilde{E}_1(\omega)\}_{\omega \in \widetilde{\Omega}}$ of subspaces of V of constant finite dimension, depending measurably on ω, such that for any $\omega \in \widetilde{\Omega}$ there holds

- $\widetilde{E}_1(\omega) \oplus \widetilde{F}_1(\omega) = V$,
- *letting $P(\omega)$ stand for the projection of V onto $\widetilde{F}_1(\omega)$ along $\widetilde{E}_1(\omega)$, we have*

$$\lim_{t \to \pm\infty} \frac{\ln \|P(\theta_t(\omega))\|}{t} = 0$$

(the family of projections $\{P(\omega)\}_{\omega \in \widetilde{\Omega}}$ is tempered),

Oseledets Decomposition, cont'd

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- $U_\omega(t)\,\widetilde{E}_1(\omega) = \widetilde{E}_1(\theta_t(\omega))$ *for all* $t \geq 0$,
- $\widetilde{E}_1(\omega) \setminus \{0\}$ *can be characterized as the set of those* $u \in V$ *for which there exists a function* $\hat{w}_\omega \colon \mathbb{R} \to V$ *satisfying*
    - $\hat{w}_\omega(0) = u$, *and* $U_{\theta_s(\omega)}(t)\,\hat{w}_\omega(s) = \hat{w}_\omega(s+t)$ *for any* $s \in \mathbb{R}$ *and any* $t \geq 0$ (*we call such functions* entire orbits),
    - $$\lim_{t\to\pm\infty} \frac{\ln \|\hat{w}_\omega(t)\|}{t} = \lambda_{\mathrm{top}}.$$

Definition of Cone

By a *cone* in a Banach space V we understand a closed convex subset $V_+ \neq \{0\}$ such that

- $[0, \infty) V_+ \subset V_+$;
- V_+ does not contain a one-dimensional linear subspace of V.

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A cone V_+ is

- *solid* if its interior, V_{++}, is nonempty,
- *reproducing* if $V_+ - V_+ = V$,
- *total* if the closure of $V_+ - V_+$ equals V.

Definition of Cone, Cont'd

A pair (V, V_+), where V_+ is a cone in V, is called an *ordered Banach space*.

For $u, v \in V$ we write

- $u \leq v$ if $v - u \in V_+$,
- $u < v$ if $v - u \in V_+$ and $u \neq v$.

If V_{++} is solid, write $u \ll v$ if $v - u \in V_{++}$.

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If V_{++} is solid, write $u \ll v$ if $v - u \in V_{++}$.
A cone V_+ is called *normal* if there is $K > 0$ such that

$$\|u\| \le K\|v\|$$

for any $0 \le u \le v$. When the cone V_+ is normal, V can be renormed so that $K = 1$.
When speaking of a normal cone we always assume such renorming.

Examples of Cones

- Let $V = \mathbb{R}^d$ with the Euclidean norm,

$$V_+ = \{\, u = (u_1, \dots, u_d) : u_i \geq 0 \text{ for all } 1 \leq i \leq d \,\}.$$

The cone V_+ is solid and normal, with

$$V_{++} = \{\, u = (u_1, \dots, u_d) : u_i > 0 \text{ for all } 1 \leq i \leq d \,\}.$$

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- Let $V = C([0,1])$ with the standard norm,

$$V_+ = \{u \in C([0,1]) : u(x) \geq 0 \text{ for all } x \in [0,1]\,\}.$$

The cone V_+ is solid and normal, with monotone norm. The interior of V_+ equals

$$V_{++} = \{u \in C([0,1]) : u(x) > 0 \text{ for all } x \in [0,1]\,\}.$$

Examples of Cones, cont'd

- Let $V = L_p((0,1))$, $1 \le p < \infty$, with the standard norm,

$$V_+ = \{u \in L_p((0,1)) : u(x) \geq 0 \text{ for Lebesgue-a.e. } x \in (0,1)\}$$

The cone V_+ is not solid but reproducing and normal, with monotone norm.

Examples of Cones, cont'd

- Let $V = C^1([0,1])$ with the standard norm,

$$V_+ = \{u \in C^1([0,1]) : u(x) \geq 0 \text{ for all } x \in [0,1]\}$$

The cone V_+ is solid but not normal.The interior of V_+ equals

$$V_{++} = \{u \in C^1([0,1]) : u(x) > 0 \text{ for all } x \in [0,1]\}.$$

Examples of Cones, cont'd

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$$V_{++} = \{u \in C^1([0,1]) : u(x) > 0 \text{ for all } x \in [0,1]\}.$$

- Let $V = C([0,1])$ with the standard norm, but now

$$V_+ = \{u \in C([0,1]) : u \text{ is nondecreasing and } u(x) \geq 0 \text{ for all } x \in [0,1]\}$$

The cone V_+ is normal and not reproducing, but total.

Definition of Monotone Linear Skew-Product Dynamical System

From now on we assume that (V, V_+) is an ordered Banach space.

A topological linear skew-product semidynamical system $((\phi_t), (U_y(t)))$ on $Y \times V$ is *monotone* if $U_y(t)\, V_+ \subset V_+$ for all $y \in Y$ and all $t \geq 0$.

A measurable linear skew-product semidynamical system $((\theta_t), (U_\omega(t)))$ on $\Omega \times V$ is *monotone* if $U_\omega(t)\, V_+ \subset V_+$ for all $\omega \in \Omega$ and all $t \geq 0$.

Entire Positive Orbits

Theorem

Let $((\theta_t), (U_\omega(t)))$ be a monotone measurable linear skew-product semidynamical system, with two-sided θ and total V_+. Assume that $\lambda_{\mathrm{top}} > -\infty$. Then for $\mathbb{P}$-a.e. $\omega \in \Omega$ there holds

- $\widetilde{E}_1(\omega) \cap V_+ \neq \{0\}$,

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- $\widetilde{E}_1(\omega) \cap V_+ \neq \{0\}$,
- *there is an entire orbit $\hat{w}_\omega$ such that $\hat{w}_\omega(t) \in (\widetilde{E}_1(\omega) \cap V_+) \setminus \{0\}$ for all $t \in \mathbb{R}$ (such $\hat{w}_\omega$ is called an* entire positive orbit*).*

Strong Monotonicity

We say a monotone topological linear skew-product semidynamical system $((\phi_t), (U_y(t)))$ is *strongly monotone* if V is strongly ordered and $U_y(1)\,(V_+ \setminus \{0\}) \subset V_{++}$ for all $y \in Y$.

Exponential Separation

Theorem on Exponential Separation

Let $((\phi_t), (U_y(t)))$ be a strongly monotone topological linear skew-product semidynamical system covering a two-sided flow ϕ on Y, such that $U_y(1)$ is compact for all $y \in Y$. Then there exist

- *a family $\mathcal{E}$ of one-dimensional linear subspaces $\{E(y)\}_{y \in Y} = \{\mathrm{span}\{w(y)\}\}_{y \in Y}$, $\|w(y)\| = 1$, continuously depending on $y \in Y$,*
- *a family $\mathcal{F}$ of one-codimensional linear subspaces $\{F(y)\}_{y \in Y}$ continuously depending on $y \in Y$,*
- *constants $C \geq 1$ and $\mu > 0$*

such that for any $y \in Y$ there holds:

Exponential Separation, cont'd

Theorem on Exponential Separation, cont'd

1. $E(y) \oplus F(y) = V$,
2. $U_y(t)\, E(y) = E(\phi_t(y))$ *for all* $t \geq 0$,
3. $U_y(t)\, F(y) \subset F(\phi_t(y))$ *for all* $t \geq 0$,
4. $w(y) \in V_{++}$,
5. $F(y) \cap V_{+} = \{0\}$,

Exponential Separation, cont'd

Theorem on Exponential Separation, cont'd

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$$\frac{\|U_y(t)\,v\|}{\|U_y(t)\,u\|} \le Ce^{-\mu t}\,\frac{\|v\|}{\|v\|}$$

for all $t \ge 0$, $v \in F(y)$ *and* $u \in E(y)$, $u \ne 0$.

The property 6 (in general, with no conditions on dimension) was called *exponential separation*. However now, especially in the theory of smooth dynamical systems, it is called *dominated splitting*. In some 1970–80s papers by Bronshteĭn and Cherniĭ a more general property was called a *Perron condition*.

Comments on Exponential Separation

The dynamical meaning of the exponential separation is the following: for any $y \in Y$ and any vector $u \in V \setminus F(y)$, the directions of $U_y(t)\,u$ converge, as $t \to \infty$, exponentially to directions in $\mathcal{E}$.

It follows, in particular, that any $y \in Y$ and any $u \in V \setminus F(y)$ the vectors $U_y(t)\,u$ eventually belong to V_{++}. Consequently, $\mathcal{F}$ can be characterized as the set of vectors which never, in the future, get into V_{++}.

Comments on Exponential Separation

Perhaps even more important property is that for each $y \in Y$ the function $\hat{w}_y \colon \mathbb{R} \to V_{++}$ defined by

$$\hat{w}_y(t) := \begin{cases} \dfrac{w(\phi_t(y))}{\|U_{\phi_t(y)}(-t)\, w(\phi_t(y))\|} & \text{for } t < 0 \\ U_y(t)\, w(y) & \text{for } t \geq 0 \end{cases}$$

is a positive entire orbit, and any positive entire orbit is of the form $\alpha w_y(\cdot)$, where $\alpha > 0$.

Exponential Separation: References

The first to prove such a theorem in the case of V finite-dimensional was D. Ruelle: *Analycity* **(sic!)** *properties of characteristic exponents of random matrix products*, Adv. Math. **32** (1979), 68–80 (discrete time).

In the infinite-dimensional case it was proved by P. Poláčik and I. Tereščák: *Exponential separation and invariant bundles for maps in ordered Banach spaces with applications to parabolic equations*, J. Dynam. Differential Equations **5** (1993), 279–303 (discrete time).

Exponential Separation: References, cont'd

J. Húska, P. Poláčik and M. V. Safonov (in various combinations) proved the existence of exponential separation in a very concrete situation of second order parabolic PDEs, making use of (a form of) the Harnack inequality, see, e.g., their joint paper *Harnack inequalities, exponential separation, and perturbations of principal Floquet bundles for linear parabolic equations*, Ann. Inst. H. Poincaré **24** (2007), 711–739.

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Their ideas were applied later in J. M. and W. Shen's monograph *Spectral Theory for Random and Nonautonomous Parabolic Equations and Applications*, Chapman & Hall/CRC Monographs and Surveys in Pure and Applied Mathematics, **139**, CRC Press, Boca Raton, 2008.

Exponential Separation: A Generalization

One should mention here an (again abstract) result proved by I. Tereščák (*Dynamics of C^1 smooth strongly monotone discrete-time dynamical systems*, unpublished manuscript), where it is assumed that there is a discrete-time **one-sided** topological semidynamical system on the base space Y, that is, the iterates of a continuous $F\colon Y \to Y$, and that $U_y(1)$, $y \in Y$, are compact linear operators from V into V, depending continuously in the norm topology on $y \in Y$, and such that $U_y(1)\,(V_+ \setminus \{0\}) \subset V_{++}$ for all $y \in Y$.

Then the family $\mathcal{F}$ is well defined, and may be characterized as the set of those (y, u) such that $U_y(k)\,u \notin V_{++}$ for any $k \in \mathbb{N}$. Instead of one $w(y)$ there is a whole family of unit vectors in V_{++}, with the property that the union of all those families over $y \in Y$ is compact. Exponential separation is retained.

The proof rests on applying the inverse limit construction.

Hilbert's Projective Metric

A useful tool is Hilbert's projective metric.

Let V_+ be a cone. We introduce an equivalence relation $\sim$ on V_+: $u \sim v$ (we say u and v are *comparable*) if and only if there are $0 < \alpha \le \beta$ such that $\alpha u \le v \le \beta u$.

Equivalence classes of $\sim$ are called *parts* of V_+. Notice that $\{0\}$ is a part of V_+, and if V_+ is solid then V_{++} is a part of V_+.

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Equivalence classes of $\sim$ are called *parts* of V_+. Notice that $\{0\}$ is a part of V_+, and if V_+ is solid then V_{++} is a part of V_+.

For comparable nonzero $u, v \in V_+$ define

$$d(u, v) := \ln \frac{\inf\{\alpha > 0 : u \le \alpha v\}}{\sup\{\alpha > 0 : \alpha u \le v\}}.$$

d satisfies all the properties of a metric, except that $d(u, v) = 0$ is equivalent to the existence of $\alpha > 0$ such that $u = \alpha v$. $d(\cdot, \cdot)$ is called *Hilbert's projective metric* (perhaps *pseudometric* would be more correct).

Properties of Hilbert's Projective Metric

- If V_+ is normal, then for any comparable $u, v \in V_+$, $\|u\| = \|v\| = 1$, there holds $\|u - v\| \leq 3(e^{d(u,v)} - 1)$. If V is a Banach lattice we can take 1 instead of 3.
- As a consequence of the above, if V_+ is normal then the intersection of any part of V_+ with the unit sphere is a complete metric space with respect to $d(\cdot, \cdot)$.

Properties of Hilbert's Projective Metric, cont'd

- For a linear operator $L \colon V \to V$ such that $L(V_+) \subset V_+$ define its *projective diameter* $\Delta(L)$ and its *Birkhoff contraction ratio* $q(L)$ as

$$\Delta(L) := \sup \left\{ d(Lu, Lv) : u, v \in V_+, Lu \sim Lv \right\},$$

$$q(L) := \sup \left\{ \frac{d(Lu, Lv)}{d(u, v)} : u, v \in V_+, u \sim v, d(u, v) > 0 \right\}.$$

Then either $\Delta(L) = \infty$ and $q(L) = 1$, or $\Delta(L) < \infty$ and $q(L) = \frac{1}{4} \tanh \Delta(L)$. Here "Birkhoff" refers to Garrett Birkhoff, the son of George David Birkhoff (of the Birkhoff ergodic theorem).

A Generic Theorem on Separation

Theorem on Separation, Measurable Case

Let $((\theta_t), (U_\omega(t)))$ be a monotone measurable linear skew-product semidynamical system, **with two-sided** θ*, with (V, V_+) an ordered Banach space, such that V_+ is normal and reproducing and V^* is separable. Assume that there exist*

- $\mathbf{e} \in V_+ \setminus \{0\}$ *and* $U_\omega(1)\,\mathbf{e} \neq 0$ *for all* $\omega \in \Omega$,
- *a measurable* $\varkappa\colon \Omega \to [1, \infty)$ *with* $\ln^+ \ln \varkappa \in L_1((\Omega, \mathfrak{F}, \mathbb{P}))$ *such that for any* $\omega \in \Omega$ *and any* $u \in V_+ \setminus \{0\}$ *there is* $\beta(\omega, u) > 0$ *with the property that*

$$\beta(\omega, u)\,\mathbf{e} \le U_\omega(1)\,u \le \varkappa(\omega)\beta(\omega, u)\,\mathbf{e}$$

(*the above property is called* focusing).

A Generic Theorem on Separation, cont'd

Theorem on Separation, Measurable Case, cont'd

Then there exist an invariant $\Omega_1 \subset \Omega$, $\mathbb{P}(\Omega_1) = 1$, *and*

- *a measurable function* $w \colon \Omega_1 \to V_+$ *with* $w(\omega) \sim \mathbf{e}$, $\|w(\omega)\| = 1$ *for all* $\omega \in \Omega_1$,
- *a measurable family of one-codimensional subspaces* $\{F_1(\omega)\}_{\omega \in \Omega_1}$

with the properties that, for each $\omega \in \Omega_1$,

- *putting* $E_1(\omega) := \mathrm{span}\{w(\omega)\}$, $U_\omega(t)\, E_1(\omega) = E_1(\theta_t(\omega))$,
- $U_\omega(t)\, F_1(\omega) \subset F_1(\theta_t(\omega))$,
- $F_1(\omega) \cap V_+ = \{0\}$,
- $$\lim_{t\to\infty} \frac{\ln \|U_\omega(t)\, u\|}{t} = \lambda_{\mathrm{top}} \qquad \text{for any } u \in V_+ \setminus \{0\},$$

A Generic Theorem on Separation, cont'd

Theorem on Separation, Measurable Case, cont'd

- *the function* $\hat{w}_\omega \colon \mathbb{R} \to V$ *defined as*

$$\hat{w}_\omega(t) := \begin{cases} \dfrac{w(\theta_t(\omega))}{\|U_{\theta_t(\omega)}(-t)\, w(\theta_t(\omega))\|} & \text{for } t < 0 \\ U_\omega(t) w(\omega) & \text{for } t \geq 0 \end{cases}$$

is a positive entire orbit, unique up to multiplication by positive scalars.

A Generic Theorem on Separation, cont'd

Theorem on Separation, Measurable Case, cont'd

If $\lambda_{\mathrm{top}} > -\infty$ *then there is* $\sigma > 0$ *such that for each* $\omega \in \Omega_0$ *there holds* $E_1(\omega) \oplus F_1(\omega) = V$ *and*

$$\lim_{t\to\infty} \frac{1}{t} \ln \frac{\|U_\omega(t)\restriction_{F_1(\omega)}\|}{\|U_\omega(t)\, w(\omega)\|} = -\sigma.$$

A Generic Theorem on Separation, Idea of Proof

$w(\omega)$ is defined as the pullback limit,

$$\lim_{t\to\infty} \frac{U_{\theta_{-t}(\omega)}(t)\,\mathbf{e}}{\|U_{\theta_{-t}(\omega)}(t)\,\mathbf{e}\|}$$

in Hilbert's projective metric.
$F_1(\omega)$ equals the nullspace of $w^*(\omega)$, the analog of $w(\omega)$ for the dual of the system on the dual space $\Omega \times V^*$.

An Aside: Finite-Dimensional Case

In: L. Arnold, V. M. Gundlach and L. Demetrius, *Evolutionary formalism for products of positive random matrices*, Ann. Appl. Probab. **4** (1994), 859–901, a proof is given, using Hilbert's projective metric, of the theorem in finite-dimensional case.

A Generic Theorem on Separation, Idea of Proof, cont'd

For a proof of the general theorem see J. M., W. Shen *Principal Lyapunov exponents and principal Floquet spaces of positive random dynamical systems. I. General theory*, Trans. Amer. Math. Soc. **365** (2013), 5329–5365, with some extensions in J. M., S. Novo, R. Obaya *Principal Floquet subspaces and exponential separations of type II with applications to random delay differential equations*, Discrete Contin. Dyn. Syst. **38**, December 2018, special issue on Llavefest, 6163–6193.

Separation vs. Oseledets Decomposition

If we assume additionally that $U_\omega(1)$ is compact, then the decomposition $E_1(\omega) \oplus F_1(\omega) = V$ obtained above is $\mathbb{P}$-a.e. equal to $\widetilde{E}_1(\omega) \oplus \widetilde{F}_1(\omega) = V$, see M. Kryspin, J. M., S. Novo, R. Obaya, *Two dynamical approaches to the notion of exponential separation for random systems of delay differential equations*, preprint, 2023, arXiv 2305.17990.

A Related Result: Top Lyapunov exponent as an Integral

Let us go back to a strongly monotone linear skew-product dynamical system generated by

$$\dot{u} = A(\theta_t(\omega))u, \quad u = (u_1, \dots, u_d) \in \mathbb{R}^d_+,$$

where $A(\cdot)$ is a $d \times d$-matrix with all entries positive. If (θ_t) is two-sided, then $w(\omega)$ is well-defined for $\mathbb{P}$-a.e. $\omega \in \Omega$, and it is a simple consequence of (G. D.) Birkhoff's ergodic theorem that

$$\lambda_{\mathrm{top}} = \int\limits_{\Omega} \langle A(\omega)w(\omega), w(\omega)\rangle \, \mathbb{P}(\mathrm{d}\omega).$$

If (θ_t) is one-sided, then, as there need be no (at least uniquely determined) function $w(\cdot)$, the above result cannot be extended word for word.

A Related Result: Top Lyapunov exponent as an Integral, cont'd

M. Benaïm, C. Lobry, T. Saari and E. Strickler in *A note on the top Lyapunov exponent of linear cooperative systems* (arXiv: 2302.05874) consider a situation where Ω is the path-space of a uniquely ergodic Feller Markov process on a compact metric space S. Then there exists a probability measure Π on $S \times \Delta$, where Δ is the standard probability simplex in $\mathbb{R}^d$, such that

$$\lambda_{\mathrm{top}} = \int\limits_{S\times\Delta} \langle A(s)\, u, \mathbb{1}\rangle \, \Pi(\mathrm{d}s\, \mathrm{d}u)$$

a.e. on Ω, where $\mathbb{1} = (1, \ldots, 1)$.

Carrying Simplices

A C^1 system of ODEs

$$\dot{x}_i = x_i f_i(x_1, \dots, x_d) \qquad x = (x_1, \dots, x_d) \in \mathbb{R}^d_+$$

is called *totally competitive* if $\mathrm{D}_j f_i(x) < 0$, $1 \le i, j \le d$. For a totally competitive systems of ODEs for which both the origin and the point at infinity are repellers (plus an additional assumption on the linearizations at equilibria) there exists an unordered invariant set Σ (the *carrying simplex*), homeomorphic to the standard probability $(d-1)$-simplex via radial projection and attracting any bounded set bounded away from the origin.

It follows from results by Müller and Kamke that the derivative dynamical system restricted to S is monotone for time reversed. Further, if we take a compact invariant $Y \subset \Sigma \cap \mathbb{R}^d_{++}$ then for the restriction to Y we have a strongly monotone linear skew-product dynamical system.

Carrying Simplices, cont'd

Here the interpretation is the following: the one-codimensional linear subspaces are regarded as tangent hyperplanes. Indeed, it can be proved that $\mathcal{V}$ equals a sort of C^1 tangent bundle of the carrying simplex Σ restricted to Y,

The main part in the construction is played by the difference between the exponential growth rates on Σ and in directions transverse to Σ. If that difference is (relatively) big then Y can be proved to be of class C^k with $k > 1$ (provided, of course, that the vector field is sufficiently regular).

An interesting sufficient condition, expressed in terms of Hilbert's projective metric, was given by Michel Benaïm in: *On invariant hypersurfaces of strongly monotone maps*, J. Differential Equations **137** (1997), 302–319.

Persistence in Population Models

Assume that a system of two populations is modeled by a system of ODEs

$$\dot{u}_i = u_i f_i(u), \qquad u = (u_1, u_2) \in \mathbb{R}^2_+,$$

satisfying appropriate dissipativity conditions. For an invariant measure μ supported on $C_j := \{\, u \in \mathbb{R}^2_+ : u_j = 0 \,\}$, $j = 1, 2$, the number

$$\int_{C_j} f_i \, \mathrm{d}\mu,$$

where $\{i, j\} = \{1, 2\}$, called the *exterior Lyapunov exponent*, is important in ascertaining the persistence.

Persistence in Population Models, cont'd

Indeed, if

$$\min_{\mu \in \mathcal{M}_1} \int_{C_1} f_2 \, \mathrm{d}\mu > 0 \quad \text{and} \quad \min_{\mu \in \mathcal{M}_2} \int_{C_2} f_1 \, \mathrm{d}\mu > 0,$$

where $\mathcal{M}_j$, $j = 1, 2$, denotes the set of all invariant measures supported on C_j, then there exits $\eta > 0$ such that any solution starting in $\mathbb{R}^2_{++}$ is eventually at a distance $> \eta$ of $\mathbb{R}^2_{+} \setminus \mathbb{R}^2_{++}$ (B. Garay, J. Hofbauer, S. J. Schreiber).

Persistence in Population Models, cont'd

Indeed, if

$$\min_{\mu \in \mathcal{M}_1} \int_{C_1} f_2 \, \mathrm{d}\mu > 0 \quad \text{and} \quad \min_{\mu \in \mathcal{M}_2} \int_{C_2} f_1 \, \mathrm{d}\mu > 0,$$

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The above property holds under some stochastic perturbation, as shown, among others, in M. Benaïm, J. Hofbauer and W. H. Sandholm, *Robust permanence and impermanence for stochastic replicator dynamics*, J. Biol. Dyn. **2** (2008), 180–195, and in S. J. Schreiber, *On persistence and extinction for randomly perturbed dynamical systems*, Discrete Contin. Dyn. Syst. Ser. B **7** (2007), 457–463.

Persistence in Population Models: Connections to Monotone Flows

Consider a model of two spatially distributed interacting populations

$$\frac{\partial u_i}{\partial t} = \Delta u_i + u_i f_i(x, u), \qquad t > 0, \; u = (u_1, u_2) \in \mathbb{R}^2_+, \; x \in D, \tag{PDE}$$

where Δ is the Laplace operator, $D \subset \mathbb{R}^N$, $N = 1, 2$ or 3, is a bounded domain with sufficiently smooth boundary ∂D, endowed with the Dirichlet boundary conditions

$$u_i(t, x) = 0 \quad i = 1, 2, \; t > 0, \; x \in \partial D. \tag{BC}$$

Persistence in Population Models: Connections to Monotone Flows, cont'd

As is usual in applications of the dynamical systems theory to (evolutionary) PDEs, a solution of (PDE)+(BC) is looked upon as a function of time variable t taking values in a Banach space $V^{(2)}$ of $\mathbb{R}^2$- (or rather $\mathbb{R}^2_+$-)valued functions (or equivalence classes of functions) defined on D.

Under suitable regularity assumptions, for any initial value $u_0 = (u_{0,1}, u_{0,2}) \in V^{(2)}$ defined on D there is a locally unique solution $u(\cdot; u_0)$ to (PDE)+(BC) defined on some $[0, T)$, $T > 0$, and satisfying $u(0; u_0) = u_0$. We will use the semiflow notation: $\phi_t(u_0)$ instead of $u(\cdot; u_0)$.

Persistence of Population Models: Connections to Monotone Flows, cont'd

Examples of such spaces $V^{(2)}$ are:

Persistence of Population Models: Connections to Monotone Flows, cont'd

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Persistence of Population Models: Connections to Monotone Flows, cont'd

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Persistence of Population Models: Connections to Monotone Flows, cont'd

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Persistence of Population Models: Connections to Monotone Flows, cont'd

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Persistence of Population Models: Connections to Monotone Flows, cont'd

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- various interpolation spaces, as Sobolev–Slobodetskiĭ spaces, Bessel potential spaces, Besov spaces, fractional power spaces,

Persistence of Population Models: Connections to Monotone Flows, cont'd

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Persistence of Population Models: Connections to Monotone Flows, cont'd

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- various interpolation spaces, as Sobolev–Slobodetskiĭ spaces, Bessel potential spaces, Besov spaces, fractional power spaces,
- via Sobolev embedding theorems, Hölder spaces;

and another thread:

- $C(\bar{D}, \mathbb{R}^2)$,
- $C^1(\bar{D}, \mathbb{R}^2)$.

Persistence in Population Models: Connections to Monotone Flows, cont'd

Put

$$V_+^{(2)} := \{\, u = (u_1, u_2) \in V : u_1(x) \geq 0 \text{ and } u_2(x) \geq 0, \text{ for (perhaps a.e.) } x \in D \,\}.$$

We consider only solutions starting from $u_0 \in V_+^{(2)}$. By the standard parabolic maximum principle they remain in $V_+^{(2)}$ (as long as they exist).

Persistence in Population Models: Connections to Monotone Flows, cont'd

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We consider only solutions starting from $u_0 \in V_+^{(2)}$. By the standard parabolic maximum principle they remain in $V_+^{(2)}$ (as long as they exist).

Assume that (PDE)+(BC) is *dissipative*: there is a bounded subset of $V_+^{(2)}$ such that each solution eventually takes values in that subset.
A first consequence of dissipativity is that each solution is defined on $[0, \infty)$.

Persistence in Population Models: Connections to Monotone Flows, cont'd

Further, it follows from dissipativity that there exists a compact and invariant set $A \subset V_+^{(2)}$ (the *global attractor*) attracting each bounded subset of $V_+^{(2)}$.

By the injectivity property for parabolic PDEs, the solution operator restricted to A is, for each $t \geq 0$, a homeomorphism, so we have a two-sided topological dynamical system on A.

Persistence in Population Models: Connections to Monotone Flows, cont'd

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By the injectivity property for parabolic PDEs, the solution operator restricted to A is, for each $t \geq 0$, a homeomorphism, so we have a two-sided topological dynamical system on A.

The compact set $\{ u \in A : u_2 = 0 \}$ is also invariant. We take this set for Y.

The topological skew-product linear semiflow is generated on $Y \times V$ by a single PDE

$$\frac{\partial v}{\partial t} = \Delta v + f_2(x, \phi_t(y)(x))\, v, \qquad t > 0,\ x \in D,\ y \in Y$$

(plus the boundary conditions).

Persistence in Population Models: Connections to Monotone Flows, cont'd

The strong parabolic maximum principle provides that the above semiflow has required monotonicity properties. For instance, when we take $V = C(\bar{D})$ the semiflow is strongly monotone, so there is exponential separation.

Taking for $\mathbb{P}$ an ergodic invariant probability measure with support contained in Y we obtain a measurable monotone skew-product semidynamical system to which we can apply Hilbert's projective metric. The principal Lyapunov exponent plays the role of the invasion rate by the second species.

Persistence in Population Models: Connections to Monotone Flows, cont'd

J. M., W. Shen and X. Zhao, *Uniform persistence for nonautonomous and random parabolic Kolmogorov systems*, J. Differential Equations **204** (2004), 471–510.

Generalizations for delay ODEs: P. L. Salceanu, *Robust uniform persistence for structured models of delay differential equations*, Discrete Contin. Dyn. Syst. Ser. B **27** (2022), 4923–4939.