# Some new trends in applied mathematics:
climate dynamics

Łukasz Płociniczak

November 26, 2018

Most of the material of this lecture is based on [15, 4, 6].

## 1 Budyko-Sellers model

We begin with a derivation of the governing equation which is the statement of the energy conservation. The initial step in the model formulation is a Budyko-Sellers type of energy conservation. At first, we assume the so-called zero-dimensional approximation meaning that we are concerned with globally averaged energy balance over sufficiently long time periods. Later, we will add latitudinal (zonal) dependence. Schematically, the energy equation has the form

$$c\frac{dT}{dt} = q_i - q_o, \tag{1}$$

where $T$ is the globally averaged temperature, $c$ is the atmosphere heat capacity while $q_i$ and $q_o$ are, respectively, incoming shortwave and outgoing longwave radiative thermal fluxes. One can directly calculate that $c$ defined in the above formula is approximately equal to $m_a c_a / A_E$, with $m_a$ being the mass of the atmosphere, $c_a$ specific heat of air and $A_E$ Earth's area (see [3]).

Our Earth receives the energy in a high quality (mostly) shortwave solar radiation. Let $Q$ denote the solar constant, i.e. the mean amount of Sun's irradiance per unit area of a plane perpendicular to the solar rays. Assuming the parallel ray approximation, Earth receives the amount of energy per unit time equal to $\pi R_E^2 Q$. Since $4\pi R_E^2$ is the total Earth's surface area we have

$$q_i = \frac{1}{4}(1-\alpha)Q, \tag{2}$$

where $\alpha$ is the mean terrestrial albedo (the fraction of reflected to absorbed radiation). It has to be noted that by no means should $Q$ and $\alpha$ be regarded as spatially independent. For example, the time-average solar irradiance (so-called insolation) has a profound meridional distribution: more light rays strike the equatorial strip than the high latitude polar region. This distribution can be calculated directly from the spherical geometry and has been done for example in [10]. One should point, however, that the horizontal distribution of heat by turbulent motion of the atmosphere takes it place at a much quicker time-scale than the one that interests us in our conceptual model (see [4]). Hence, it is not unreasonable to assume that the whole planet has a well-defined average uniform temperature.

The outgoing longwave radiation (OLR) is Earth's main mean of energy loss. Because the atmosphere contains many absorbers sensitive to the long electromagnetic waves (hence the Greenhouse effect), the precise quantitative description of the OLR is a complex task (see for ex. [14]). Staying at the conceptual level we will adopt a semi-empirical approach in determining $q_o$. Since the magnitude of OLR closely agrees with incoming solar radiation, the planet is in a energetic equilibrium. We could thus use the Stefan-Boltzmann's law at state that the outgoing flux is equal to $\sigma e^{-\Gamma} T^4$. Here, $\sigma$ is the Stefan-Boltzmann constant while $\Gamma$ is an empirical parameter modelling the radiation absorption in the atmosphere - the so-called greenhouse parameter. This approach had been taken in a number of works, for example in [4, 3]. We will, however, model the outgoing flux in a supposedly simpler way which dates back to Budyko's original work [1]. Mostly due to the short interval of relevant temperatures, say $250 - 320$K, the sensible choice is to propose the following linear dependence

$$q_o = A + BT, \tag{3}$$

where $A$ and $B$ are constants determined from the data (to be given below in Tab. **??**). It has also been previously argued that $A$ can be related to the concentration of $CO_2$ in the atmosphere (see [17]). Finally, putting (2) and (3) in (1) we obtain the equation of energy conservation

$$c\frac{dT}{dt} = \frac{Q}{4}\left(1 - \alpha(T)\right) - A - BT, \tag{4}$$

where we noted that $\alpha$ can depend on the temperature. A careful examination of the corresponding bifurcation diagram shows hysteretic behaviour thanks to which the so-called tipping points of abrupt climate change are present [8, 4].

The simple equation (4) exhibits a remarkable behaviour that has possible profound significance for the climate. To see this we have to impose some parametrisation of the albedo. It cannot be measured exactly but we know that it decreases with temperature since lower temperatures favour more ice sheets and hence larger albedo (we will improve this argument later). Of course, the function $\alpha = \alpha(T)$ should be bounded and hence it is sensible to assume it has a sigmoidal shape.

**Definicja 1.** *A function $\sigma : \mathbb{R} \to \mathbb{R}$ is called **sigmoid** if it is bounded and differentiable with a non-negative derivative. As a normalization one can take* $\lim\limits_{x\to\pm\infty} \sigma(x) = \pm 1$.

Therefore, we introduce

$$\alpha(T) = \frac{1}{2}\left(\alpha_+ + \alpha_- + (\alpha_+ - \alpha_-)\,\sigma_\alpha\left(\frac{T - T_\alpha}{\Delta\alpha}\right)\right), \quad \alpha_- \geq \alpha_+, \tag{5}$$

and a typical plot of the above is presented on Fig. 1. This model describes what is called the *ice-albedo feedback*.

The equation (4) is separable and thus easy to integrate (but maybe not exactly). However, it is much more informative to look for its critical points, that is those temperatures $T_c$ that satisfy

$$\frac{Q}{4}\left(1 - \alpha(T_c)\right) = A + BT_c. \tag{6}$$

Choosing appropriate parametrisation $\sigma$ may allow the above to be solved exactly but for our needs it is only needed to proceed geometrically. The relevant plot of $f(T) := \frac{Q}{4}\left(1 - \alpha(T_c)\right) - (A + BT_c)$ is depicted on Fig. 2.

We can see that there are three possible critical points. The left and right ones are stable ($f$ decreasing) while the middle is unstable ($f$ increasing). Our present climate is, presumably, represented by the warm right critical point. It is known from the astrophysics that the Sun's energy output increases with time. Therefore, it is meaningful to consider the change of the solar constant $Q$. For our function $f$ changing $Q$ is merely responsible for translating the graph vertically. The position of the critical points can then be depicted on the bifurcation diagram presented on Fig. 3.

We can see a standard hysteretic behaviour when a critical points undergo saddle-node bifurcation. That is to say, when $Q$ increases a pair of stable-unstable critical points is born. On the

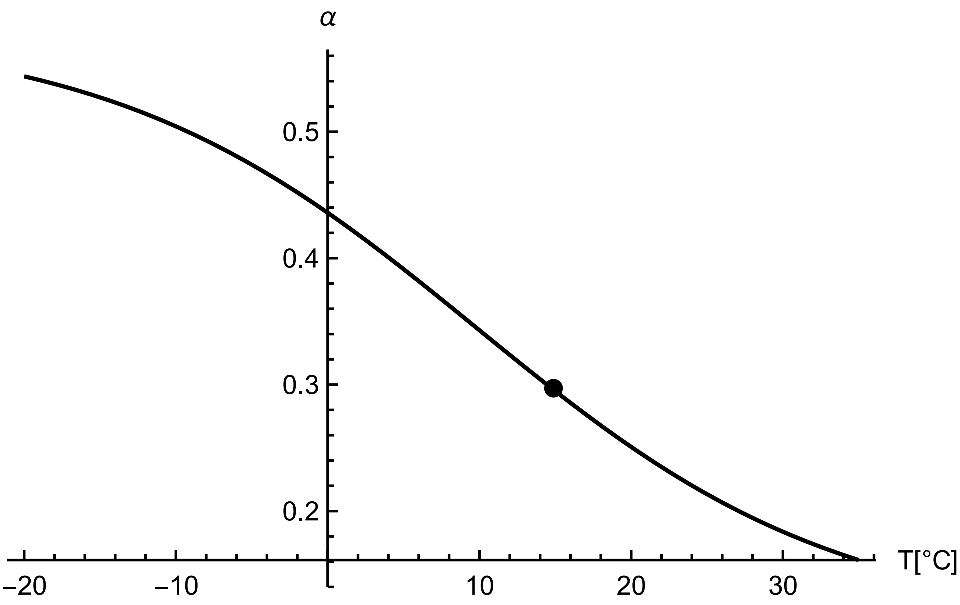


Figure 1: A typical profile for albedo. Present state of the climate is marked with a point.

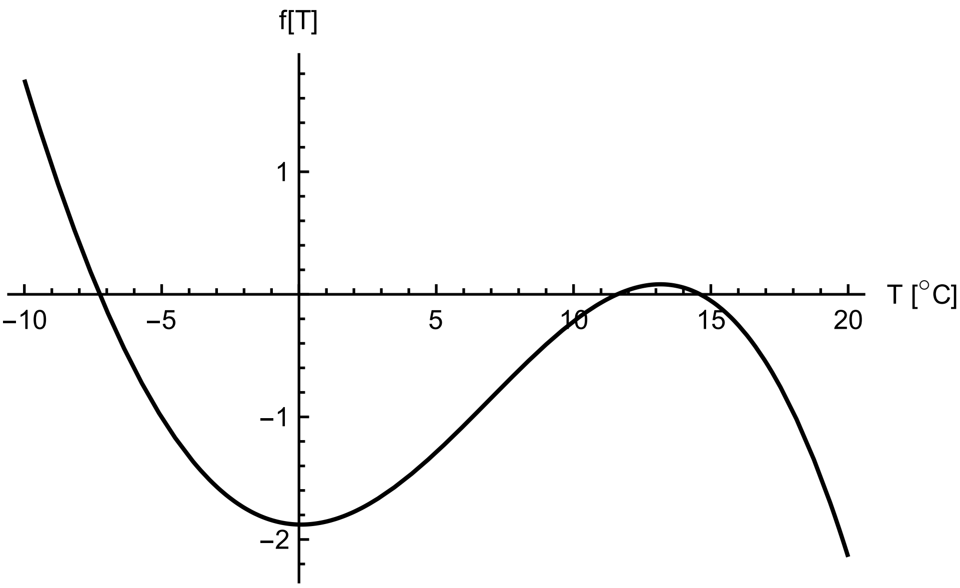


Figure 2: Critical points of (4).

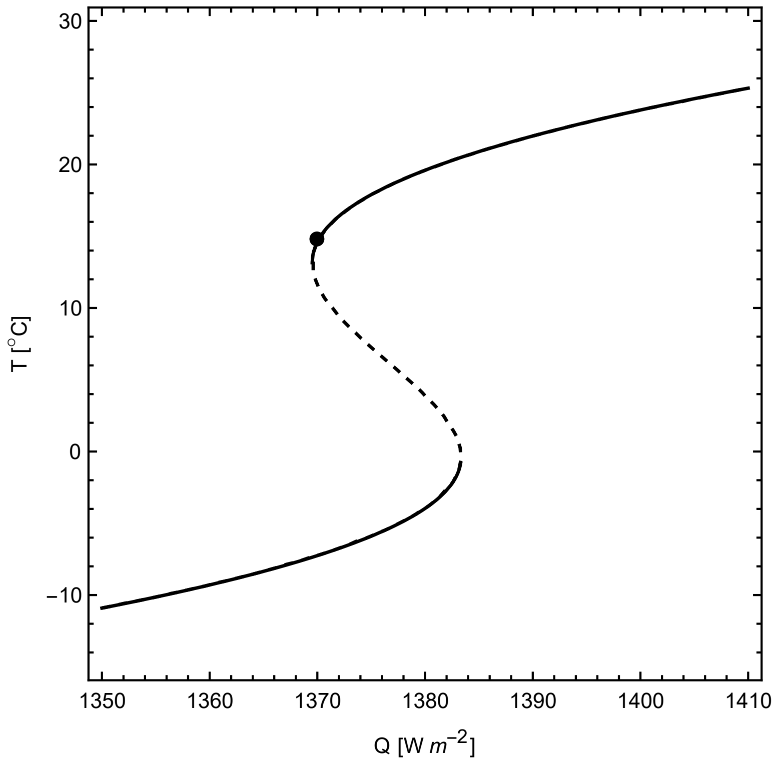


Figure 3: Bifurcation diagram of (4). The present state of the climate is marked with a point.

other hand, suppose that the solar constant slowly starts to decrease. Then, the point representing our climate marked on Fig. 3 decreases its temperature and suddenly drops to a lower stable branch of the diagram. This is accompanied by a abrupt change of the temperature to frosty range. This is called a *catastrophe* and the last climate state before the drop is known as the *tipping point*. We can thus see that, at least according to that simple model, the climate can suddenly change its state.

There is an active and important meridional transport of energy caused by the temperature gradient between the equator and poles. This causes the air to rise near the equator, then cool down when travelling northward, finally sink near the tropics. This is called the Hadley cell and similar flow happens further forth in mid-latitude and polar cells. Therefore, a more realistic model of the climate could include this transport in which the temperature would be averaged longitudinally (zonally)

$$T = T(x, y), \quad \text{where} \quad y = \sin \varphi, \tag{7}$$

where $\varphi$ is the latitude. Then, the previously considered globally averaged temperature would be

$$\overline{T}(t) := \int_0^1 T(t, y)\, dy. \tag{8}$$

The ice sheets are modelled by introducing the ice line $\eta$, that is the Earth is frozen for $y \in (\eta, 1)$ and no ice resides below $\eta$. The incoming solar energy is now distributed according to the law $s = s(y)$, i.e. the flux now becomes

$$Qs(y)(1 - \alpha(T, \eta)), \tag{9}$$

where we have added the fact that albedo for the ice sheets is different than for the rest of Earth. Moreover, the heat distribution has to be normalized

$$\int_0^1 s(y)\, dy = 1. \tag{10}$$

Although the functional form for $s$ can be computed explicitly (see [9]), its very good approximation is $s(y) = 1.241 - 0.723y^2$.

The last ingredient to the model is the horizontal heat transport which, following Budyko, can be included by a term responsible for relaxation to the average. Hence, the equation (4) now becomes

$$c\frac{\partial T}{\partial t} = Qs(y)(1-\alpha(y,\eta)) - (A+BT) - C(T-\overline{T}). \tag{11}$$

We see that the greatest the difference between $T$ and its average, the biggest the horizontal flux. The transport term can be also modelled by a turbulent eddy diffusivity [11]. Notice also that we have taken the albedo to be dependent on the latitude and the ice line which encodes the temperature changes.

Let us see what are the equilibrium states of (11). Setting the derivative to zero we have

$$Qs(y)(1-\alpha(y,\eta)) - (A+BT_c) - C(T_c-\overline{T}_c) = 0, \tag{12}$$

which, after integrating $y$ from 0 to 1, reduces to

$$\overline{T}_c(\eta) = \frac{1}{B}\left(Q\left(1-\overline{\alpha}(\eta)\right) - A\right), \tag{13}$$

where we have the averaged albedo

$$\overline{\alpha}(\eta) = \int_0^1 s(y)\alpha(y,\eta)dy. \tag{14}$$

Hence, the critical point of (11) is

$$T_c(y,\eta) = \frac{Q}{B+C}\left(s(y)\left(1-\alpha(y,\eta)\right) + \frac{C}{B}\left(1-\overline{\alpha}(\eta)\right)\right) - \frac{A}{B}. \tag{15}$$

As it can be seen, this equilibrium solution is a function of two variables: the latitude $y$ and the ice line $\eta$. However, the latter can be thought as a parameter. To interpret this result we have to impose the appropriate form of the albedo and, following Budyko, we can take the simplest step function

$$\alpha(y,\eta) = \begin{cases} \alpha_w, & 0 \le y \le \eta, \\ \alpha_s, & \eta < y \le 1, \end{cases} \tag{16}$$

where the numerical values can be taken as $\alpha_w = 0.32$ and $\alpha_s = 0.62$. This form of albedo is not of the sigmoid type but easily can be approximated by a function from such family.

There is one more free parameter that have to be fixed - the ice line $\eta$. Data shows that ice sheets start to form where the mean temperature drops below $-10°C$. Thus, we add additional requirement that

$$T_c(\eta) = -10°\mathrm{C}. \tag{17}$$

We can solve the above equation graphically, by defining

$$h(\eta) = \frac{1}{2}\left(\lim_{y\to\eta^-} T_c(y,\eta) + \lim_{y\to\eta^+} T_c(y,\eta)\right), \tag{18}$$

begin the average temperature on the ice line. The albedo is not continuous and hence the above is neither. The plot of $h = h(\eta)$ is presented on Fig. 4.

Notice that we have two meaningful steady states: one predicting a very large ice sheet with a global mean temperature $\overline{T}_c(\eta_1) = -21.4°C$ and the other $\overline{T}_c(\eta_2) = 14.9°C$. The latter resembles the present state of the climate very closely! Another pleasant feature of such a simple model!

So far we have not considered any dynamical features of the model such as bifurcation and stability of found critical points. This is not so difficult but we will refrain from that pointing the Reader to [17] and other works of McGehee et al.

Now, we will see how the simple energy balance model can be coupled with a dynamics of ice sheets to represent the climate more fully.

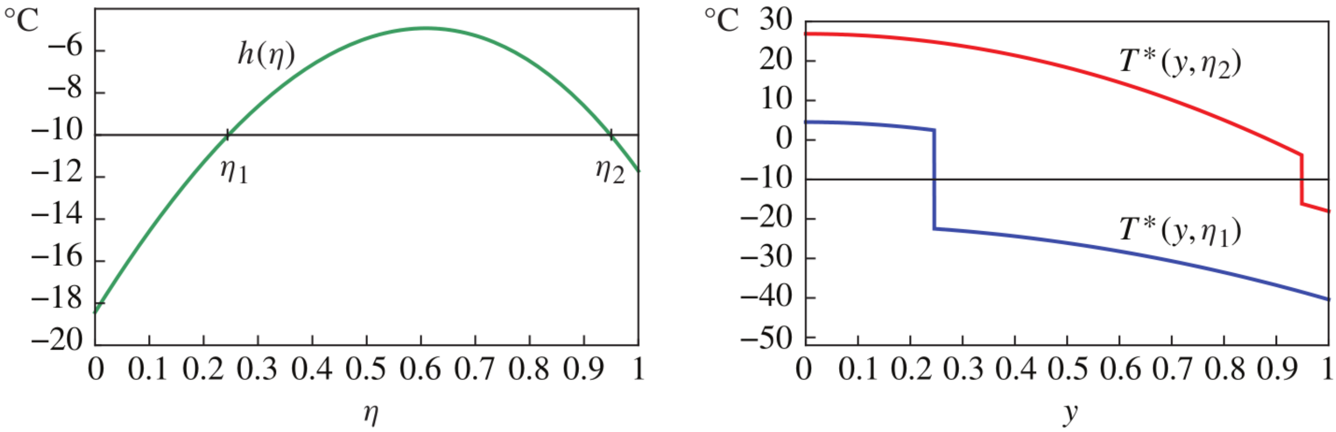


Figure 4: The plot of function $h$ defined in (18) and equilibrium temperature profiles for two solutions for which $h(\eta) = -10°C$. Plot taken from [17].

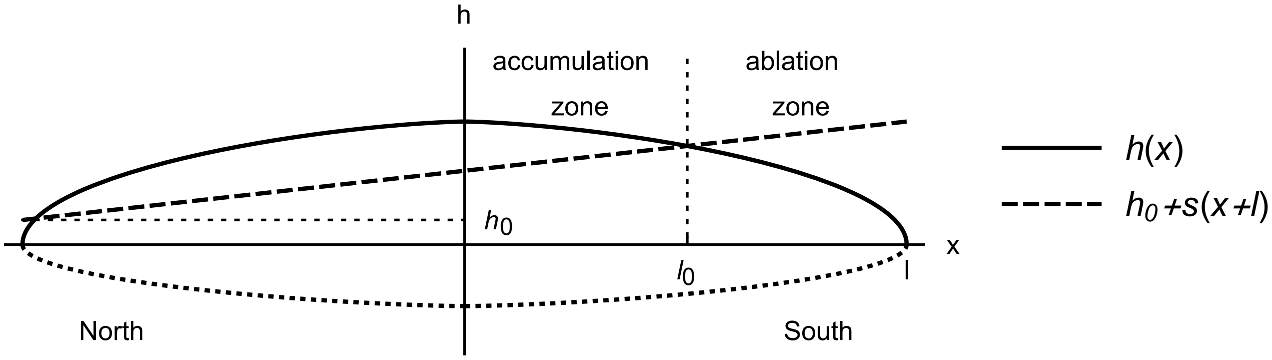


Figure 5: Schematic view of an ice sheet. Adapted from [18].

## 2 KCG (Källén, Crafoord, Ghil) generalized model

At the beginning we will use yet another version of the albedo function much more realistic than just a step function

$$\alpha(T, l) = \gamma\alpha_c(l) + (1-\gamma)\alpha_o(T), \tag{19}$$

where $l$ is the ice sheet extent (how far an ice sheet went to the south), $\alpha_c$ is the albedo due to the continents, $\alpha_o$ is the oceanic albedo and $\gamma$ is the continents to oceans area ratio.

By the mass balance we understand the growth and retreat of ice sheets under the influence of the climate variation. We will derive an equation describing the average behaviour of those great masses of ice. Our reasoning is based on the ice sheet model proposed in [18] (but see also [12, 6]). This is a simplified model based on the plastic flow and isostasy. The more accurate and elaborated models based on Glen's law are surveyed in [16, 13, 2].

The overall picture of the ice sheet is presented on Fig. 5. The ice sheet is assumed to be zonally symmetric, i.e. have an essentially two-dimensional latitudinal profile. Moreover, we orient the $x$-axis pointing southward with origin at the ice sheet's half-width $l$. The northward rim $x = -l$ is located at the Artic Ocean where the ice shelves form. Assume that the mass of ice of height $h(x)$ above the sea level rests on the isostatically depressed bedrock. Following [18] we can assume that the rock density is three times the density of ice. Then, the buoyancy law states that the total ice thickness is equal to $\frac{3}{2}h(x)$.

The ice sheet moves according to the plastic flow law, i.e. the whole mass behaves as a plastic fluid experiencing the yield stress $\tau_0$. Since the normal stresses are hydrostatic, in the equilibrium we must have

$$\tau_0 = \frac{3}{2}\rho_i g h \left|\frac{dh}{dx}\right|, \tag{20}$$

where $\rho_i$ is the density of ice. This equation can be immediately integrated giving the parabolic profile

$$h(x) = \sqrt{\frac{4\tau_0 l}{3\rho_i g}\left(1-\frac{|x|}{l}\right)} = H\sqrt{l}\sqrt{1-\frac{|x|}{l}}, \quad H := \sqrt{\frac{4\tau_0}{3\rho_i g}}. \tag{21}$$

The typical value of the yield stress and the corresponding height scale is given.

The dynamics of the ice sheet is governed by an interplay of snow accumulation and melting. We assume that the northern part of the ice sheet is in equilibrium with its southern counterpart. Therefore, the mass balance of the whole sheet is governed by the ablation and accumulation in the region $[0, l]$. The amount of snowfall is closely associated with the temperature via the so-called snow line. The temperature falls with the height and thus there is a well-defined 0°C isotherm under which the snow, if fallen, will melt. In the cited works this isotherm has been taken to be linear, i.e.

$$h_{iso}(x) = h_0 + s(x+l), \tag{22}$$

where $h_0$ is the height at the Arctic ocean while $s$ is the slope parameter. The northern height of the isotherm depends generally on the temperature or astronomical forcing. This has been taken into account in the literature. For example in [18, 3] Authors correlated the variation in $h_0$ with the Milankovitch oscillations while in [6] a linear dependence on the global temperature has been prescribed.

The snowline position $x = l_0$ defining the part of the sheet which is nourished by the snowfall is defined as a solution of $h(x) = h_{iso}(x)$. The mass balance can then be written as

$$\frac{d}{dt}\int_0^l \frac{3}{2}h(x)dx = a l_0 - m(l - l_0), \tag{23}$$

where $a$ and $m$ are respectively accumulation and melting rates. Now, if we assume that the ice sheet can grow or retreat, that is $l = l(t)$, we can obtain

$$\frac{d}{dt}\int_0^l \frac{3}{2}h(x)dx = \frac{3}{2}H\frac{d}{dt}\int_0^l \sqrt{l}\sqrt{1-\frac{x}{l}}dx = \frac{3}{2}H\sqrt{l}\frac{dl}{dt}. \tag{24}$$

Furthermore, the value of $l_0$ can be found by equating (21) and (22)

$$h_0 + s(l_0 + l) = H\sqrt{l}\sqrt{1-\frac{l_0}{l}}, \tag{25}$$

which is a quadratic equation with the meaningful solution

$$l_0 = \frac{H^2}{s^2}\left[-\left(\frac{h_0 s}{H^2} + \frac{s^2}{H^2}l + \frac{1}{2}\right) + \sqrt{\frac{h_0 s}{H^2} + 2\frac{s^2}{H^2}l + \frac{1}{4}}\right], \tag{26}$$

which together with (23) gives the second dynamic equation

$$\frac{3}{2}H\sqrt{l}\frac{dl}{dt} = (a+m)\frac{H^2}{s^2}\left[-\left(\frac{h_0 s}{H^2} + \frac{s^2}{H^2}l + \frac{1}{2}\right) + \sqrt{\frac{h_0 s}{H^2} + 2\frac{s^2}{H^2}l + \frac{1}{4}}\right] - ml \quad \text{for } l_0 \geq 0. \tag{27}$$

This is a rather complicated nonlinear equation but we will shortly see that an appropriate scaling and approximation will yield its accurate simplification. Also notice that nowhere in the above discussion we have mentioned the response of the bedrock to the evolution of the ice sheet. The moving mass will produce a delayed feedback from the lithosphere and hence, might provide an essential ingredient of the dynamics. This requires adding an additional equation to our system and a simplified version of it have been introduced in [5, 7]. In this paper, however, we will consider only the instantaneous adjustment of the bedrock leaving the more general problem for the future work.

Two adjustments are necessary for the well-posedness of the model. First, it may happen that the whole southern side of the ice sheet is in the ablation zone. This can happen for sufficiently large sheets or by raising the 0°C isotherm. In this case the whole mass of ice becomes stagnant and our derivation is not valid (equation (25) does not have a positive solution). Authors of [18] and [3] propose to assume that we can take $l_0 = 0$ and then, the mass balance can be written as

$$\frac{3}{2}H\sqrt{l}\,\frac{dl}{dt} = -ml \quad \text{for} \quad l_0 < 0. \tag{28}$$

Another situation of (25) failing to have a solution is when the negative snowline elevation $h_0$ becomes smaller than $-sL$. Then, the snow will accumulate on the ground and if $l = 0$ the ice sheet will nucleate. For a sufficiently high snowfall rates $a$ (or sufficiently small $-\frac{h_0}{s}$) the reasonable approximation is to assume (see [18]) that $l = -\frac{h_0}{2s}$ and

$$\frac{3}{2}H\sqrt{l}\,\frac{dl}{dt} = -\frac{ah_0}{2s} \quad \text{for} \quad l < -\frac{h_0}{2s}, \tag{29}$$

while for $l > -\frac{h_0}{s}$ the main equation (27) still holds.

The above equations are much more complicated than the simple Budyko-Sellers energy balance model, but they describe the climate in much more detail. Without any further details (which can be found in [15]) we can nondimensionalize the energy and mass balance to obtain

$$\begin{cases} \dfrac{d\theta}{d\tau} = \mu\left[1+\beta-\gamma\left(\alpha_1+\alpha_2\lambda\right)-(1-\gamma)\alpha_o(\theta)-\theta\right] =: F(\theta,\lambda), \\ \dfrac{d\lambda}{d\tau} = \sqrt{\lambda}\left(\left(1+\xi(\theta)\right)\left(1-4\lambda\right)-1\right) =: G(\theta,\lambda), \end{cases} \quad \text{for} \quad \theta > 0, \quad 0 < \lambda \le \frac{1}{4}, \tag{30}$$

where $\xi = \xi(\theta)$ is the accumulation to ablation ratio which can depend on the temperature. Apart from the already mentioned ice-albedo feedback encoded in the temperature dependence of $\alpha$, the function $\xi$ depicts the second fundamental feedback mechanics, i.e. the *precipitation-temperature feedback*. Very cold climates affect the evaporation of the oceans reducing the amount of vapour in the atmosphere. Hence, the meridional transport of air moves only a little water to the higher latitudes. The drier conditions in the Pole regions inhibit snowfall and, therefore, nourishment of the ice sheets. Thanks to that, the functional form of $\xi = \xi(\theta)$ should be a monotone increasing.

There is a lot to be done with our model (30) but the most important fact is that is possesses a supercritical Hopf bifurcation. In this case, when $\mu$ increases through a critical value $\mu_c$, a stable limit cycle is born and the climate oscillates. This has a profound meaning for the planet - the ice sheets can advance and retreat without any external astronomical forcing! Hence, the Milankovich theory may not be the decisive in predicting the ice ages. The climate acts as a giant internal nonlinear oscillator!

# References


[1] Mikhail I Budyko. The effect of solar radiation variations on the climate of the earth. *Tellus*, 21(5):611–619, 1969.

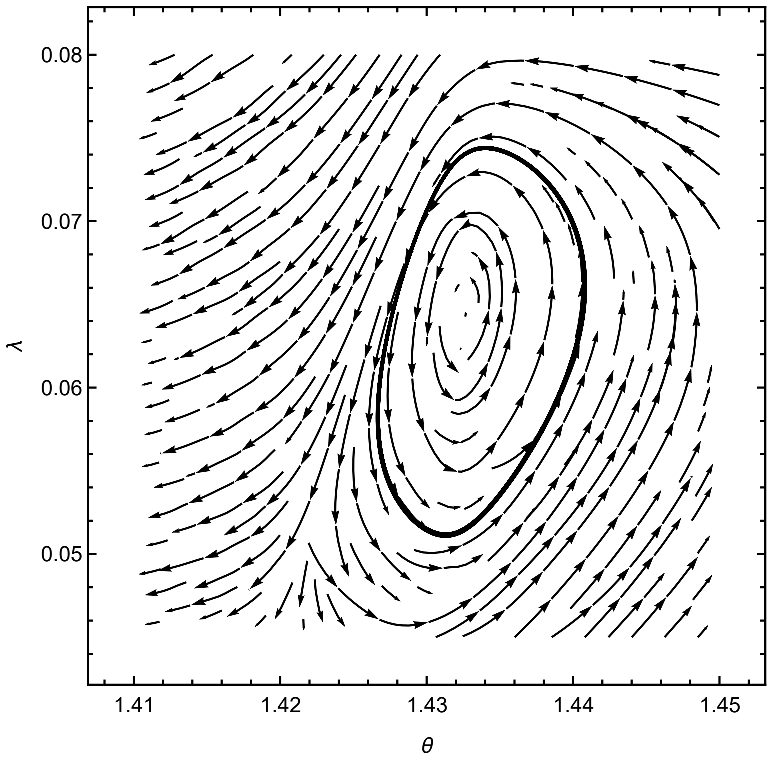


Figure 6: The phase plane of (30) with a limit cycle.

[2] AC Fowler. Glaciers and ice sheets. In *The Mathematics of Models for Climatology and Environment*, pages 301–336. Springer, 1997.

[3] AC Fowler, REM Rickaby, and EW Wolff. Exploration of a simple model for ice ages. *GEM-International Journal on Geomathematics*, 4(2):227–297, 2013.

[4] Andrew Fowler. *Mathematical geoscience*, volume 36. Springer Science & Business Media, 2011.

[5] Michael Ghil and Hervé Le Treut. A climate model with cryodynamics and geodynamics. *Journal of Geophysical Research: Oceans*, 86(C6):5262–5270, 1981.

[6] E Källén, C Crafoord, and M Ghil. Free oscillations in a climate model with ice-sheet dynamics. *Journal of the Atmospheric Sciences*, 36(12):2292–2303, 1979.

[7] Hervé Le Treut and Michael Ghil. Orbital forcing, climatic interactions, and glaciation cycles. *Journal of Geophysical Research: Oceans*, 88(C9):5167–5190, 1983.

[8] Timothy M Lenton. Early warning of climate tipping points. *Nature Climate Change*, 1(4):201–209, 2011.

[9] Richard McGehee and Clarence Lehman. A paleoclimate model of ice-albedo feedback forced by variations in earth's orbit. *SIAM Journal on Applied Dynamical Systems*, 11(2):684–707, 2012.

[10] Alice Nadeau and Richard McGehee. A simple formula for a planet's mean annual insolation by latitude. *Icarus*, 291:46–50, 2017.

[11] Gerald R North. Theory of energy-balance climate models. *Journal of the Atmospheric Sciences*, 32(11):2033–2043, 1975.

[12] J Oerlemans. Model experiments on the 100,000-yr glacial cycle. *Nature*, 287(5781):430–432, 1980.

[13] Johannes Oerlemans and Cornelis J Van Der Veen. *Ice sheets and climate*, volume 21. Springer, 1984.

[14] Raymond T Pierrehumbert. *Principles of planetary climate*. Cambridge University Press, 2010.

[15] Łukasz Płociniczak. Hopf bifurcation in a conceptual climate model with ice-albedo and precipitation-temperature feedbacks. *arXiv preprint arXiv:1801.09087*, 2018.

[16] Cornelis J Van der Veen. *Fundamentals of glacier dynamics*. CRC Press, 2013.

[17] James Walsh and Richard McGehee. Modeling climate dynamically. *The College Mathematics Journal*, 44(5):350–363, 2013.

[18] Johannes Weertman. Milankovitch solar radiation variations and ice age ice sheet sizes. *Nature*, 261:17–20, 1976.