Some new trends in applied mathematics: climate dynamics

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[1]Support from Erasmus+

Climate

Statistical long-term description of the quantities associated with weather (from months to milions of years)[2]

- Why?
  - To know the world.
  - To save the world.
- Who?
  - Climatologists.
  - Physicists.
  - **Mathematicians?!**
- Why?
  - Global measurements.
  - Climate proxy.
  - Mathematical models.

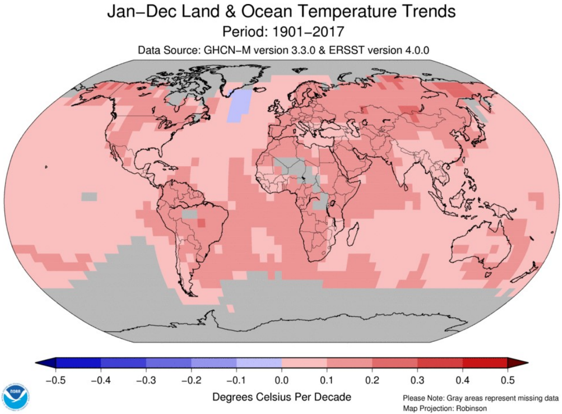


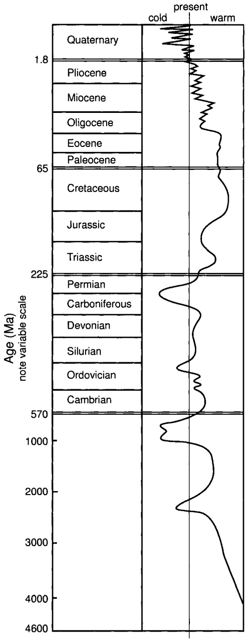
Figure: Global temperature trend, NOAA.

[2] Intergovernmental Panel on Climate Change, Annex III: Glossary [Planton, S. (ed)], 2013

Palaeoclimatology

There are several ingenious ways to find what was the temperature millions of years ago!

- Climate proxy - indirect inference.
  - Ice cores (isotopes: ^{18}O and D (and many more)).
  - Sediments.
  - Tree rings and leaves.
  - Corals.
  - Foraminifera.
  - Pollen.
  - ...
- Figure: changes of global temperature. Source: Saltzman, Dynamical Paleoclimatology (2002).



Palaeoclimatology

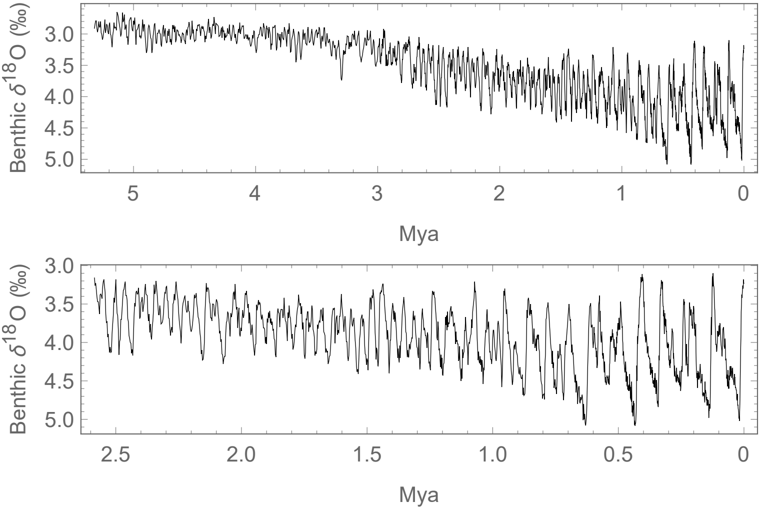


Figure: Climate proxy for the temperature. Source: Lisiecki and Raymo (2005).

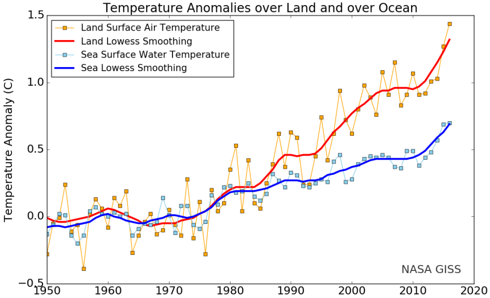
Ice ages

- Ice ages: huronian (early Proterozoic), cryogenian (late Proterozoic), andyjsko-saharyjskie (late Ordovician - Silurian), Karoo (Carbon) and quaternary (present).
- What is the reason for the climate oscillations-relaxations in the quaternary?
  - Astronomical forcing (Milankovitch theory?).
  - Changes in the atmosphere content?
  - Plate tectonics?
  - Internal nonlinear mechanisms?
- Snowball Earth
  - Hypothetical state of the climate in which almost the whole planet is under the ice (Neoproterozoic)
  - What is the mechanism responsible for initialization?
  - Positive feedback: colder $\rightarrow$ more ice $\rightarrow$ higher albedo $\rightarrow$ colder.
  - What is the mechanism responsible for termination? (volcanism?)

Human influence

It is extremely likely that human influence has been the dominant cause of the observed warming since the mid-20th century.[3]

- Humans emit enormous amounts of CO_2 - a greenhouse gas.
- The predicted increase in tempera in XXI century: 0.3 – 1.7°C (lowe emission); 2.6 – 4.8°C (highest emission).
- Consequences:
  - increase of global temperature,
  - raising sea level,
  - changes in the precipitation distribution,
  - extreme weather phenomena,
  - extinction of species.



[3] IPCC, 2013: The Physical Science Basis - Summary for Policymakers

Climate models

Complexity of hierarchy.

- Global Climate Models (GCMs)
  - Geophysical Navier-Stokes on a sphere + equation of state + conservation of energy + conservation of salinity + other parametrisations

$$\rho\left(\frac{D\mathbf{u}}{Dt} + 2\Omega \times \mathbf{u}\right) = -\nabla p + \nu\nabla^2\mathbf{u} + \mathbf{f}, \quad \frac{\partial\rho}{\partial t} + \nabla\cdot(\rho\mathbf{u}) = 0,$$
$$p = p(\rho, T, s), \quad \frac{DT}{Dt} = q_T, \quad \frac{Ds}{Dt} = q_e.$$

- Earth System Models of Intermediate Complexity (EMICs)[4]
  - GCM with more parametrisations/simplifications.
- Conceptual models
  - Box models.
  - **Dynamical systems**.

[4] Claussen, Martin, et al. „Earth system models of intermediate complexity: closing the gap in the spectrum of climate system models." Climate dynamics 18.7 (2002): 579-586.

The simplest model

The simplest zero-dimensional model (Energy Balance Model)

- Let $T = T(t)$ be the globally averaged temperature. Then

$$c\frac{dT}{dt} = \underbrace{\frac{1}{4}(1-\alpha(T))\,Q}_{\text{incoming shortwave radiation}} - \underbrace{\gamma\sigma T^4}_{\text{outgoing longwave radiation}} =: f(T),$$

  where c - specific heat, Q - solar constant, γ - greenhouse coefficient, σ - Stefan-Boltzmann constant and α - temperature dependent albedo.
- Albedo-temperature feedback: colder $\rightarrow$ more ice $\rightarrow$ larger albedo $\rightarrow$ colder

$$\lim_{T\to 0^+}\alpha(T) = \alpha_-, \quad \lim_{T\to\infty}\alpha(T) = \alpha_+ \quad \text{oraz} \quad \alpha'(T) \leq 0,$$

  where $\alpha_- > \alpha_+$.

Critical points

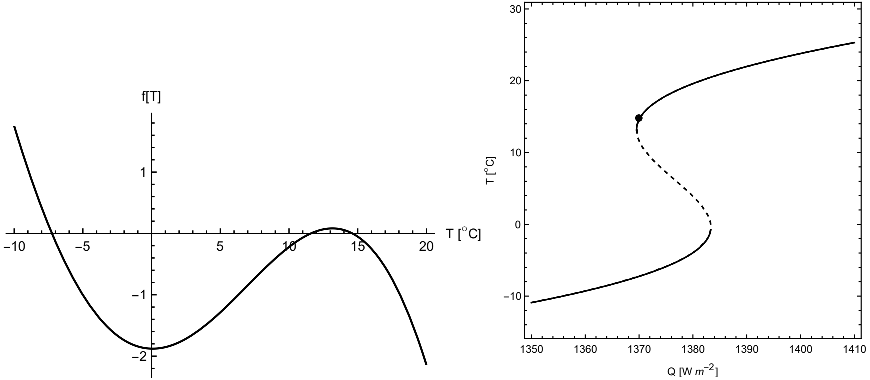


Figure: Exemplary simulation (data taken from A.C.Fowler, Mathematical Geoscience, 2011). On the left: zeros of f are the stationary points of the climate. On the right: bifurcation diagram with hysteresis.

Another dimension: Budyko-Sellers

- The previous model is based on the seminal papers of Budyko and Sellers[5,6].
- This time $T = T(y, t)$, where y is the sine of the latitude.
- Conservation of energy with the meridional transport

$$c\frac{\partial T}{\partial t} = \underbrace{Qs(y)(1-\alpha(T))}_{\text{promieniowanie krótkofalowe}} - \underbrace{A+BT}_{\text{promieniowanie długofalowe}} - \underbrace{C(T-\bar{T})}_{\text{transport poziomy}},$$

where $s = s(y)$ is the meridional distribution of solar radiation, and $\bar{T}$ is the mean temperature on Earth

$$\int_0^1 s(y)dy = 1, \quad \bar{T}(t) := \int_0^1 T(t, y)dy.$$

- North[7] proposed diffusive transport.
- Modern and **mathematical** approach: McGehee and collaborators [8].

[5] Budyko, M. I. (1969), Tellus 21(5), 611-619.

[6] Sellers, W. D. (1969), Journal of Applied Meteorology 8(3), 392-400.

[7] North, G. R. et al. (1981), Reviews of Geophysics 19(1), 91-121.

[8] McGehee, R. et al. (2012), SIADS 11(2), 684-707.

Further degrees of freedom

- Essential addition: **ice sheet extent**
  - At present: Greenland and Antarctica.
  - Last maximal glaciation: Laurentide, Patagonian and Vistulian.
- Model Källén, Crafoord, Ghil[9] based on Weertman[10]

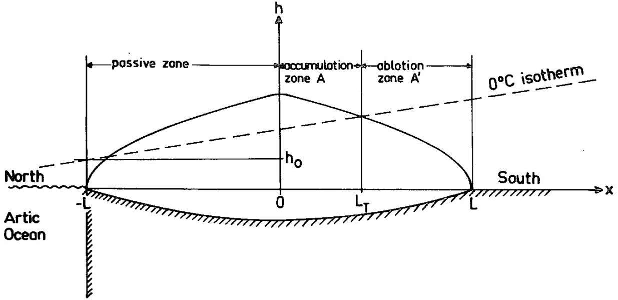


Figure: Ice sheet model according to KCG and Weertman.

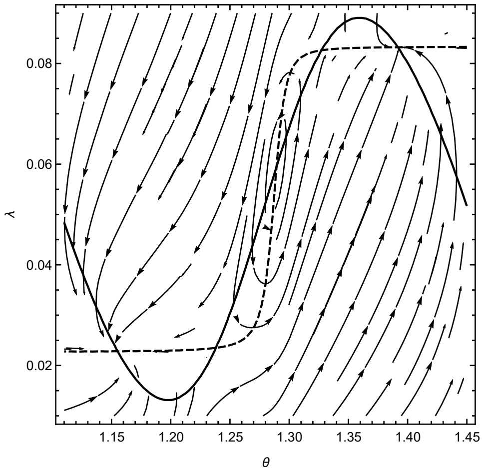
[9] KCG (1979), Journal of the Atmospheric Sciences 36(12), 2292-2303.
[10] Weertman, J. (1976), Nature 261, 17-20.

General KCG model

- KCG model implies internal climate oscillations (Hopf bifurcation).
- General KCG model **exhibits them also!** Is the climate an internal oscillator?

$$\frac{dV_I}{dT} = aL_T - m\,(L - L_T)\,.$$

- Precipitation-temperature feedback: colder → lower evaporation → lower precipitation → less ice → lower albedo → warmer.



What else?

- CO_2: the most important (but not the strongest) greenhouse gas.

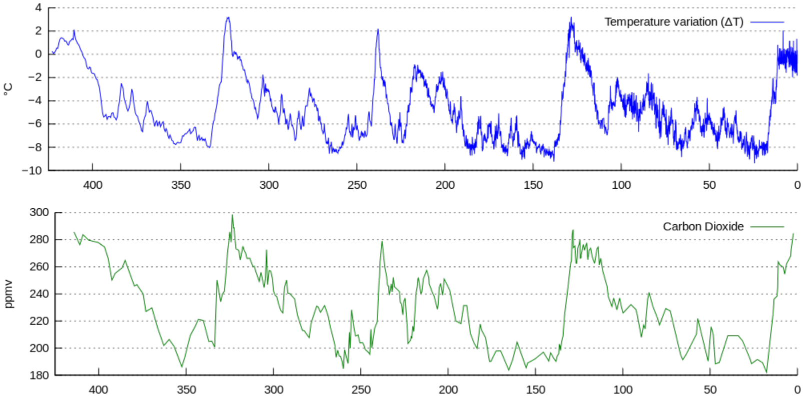


Figure: Historical CO_2 and the temperature at Vostok station.

Carbon cycle

- Fowler's „simple model"[11]

$$\frac{dm_{CO_2}}{dt} = -A_L W(T, m_{CO_2}) + v,$$

where A_L - area of continents, W - weathering rate and v - volcanism.

- Similarly to water, carbon has also its cycle:
  - Earth's mantle $\rightarrow$ volcanoes (v) $\rightarrow$ atmosphere $\rightarrow$ weathering (W) $\rightarrow$ rivers $\rightarrow$ oceans $\rightarrow$ subduction.
- We need to couple that model with other equations[12]

$$W(T, m_{CO_2}) = W_0 m_{CO^2}^{\mu} \exp\left(\frac{T - T_0}{\Delta T}\right),$$

where $\mu \approx 0.3$.

[11] Fowler, A. C., et al. (2013), International Journal on Geomathematics 4(2), 227-297.
[12] Walker, J. C. et al. (1981), Journal of Geophysical Research: Oceans, 86(C10), 9776-9782.

The number of equations grows!

- What else?
  - The meaning of the oceans: CO_2.
  - We need to describe the chemistry of all oxidation reactions.
  - Mechanism for immediate entering into the interglacial period.
  - ...
- Model by Maasch and Saltzman[13]: ice mass, concentration of CO_2, temperature at the bottom od Atlantic Ocean.
  - Leading order of conservation laws.
  - Milankovitch + tectonic „trend".
  - A change in the period of oscillations - bifurcation.
  - Mathematically: Bogdanov-Takens bifurcation[14].
- Many more approaches...[15]
- **...and a lot of interesting mathematics!**

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[13] M-S (1990), Journal of Geophysical Research: Atmospheres, 95(D2), 1955-1963.
[14] Engler, H. et al. (2017), arXiv preprint arXiv:1705.07387.
[15] Crucifix, M. (2012), Phil. Trans. R. Soc. A, 370(1962), 1140-1165.

Summary of the lecture

1. Introduction (now it ends!).
2. Budyko-Sellers energy balance model.
3. KCG precipitation-temperature feedback model.

THANK YOU!

Literature

- This lecture is based on

1. Płociniczak, Ł. (2018). Hopf bifurcation in a conceptual climate model with ice-albedo and precipitation-temperature feedbacks. arXiv preprint arXiv:1801.09087.
2. Walsh, J., McGehee, R. (2013). Modeling climate dynamically. The College Mathematics Journal, 44(5), 350-363.
3. Fowler, A. (2011). Mathematical geoscience (Vol. 36). Springer Science & Business Media.

- Models

1. Budyko, M. I. (1969). The effect of solar radiation variations on the climate of the earth. tellus, 21(5), 611-619.
2. Sellers, W. D. (1969). A global climatic model based on the energy balance of the earth-atmosphere system. Journal of Applied Meteorology, 8(3)
3. Källén, E., Crafoord, C., & Ghil, M. (1979). Free oscillations in a climate model with ice-sheet dynamics. Journal of the Atmospheric Sciences, 36(12), 2292-2303.

- Textbooks on climate

1. Ruddiman, W. F. (2001). Earth's Climate: past and future. Macmillan.
2. Pierrehumbert, R. T. (2010). Principles of planetary climate. Cambridge Univer