Maximum principle for time-fractional diffusion equations with the Caputo-Katugampola derivative

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Introduction

Extremum principles are a very important tool in the theory of differential equations and partial differential equations. Among other things, using these principles one can obtain some results on the uniqueness of solutions for partial differential equations. Recently, some papers have considered these principles for partial differential equations involving fractional derivatives (see [1, 2, 3, 4, 5, 6, 7, 8]).

Introduction

In particular, in [7] the authors proved some maximum principles for time-fractional diffusion equations involving Caputo-Katugampola derivative. More precisely, they consider the following equation

$$
{}^{c}D_{0^+}^{\alpha,\rho} u(x,t) = a(x,t)u_{xx} + b(x,t)u_x + c(x,t)u + F(t,x,u),
$$

for $(t,x) \in \Omega_T = (0,\rho) \times (0,T)$, with the boundary conditions

$$
\left.\begin{array}{ll}
u(0,t) = g_1(t), & t \in [0,T] \\
u(\rho,t) = g_2(t), & t \in [0,T] \\
u(x,0) = \phi(x), & x \in [0,\rho]
\end{array}\right\}
$$

where ${}^{c}D_{0^+}^{\alpha,\rho}$ denotes the Caputo-Katugampola derivative, $\alpha \in (0,1)$ and $\rho > 0$.

Introduction

Motivated by [7], in this talk, we consider the following time-fractional diffusion equation

$$\left({}^{c}D_{0^+}^{\alpha,\rho} u\right)(t,x) = div\left(p(x)\nabla u(t,x)\right) + F(t,x), \tag{1}$$

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for $(t,x) \in (a,T] \times \Omega$, where $a \geq 0$, $\alpha \in (0,1)$, $\rho > 0$ and Ω is an open bounded subset of $\mathbb{R}^n$, under the following boundary conditions

$$\left.\begin{array}{ll} u(a,x) = \varphi(x), & x \in \overline{\Omega} \\ u(t,x) = \gamma(t,x), & (t,x) \in [a,T] \times \partial\Omega \end{array}\right\} \tag{2}$$

where *div* is the divergence operator with respect to the space variable x, ∇ is the gradient operator with respect to the variable x, $p \in C^{(1)}(\overline{\Omega})$ verifies $p(x) > 0$ for $x \in \Omega$, $F \in \mathcal{C}([a,T] \times \overline{\Omega})$, $\varphi \in \mathcal{C}(\overline{\Omega})$ and $\gamma \in \mathcal{C}([a,T] \times \partial\Omega)$. S10

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Preliminaries

Firstly, we recall some definitions on fractional calculus ([9, 10, 11]).

Definition 1

Suppose that $f \in L^1[a,b]$, $\alpha > 0$ and $\rho > 0$. The Riemann-Katugampola (R-K) fractional integral of order α with respect to the parameter ρ of the function f is defined as

$$I_{a^+}^{\alpha,\rho} f(t) = \frac{\rho^{1-\alpha}}{\Gamma(\alpha)} \int_a^t \frac{s^{\rho-1} f(s)}{(t^\rho - s^\rho)^{1-\alpha}} ds.$$

Definition 2

Suppose that $f \in AC[a,b]$. The Caputo-Katugampola (C-K) fractional derivative of order α with respect to the parameter ρ of the function f is given by

$${}^c D_{a^+}^{\alpha,\rho} f(t) = \frac{\rho^\alpha t^{1-\rho}}{\Gamma(1-\alpha)} \frac{d}{dt} \left(\int_a^t \frac{s^{\rho-1}}{(t^\rho - s^\rho)^\alpha} (f(s) - f(a)) ds \right).$$

Note: Hereafter, we will refer to it as the C-K fractional derivative.

Moreover, we recall Theorem 1 of [11]

Lemma 1

Suppose that $f \in C^{(1)}([a,b])$ *and* $\alpha \in (0,1)$ *then*

$$ {}^{c}D_{a^+}^{\alpha,\rho} f(t) = \frac{\rho^{\alpha}}{\Gamma(1-\alpha)} \int_a^t \frac{f'(s)}{(t^{\rho}-s^{\rho})^{\alpha}} ds. $$

Preliminaries

And the following theorem, which appears in [4] and [7], and plays a significant role to obtain the maximum principle for equation (1), under boundary conditions (2).

Theorem 1

Let $f \in C^{(1)}[a,b]$, $\alpha \in (0,\ 1)$ and suppose that f attains its maximum on $[a,b]$ at a point $t_0 \in (a,b)$. Then we have that

$$ {}^{c}D^{\alpha,\rho}_{a^+} f(t_0) \geq \frac{\rho^{\alpha}(t_0^{\rho} - a^{\rho})^{-\alpha}}{\Gamma(1-\alpha)}(f(t_0) - f(a)) \geq 0. $$

Changing f to -f in Theorem 1, it becomes to

Theorem 2

Suppose that $f \in C^{(1)}([a,b])$ and that f attains its minimum at $t_0 \in (a,b)$. Then

$$ {}^{c}D^{\alpha,\rho}_{a^+} f(t_0) \leq \frac{\rho^{\alpha}(t_0^{\rho} - a^{\rho})^{-\alpha}}{\Gamma(1-\alpha)}(f(t_0) - f(a)) \leq 0. $$

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Main results - Operator L

From this point onwards, Ω denotes an open bounded subset of $\mathbb{R}^n$ and $[a, T]$ an interval in $\mathbb{R}$ where $a \geq 0$. Now, we consider the following operator

$$Lu(t,x) =^c D_{a^+}^{\alpha,\rho} u(t,x) - div(p(x)\nabla u(t,x)), \tag{3}$$

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for $(t,x) \in (a, T] \times \Omega$, satisfying the following boundary conditions

$$\left.\begin{array}{ll} u(a,x) \geq 0, & x \in \overline{\Omega} \\ u(t,x) \geq 0, & (t,x) \in [a, T] \times \partial\Omega \end{array}\right\} \tag{4}$$

where $\alpha \in (0,1)$, $\rho > 0$, the C-K derivative is considered with respect to the time variable t, div and ∇ are the divergence and the gradient operators with respect to the variable x, respectively, and $p \in C^{(1)}(\overline{\Omega})$ with $p(x) > 0$ for $x \in \Omega$.

Main results - Operator L

Our initial results are as follows

Theorem 3

Let $u \in C^{(2)}([a, T] \times \overline{\Omega})$ *be such that* $Lu(t,x) \geq 0$ *for* $(t,x) \in (a, T] \times \Omega$ *and* u *satisfies the boundary conditions (4). Then*

$$u(t,x) \geq 0, \quad (t,x) \in [a, T] \times \overline{\Omega}.$$

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$$u(t,x) \geq 0, \quad (t,x) \in [a, T] \times \overline{\Omega}.$$

By changing u to $-u$ in Theorem 3, we get

Theorem 4

Let $u \in \mathcal{C}^2([a, T] \times \overline{\Omega})$ *be such that* $Lu(t,x) \leq 0$ *for* $(t,x) \in (a, T] \times \Omega$*, and*

$$\left.\begin{array}{ll} u(a,x) \leq 0, & x \in \overline{\Omega} \\ u(t,x) \leq 0, & (t,x) \in [a, T] \times \partial\Omega \end{array}\right\}$$

Then

$$u(t,x) \leq 0, \quad (t,x) \in [a, T] \times \overline{\Omega}.$$

Proof of Theorem 3: Suppose the contrary, that is, there exists $(t_0, x_0) \in [a, T] \times \overline{\Omega}$ such that $u(t_0, x_0) < 0$. Then considering that $u \in \mathcal{C}([a, T] \times \overline{\Omega})$, we find $(t_1, x_1) \in [a, T] \times \overline{\Omega}$ such that

$$u(t_1, x_1) = \min_{(t,x) \in [a,T] \times \overline{\Omega}} u(t, x) \leq u(t_0, x_0) < 0. \tag{5}$$

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According to boundary conditions (4), from (5) we infer that $(t_1, x_1) \in (a, T] \times \Omega$. So that, from $Lu(t, x) \geq 0$ for $(t, x) \in (a, T] \times \Omega$ we get $Lu(t_1, x_1) \geq 0$.

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On the other hand, by Theorem 2 and, as $u(a, x_1) \geq 0$, from (5) it follows

$${}^{c}D_{a^+}^{\alpha,\rho} u(t_1, x_1) \leq \frac{\rho^{\alpha}(t_1^{\rho} - a^{\rho})^{-\alpha}}{\Gamma(1 - \alpha)} u(t_1, x_1) < 0. \tag{6}$$

Proof of Theorem 3: Suppose the contrary, that is, there exists $(t_0, x_0) \in [a, T] \times \overline{\Omega}$ such that $u(t_0, x_0) < 0$. Then considering that $u \in \mathcal{C}([a, T] \times \overline{\Omega})$, we find $(t_1, x_1) \in [a, T] \times \overline{\Omega}$ such that

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On the other hand, by Theorem 2 and, as $u(a, x_1) \geq 0$, from (5) it follows

$${}^{c}D^{\alpha,\rho}_{a^+} u(t_1, x_1) \leq \frac{\rho^\alpha (t_1^\rho - a^\rho)^{-\alpha}}{\Gamma(1-\alpha)} u(t_1, x_1) < 0. \tag{6}$$

Moreover, we have that (t_1, x_1) is a minimum of $u(t_1, -)$ in Ω, so that $\Delta u(t_1, x_1) \geq 0$ and $\nabla u(t_1, x_1) = 0_{\mathbb{R}^n}$, so that

$$div(p(x_1)\nabla u(t_1, x_1)) = p(x_1)\Delta u(t_1, x_1) + \langle \nabla p(x_1), \nabla u(t_1, x_1)\rangle \geq 0, \tag{7}$$

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Combining (6) and (7), we deduce

$$Lu(t_1, x_1) = {}^{c}D^{\alpha,\rho}_{a^+} u(t_1, x_1) - div(p(x_1)\nabla u(t_1, x_1)) < 0$$

which is a contradiction and proves our result. ■

Theorem 3 implies the following corollary on extremum principles

Corollary 1

Suppose the following time-fractional diffusion equation

$$Lu(t,x) = F(t,x), \quad \textit{for} \quad (t,x) \in (a,T] \times \Omega, \tag{8}$$

under the following boundary conditions S18 S21 S24

$$\left.\begin{array}{ll} u(a,x) = \varphi(x), & x \in \overline{\Omega} \\ u(t,x) = \gamma(t,x), & (t,x) \in [a,T] \times \partial\Omega \end{array}\right\} \tag{9}$$

where $F \in \mathcal{C}([a,T] \times \overline{\Omega})$, $\varphi \in \mathcal{C}(\overline{\Omega})$ *and* $\gamma \in \mathcal{C}([a,T] \times \partial\Omega)$.

Suppose that $F(t,x) \geq 0$ *for* $(t,x) \in (a,T] \times \Omega$. *If* $u \in C^{(2)}([a,T] \times \overline{\Omega})$ *satisfies (8) and (9), then for* $(t,x) \in [a,T] \times \overline{\Omega}$

$$u(t,x) \geq \min\left\{ \min_{(t,x)\in[a,T]\times\partial\Omega} \gamma(t,x), \min_{x\in\overline{\Omega}} \varphi(x) \right\}$$

Proof of Corollary 1: Let us consider the value

$$m = \min\left\{ \min_{(t,x)\in[a,T]\times\partial\Omega} \gamma(t,x),\ \min_{x\in\overline{\Omega}} \varphi(x) \right\}$$

whose existence is guaranteed by our assumptions, and the following function

$$v(t,x) = u(t,x) - m, \quad (t,x) \in [a,T] \times \overline{\Omega}.$$

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whose existence is guaranteed by our assumptions, and the following function

$$v(t,x) = u(t,x) - m, \quad (t,x) \in [a,T] \times \overline{\Omega}.$$

In such a case, it follows that

$$v(t,x) \geq 0, \quad (t,x) \in [a,T] \times \partial\Omega, \tag{10}$$

and

$$v(a,x) \geq 0, \quad x \in \overline{\Omega}. \tag{11}$$

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Moreover, v satisfies

$$Lv(t,x) = F(t,x), \quad \text{for} \quad (t,x) \in (a,T]\times\Omega.$$

Therefore, as $F(t,x) \geq 0$ for $(t,x) \in (a,T]\times\Omega$,

$$Lv(t,x) \geq 0. \tag{12}$$

and by Theorem 3,

$$v(t,x) \geq 0, \quad \text{for} \quad (t,x) \in [a,T]\times\overline{\Omega},$$

which gives us the desired result. ■

Main results - Extremum principles

Analogously, Theorem 4 implies the following corollary on extremum principles

Corollary 2

Suppose that $F(t,x) \leq 0$ for $(t,x) \in (a, T] \times \Omega$. If $u \in C^{(2)}([a, T] \times \overline{\Omega})$ satisfies (8) and (9) then

$$u(t,x) \leq \max \left\{ \max_{(t,x)\in[a,T]\times\partial\Omega} \gamma(t,x),\ \max_{x\in\overline{\Omega}} \varphi(x) \right\},$$

for $(t,x) \in [a, T] \times \overline{\Omega}$.

Main results - Uniqueness of solutions

Corollaries 1 and 2 lead to the following uniqueness result.

Corollary 3

The equation (8) under the boundary conditions (9) has at most one solution $u \in \mathcal{C}^2([a, b] \times \overline{\Omega})$.

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Proof of Corollary 3: Suppose that $u_1, u_2 \in \mathcal{C}^2([a,b] \times \overline{\Omega})$ and satisfy equation (8) and conditions (9). This means that $u = u_1 - u_2$ verifies

$$Lu(t,x) = 0 \quad \text{for} \quad (t,x) \in (a,T] \times \Omega$$

and

$$\begin{array}{ll} u(a,x) = 0, & x \in \overline{\Omega} \\ u(t,x) = 0, & (t,x) \in [a,T] \times \partial\Omega. \end{array}$$

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$$Lu(t,x) = 0 \quad \text{for} \quad (t,x) \in (a,T] \times \Omega$$

and

$$\begin{aligned} u(a,x) &= 0, \quad x \in \overline{\Omega} \\ u(t,x) &= 0, \quad (t,x) \in [a,T] \times \partial\Omega. \end{aligned}$$

Now, using corollaries 1 and 2, it follows

$$u(t,x) = 0 \quad \text{for} \quad (t,x) \in [a,T] \times \overline{\Omega}.$$

Therefore, $u_1 = u_2$ in $[a,T] \times \overline{\Omega}$. This finishes the proof. ■

Main results - Supersolutions and subsolutions

Definition 3

Suppose that $F \in \mathcal{C}([a,T]\times\overline{\Omega})$, $\varphi \in \mathcal{C}(\overline{\Omega})$ and $\gamma \in \mathcal{C}([a,T]\times\partial\Omega)$. A function $\omega \in \mathcal{C}^2([a,T]\times\overline{\Omega})$ is called a *supersolution* (resp. *subsolution*) for

$$\left.\begin{array}{rcll} Lu(t,x) & = & F(t,x), & (t,x)\in(a,T]\times\Omega \\ u(a,x) & = & \varphi(x), & x\in\overline{\Omega} \\ u(t,x) & = & \gamma(t,x), & (t,x)\in[a,t]\times\partial\Omega \end{array}\right\} \tag{13}$$

if it satisfies

$$\left.\begin{array}{rcll} L\omega(t,x) & \geq^{(resp.\ \leq)} & F(t,x), & (t,x)\in(a,T]\times\overline{\Omega} \\ \omega(a,x) & \geq^{(resp.\ \leq)} & \varphi(x), & x\in\overline{\Omega} \\ \omega(t,x) & \geq^{(resp.\ \leq)} & \gamma(t,x), & (t,x)\in[a,T]\times\partial\Omega. \end{array}\right\}$$

Main results - Supersolutions and subsolutions

Corollary 4

Suppose that $\omega \in \mathcal{C}^2([a, T] \times \overline{\Omega})$ *is a supersolution (resp. a subsolution) for (13). Then, any solution* $u \in \mathcal{C}^2([a, T] \times \overline{\Omega})$ *for (13) satisfies*

$$u(t, x) \leq \omega(t, x) \quad (resp. \;\; u(t, x) \geq \omega(t, x))$$

for $(t, x) \in [a, T] \times \overline{\Omega}$.

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$$u(t,x) \leq \omega(t,x) \quad (\text{resp. } u(t,x) \geq \omega(t,x))$$

for $(t,x) \in [a, T] \times \overline{\Omega}$.

Proof of Corollary 4: Consider the function

$$v(t,x) = \omega(t,x) - u(t,x) \quad \text{for} \quad (t,x) \in [a, T] \times \overline{\Omega}.$$

If ω is a supersolution, then

$$\begin{aligned} Lv(t,x) = &\quad L\omega(t,x) - Lu(t,x) = &\quad L\omega(t,x) - F(t,x) \geq 0, &\quad (t,x) \in (a, T] \times \overline{\Omega} \\ v(a,x) = &\quad \omega(a,x) - u(a,x) = &\quad \omega(a,x) - \varphi(x) \geq 0, &\quad x \in \overline{\Omega} \\ v(t,x) = &\quad \omega(t,x) - u(t,x) = &\quad \omega(t,x) - \gamma(t,x) \geq 0, &\quad (t,x) \in [a, T] \times \partial\Omega. \end{aligned}$$

and Theorem 3 gives us the desired result. Analogous process for subsolutions, by Theorem 4, give us the conclusion. ■

Main results - Continuity respect to a boundary condition

The following theorem proves the continuity of the solution to equation (8), when (9) is verified, with respect to one of the boundary conditions.

Theorem 5

Suppose $u_i \in C^{(2)}([a,T] \times \overline{\Omega})$ $(i = 1,2)$ *are solutions to equation (8), satisfying, respectively, the following boundary conditions for* $i = 1,2$

$$\left.\begin{array}{ll} u_i(a,x) = \varphi_i(x), & x \in \overline{\Omega} \\ u_i(t,x) = \gamma(t,x), & (t,x) \in [a,T] \times \partial\Omega \end{array}\right\}$$

where $\varphi_i \in \mathcal{C}(\overline{\Omega})$ $(i = 1,2)$ *and* $\gamma \in \mathcal{C}([a,T] \times \partial\Omega)$. *Then*

$$\|u_1 - u_2\|_{\mathcal{C}([a,T]\times\overline{\Omega})} \leq \|\varphi_1 - \varphi_2\|_{\mathcal{C}(\overline{\Omega})},$$

being

$$\begin{aligned} \|u\|_{\mathcal{C}([a,T]\times\overline{\Omega})} = &\quad \max\{|u(t,x)| : (t,x) \in [a,T] \times \overline{\Omega}\} \\ \|\varphi\|_{\mathcal{C}(\overline{\Omega})} = &\quad \max\{|\varphi(x)| : x \in \overline{\Omega}\}. \end{aligned}$$

Proof of Theorem 5: Put $v(t,x) = u_1(t,x) - u_2(t,x)$, for $(t,x) \in [a, T] \times \overline{\Omega}$.
It is clear that $Lv(t,x) = 0$ for $(t,x) \in (a, T] \times \Omega$ and

$$v(a,x) = \varphi_1(x) - \varphi_2(x), \quad \text{for} \quad x \in \overline{\Omega},$$

$$v(t,x) = 0, \quad \text{for} \quad (t,x) \in [a, T] \times \partial\Omega.$$

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$$v(t,x) = 0, \quad \text{for} \quad (t,x) \in [a,T] \times \partial\Omega.$$

Using Corollary 1, we infer

$$v(t,x) \geq \min\{0, \min_{y \in \overline{\Omega}} (\varphi_1(y) - \varphi_2(y))\}, \quad \text{for} \quad (t,x) \in [a,T] \times \overline{\Omega}.$$

This gives us

$$v(t,x) \geq -\|\varphi_1 - \varphi_2\|_{\mathcal{C}(\overline{\Omega})}, \quad \text{for} \quad (t,x) \in [a,T] \times \overline{\Omega}. \tag{14}$$

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$$v(t,x) \geq -\|\varphi_1 - \varphi_2\|_{C(\overline{\Omega})}, \quad \text{for} \quad (t,x) \in [a,T] \times \overline{\Omega}. \tag{14}$$

Now using Corolary 2, it follows

$$v(t,x) \leq \max\{0, \max_{y \in \overline{\Omega}} (\varphi_1(y) - \varphi_2(y))\}, \quad \text{for} \quad (t,x) \in [a,T] \times \overline{\Omega},$$

and, from this inequality, we infer

$$v(t,x) \leq \|\varphi_1 - \varphi_2\|_{C(\overline{\Omega})}, \quad \text{for} \quad (t,x) \in [a,T] \times \overline{\Omega}. \tag{15}$$

Proof of Theorem 5: Put $v(t,x) = u_1(t,x) - u_2(t,x)$, for $(t,x) \in [a,T] \times \overline{\Omega}$. It is clear that $Lv(t,x) = 0$ for $(t,x) \in (a,T] \times \Omega$ and

$$v(a,x) = \varphi_1(x) - \varphi_2(x), \quad \text{for} \quad x \in \overline{\Omega},$$

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Using Corollary 1, we infer

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$$v(t,x) \geq -\|\varphi_1 - \varphi_2\|_{\mathcal{C}(\overline{\Omega})}, \quad \text{for} \quad (t,x) \in [a,T] \times \overline{\Omega}. \tag{14}$$

Now using Corolary 2, it follows

$$v(t,x) \leq \max\{0, \max_{y \in \overline{\Omega}} (\varphi_1(y) - \varphi_2(y))\}, \quad \text{for} \quad (t,x) \in [a,T] \times \overline{\Omega},$$

and, from this inequality, we infer

$$v(t,x) \leq \|\varphi_1 - \varphi_2\|_{\mathcal{C}(\overline{\Omega})}, \quad \text{for} \quad (t,x) \in [a,T] \times \overline{\Omega}. \tag{15}$$

Finally, from (14) and (15), we deduce

$$|v(t,x)| \leq \|\varphi_1 - \varphi_2\|_{\mathcal{C}(\overline{\Omega})}, \quad \text{for} \quad (t,x) \in [a,T] \times \overline{\Omega},$$

which is the desired result. ■

Main results - Continuity respect to a boundary condition

By using a similar argument, we get the following result

Theorem 6

Suppose $u_i \in C^{(2)}([a, T] \times \overline{\Omega})$ *(*$i = 1, 2$*) are solutions to equation (8), satisfying, respectively, the following boundary conditions for* $i = 1, 2$

$$\left.\begin{array}{ll} u_i(a, x) = \varphi(x), & x \in \overline{\Omega} \\ u_i(t, x) = \gamma_i(t, x), & (t, x) \in [a, T] \times \partial\Omega \end{array}\right\}$$

where $\gamma_i(t, x) \in \mathcal{C}([a, T] \times \partial\Omega)$ *(*$i = 1, 2$*) and* $\varphi \in \mathcal{C}(\overline{\Omega})$*. Then*

$$\|u_1 - u_2\|_{\mathcal{C}([a,T]\times\overline{\Omega})} \leq \|\gamma_1 - \gamma_2\|_{\mathcal{C}([a,T]\times\partial\Omega)}.$$

Main results - Extension of our equation

Next, we complete our study by considering equation (8) when the term $F(t,x)$ is replaced by $F(t,x,u(t,x))$, i.e.,

$$Lu(t,x) = F(t,x,u(t,x)), \qquad (t,x) \in (a,T] \times \Omega, \tag{16}$$

under the same boundary conditions (9).

Theorem 7

Suppose that $F \in C^{(1)}([a,T] \times \overline{\Omega} \times \mathbb{R})$ and

$$\frac{\partial F}{\partial y}(t,x,y) \leq 0 \;\; \text{for} \;\; (t,x,y) \in [a,T] \times \overline{\Omega} \times \mathbb{R}$$

Then, equation (16) under conditions (9) has at most one solution $u \in C^{(2)}([a,T] \times \overline{\Omega})$.

Proof of Theorem 7: Let $u_i \in C^{(2)}([a,T] \times \overline{\Omega})$ $(i = 1, 2)$ be two solutions to (16) satisfying conditions (9).

Proof of Theorem 7: Let $u_i \in C^{(2)}([a, T] \times \overline{\Omega})$ $(i = 1, 2)$ be two solutions to (16) satisfying conditions (9).

If we consider $u = u_1 - u_2$, then u is solution of the equation

$$Lu(t,x) = F(t,x,u_1(t,x)) - F(t,x,u_2(t,x)), \quad \text{for} \quad (t,x) \in (a, T] \times \Omega, \tag{17}$$

and u satisfies

$$\begin{array}{lll} u(a,x) = 0, & \text{for} & x \in \overline{\Omega}, \\ u(t,x) = 0, & \text{for} & (t,x) \in [a, T] \times \partial\Omega. \end{array} \tag{18}$$

Proof of Theorem 7: Let $u_i \in C^{(2)}([a,T] \times \overline{\Omega})$ $(i = 1, 2)$ be two solutions to (16) satisfying conditions (9).

If we consider $u = u_1 - u_2$, then u is solution of the equation

$$Lu(t,x) = F(t,x,u_1(t,x)) - F(t,x,u_2(t,x)), \quad \text{for} \quad (t,x) \in (a,T] \times \Omega, \tag{17}$$

and u satisfies

$$\begin{aligned} u(a,x) &= 0, \quad \text{for} \quad x \in \overline{\Omega}, \\ u(t,x) &= 0, \quad \text{for} \quad (t,x) \in [a,T] \times \partial\Omega. \end{aligned} \tag{18}$$

Now, using the mean value theorem, for $(t,x) \in (a,T] \times \Omega$, we have

$$F(t,x,u_1(t,x)) - F(t,x,u_2(t,x)) = \frac{\partial F}{\partial y}(t,x,y)\Big|_{(t,x,\theta(t,x))}(u_1(t,x) - u_2(t,x)),$$

where $\theta(t,x) = \lambda(t,x)u_1(t,x) + (1 - \lambda(t,x))u_2(t,x)$, for certain $\lambda(t,x) \in (0,1)$.

Proof of Theorem 7: Let $u_i \in C^{(2)}([a, T] \times \overline{\Omega})$ $(i = 1, 2)$ be two solutions to (16) satisfying conditions (9).

If we consider $u = u_1 - u_2$, then u is solution of the equation

$$Lu(t,x) = F(t,x,u_1(t,x)) - F(t,x,u_2(t,x)), \quad \text{for} \quad (t,x) \in (a, T] \times \Omega, \tag{17}$$

and u satisfies

$$\begin{array}{lll} u(a,x) = 0, & \text{for} & x \in \overline{\Omega}, \\ u(t,x) = 0, & \text{for} & (t,x) \in [a, T] \times \partial\Omega. \end{array} \tag{18}$$

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$$F(t,x,u_1(t,x)) - F(t,x,u_2(t,x)) = \frac{\partial F}{\partial y}(t,x,y)\Big|_{(t,x,\theta(t,x))}(u_1(t,x) - u_2(t,x)) \,,$$

where $\theta(t,x) = \lambda(t,x)u_1(t,x) + (1 - \lambda(t,x))u_2(t,x)$, for certain $\lambda(t,x) \in (0,1)$.

Taking into account this fact, (17) can be rewritten as

$$Lu(t,x) = \frac{\partial F}{\partial y}(t,x,y)\Big|_{(t,x,\theta(x,y))} u(t,x), \quad \text{for} \quad (t,x) \in (a, T] \times \Omega. \tag{19}$$

In order to prove our result, we proceed by contradiction. Suppose that there exists some $(t_0, x_0) \in [a, T] \times \overline{\Omega}$ such that $u(t_0, x_0) < 0$. Then, from (18), $(t_0, x_0) \in (a, T] \times \Omega$.

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Considering that $u \in \mathcal{C}([a, T] \times \overline{\Omega})$, we are able to find $(t_1, x_1) \in [a, T] \times \overline{\Omega}$, satisfying

$$u(t_1, x_1) = \min_{(t,x)\in[a,T]\times\overline{\Omega}} u(t, x) \leq u(t_0, x_0) < 0, \tag{20}$$

which implies that $(t_1, x_1) \in (a, T] \times \Omega$, and, by (19), we have

$$Lu(t_1, x_1) = \frac{\partial F}{\partial y}(t, x, y)\Big|_{(t_1,x_1,\theta(t_1,x_1))} u(t_1, x_1). \tag{21}$$

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Then, from our hypothesis and (20), we infer that the right hand side of (21) is nonnegative.

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Then, from our hypothesis and (20), we infer that the right hand side of (21) is nonnegative.

On the other hand, following a procedure similar to the one used in the proof of Theorem 3, we get that $Lu(t_1, x_1) < 0$, which is a contradiction. Therefore, $u(t, x) \geq 0$ for $(t, x) \in [a, T] \times \overline{\Omega}$.

In order to prove our result, we proceed by contradiction. Suppose that there exists some $(t_0, x_0) \in [a, T] \times \overline{\Omega}$ such that $u(t_0, x_0) < 0$. Then, from (18), $(t_0, x_0) \in (a, T] \times \Omega$.

Considering that $u \in \mathcal{C}([a, T] \times \overline{\Omega})$, we are able to find $(t_1, x_1) \in [a, T] \times \overline{\Omega}$, satisfying

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Now, repeating a similar procedure we get that $u(t, x) \leq 0$ for $(t, x) \in [a, T] \times \overline{\Omega}$.

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which implies that $(t_1, x_1) \in (a, T] \times \Omega$, and, by (19), we have

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Then, from our hypothesis and (20), we infer that the right hand side of (21) is nonnegative.

On the other hand, following a procedure similar to the one used in the proof of Theorem 3, we get that $Lu(t_1, x_1) < 0$, which is a contradiction. Therefore, $u(t, x) \geq 0$ for $(t, x) \in [a, T] \times \overline{\Omega}$.

Now, repeating a similar procedure we get that $u(t, x) \leq 0$ for $(t, x) \in [a, T] \times \overline{\Omega}$.

Consequently, $u(t, x) = 0$ and $u_1(t, x) = u_2(t, x)$ for $(t, x) \in [a, T] \times \overline{\Omega}$. ■

Main results - Extension of our equation

Theorem 8

Under assumptions of Theorem 7, suppose $u_i \in C^{(2)}([a,T] \times \overline{\Omega})$ $(i = 1, 2)$ *are solutions to equation (16), satisfiying, respectively, the following conditions for* $i = 1, 2$

$$\left.\begin{array}{ll} u_i(a,x) = \varphi_i(x), & x \in \overline{\Omega} \\ u_i(t,x) = \gamma(t,x), & (t,x) \in [a,T] \times \partial\Omega \end{array}\right\}$$

where $\varphi_i \in \mathcal{C}(\overline{\Omega})$ $(i = 1, 2)$ *and* $\gamma \in \mathcal{C}([a,T] \times \partial\Omega)$. *Then*

$$\|u_1 - u_2\|_{\mathcal{C}([a,T]\times\overline{\Omega})} \leq \|\varphi_1 - \varphi_2\|_{\mathcal{C}(\overline{\Omega})}.$$

Proof of Theorem 8: Let us consider $M = \|\varphi_1 - \varphi_2\|_{\mathcal{C}(\overline{\Omega})}$ and $u(t,x) = u_1(t,x) - u_2(t,x) + M$, for $(t,x) \in [a, T] \times \overline{\Omega}$.

Proof of Theorem 8: Let us consider $M = \|\varphi_1 - \varphi_2\|_{\mathcal{C}(\overline{\Omega})}$ and $u(t,x) = u_1(t,x) - u_2(t,x) + M$, for $(t,x) \in [a, T] \times \overline{\Omega}$.

Then, by using exactly the same process followed in the proof of Theorem 7, we get

$$Lu(t,x) = \frac{\partial F}{\partial y}(t,x,y)\Big|_{(t,x,\theta(t,x))} u(t,x), \quad \text{for} \quad (t,x) \in (a, T] \times \Omega, \tag{22}$$

and

$$\begin{aligned} u(a,x) &= \varphi_1(x) - \varphi_2(x) + M \geq 0 && \text{for} \quad x \in \overline{\Omega} \\ u(t,x) &= M && \text{for} \quad (t,x) \in [a, T] \times \partial\Omega. \end{aligned}$$

Proof of Theorem 8: Let us consider $M = \|\varphi_1 - \varphi_2\|_{\mathcal{C}(\overline{\Omega})}$ and $u(t,x) = u_1(t,x) - u_2(t,x) + M$, for $(t,x) \in [a, T] \times \overline{\Omega}$.

Then, by using exactly the same process followed in the proof of Theorem 7, we get

$$Lu(t,x) = \frac{\partial F}{\partial y}(t,x,y)\Big|_{(t,x,\theta(t,x))} u(t,x), \quad \text{for} \quad (t,x) \in (a, T] \times \Omega, \tag{22}$$

and

$$\begin{aligned} u(a,x) &= \varphi_1(x) - \varphi_2(x) + M \geq 0 && \text{for} \quad x \in \overline{\Omega} \\ u(t,x) &= M && \text{for} \quad (t,x) \in [a, T] \times \partial\Omega. \end{aligned}$$

Moreover, if we suppose that there exists $(t_0, x_0) \in [a, T] \times \overline{\Omega}$ such that $u(t_0, x_0) < 0$, then, as in the proof of Theorem 7, we find $(t_1, x_1) \in (a, T] \times \Omega$ such that

$$\begin{aligned} &Lu(t_1,x_1) < 0 \\ &Lu(t_1,x_1) = \frac{\partial F}{\partial y}(t,x,y)\Big|_{(t_1,x_1,\theta(t_1,x_1))} u(t_1,x_1) \geq 0. \end{aligned}$$

which is a contradiction. Therefore, $u(t,x) \geq 0$ for $(t,x) \in [a, T] \times \overline{\Omega}$. That is,

$$-M \leq u_1(t,x) - u_2(t,x), \quad \text{for} \quad (t,x) \in [a, T] \times \overline{\Omega}. \tag{23}$$

Proof of Theorem 8: Let us consider $M = \|\varphi_1 - \varphi_2\|_{\mathcal{C}(\overline{\Omega})}$ and $u(t,x) = u_1(t,x) - u_2(t,x) + M$, for $(t,x) \in [a, T] \times \overline{\Omega}$.

Then, by using exactly the same process followed in the proof of Theorem 7, we get

$$Lu(t,x) = \frac{\partial F}{\partial y}(t,x,y)\Big|_{(t,x,\theta(t,x))} u(t,x), \quad \text{for} \quad (t,x) \in (a, T] \times \Omega, \tag{22}$$

and

$$\begin{aligned} u(a,x) &= \varphi_1(x) - \varphi_2(x) + M \geq 0 && \text{for} \quad x \in \overline{\Omega} \\ u(t,x) &= M && \text{for} \quad (t,x) \in [a, T] \times \partial\Omega. \end{aligned}$$

Moreover, if we suppose that there exists $(t_0, x_0) \in [a, T] \times \overline{\Omega}$ such that $u(t_0, x_0) < 0$, then, as in the proof of Theorem 7, we find $(t_1, x_1) \in (a, T] \times \Omega$ such that

$$\begin{aligned} &Lu(t_1, x_1) < 0 \\ &Lu(t_1, x_1) = \frac{\partial F}{\partial y}(t,x,y)\Big|_{(t_1,x_1,\theta(t_1,x_1))} u(t_1, x_1) \geq 0. \end{aligned}$$

which is a contradiction. Therefore, $u(t,x) \geq 0$ for $(t,x) \in [a, T] \times \overline{\Omega}$. That is,

$$-M \leq u_1(t,x) - u_2(t,x), \quad \text{for} \quad (t,x) \in [a, T] \times \overline{\Omega}. \tag{23}$$

Now, repeating the same process for the function

$$w(t,x) = u_2(t,x) - u_1(t,x) + M, \quad \text{for} \quad (t,x) \in [a, T] \times \overline{\Omega},$$

we get $w(t,x) \geq 0$ for $(t,x) \in [a, T] \times \overline{\Omega}$. And, that is,

$$-M \leq u_2(t,x) - u_1(t,x), \quad \text{for} \quad (t,x) \in [a, T] \times \overline{\Omega}. \tag{24}$$

Proof of Theorem 8:
Finally, combining (23) and (24), we have

$$|u_1(t,x) - u_2(t,x)| \leq M, \quad \text{for} \quad (t,x) \in [a,T] \times \overline{\Omega}.$$

which is the desired result.

Main results - Extension of our equation

An analogous reasoning leads to the following result.

Theorem 9

Under assumptions of Theorem 7, suppose $u_i \in \mathcal{C}^2([a,T] \times \overline{\Omega})$ $(i = 1,2)$ are solutions to equation (16), satisfying, respectively, the following conditions for $i = 1,2$

$$\left.\begin{array}{ll} u_i(a,x) = \varphi(x), & x \in \overline{\Omega} \\ u_i(t,x) = \gamma_i(t,x), & (t,x) \in [a,T] \times \partial\Omega \end{array}\right\}$$

where $\gamma_i(t,x) \in \mathcal{C}([a,T] \times \partial\Omega)$ $(i = 1,2)$ and $\varphi \in \mathcal{C}(\overline{\Omega})$. Then

$$\|u_1 - u_2\|_{\mathcal{C}([a,t]\times\overline{\Omega})} \leq \|\gamma_1 - \gamma_2\|_{\mathcal{C}([a,T]\times\partial\Omega)}$$

Main results - Generalisation of the results obtained in [12]

Finally, we notice that when $f \in \mathcal{C}^{(1)}([a,b])$ and $\alpha \in (0,1)$, if $\rho \to 0^+$ in the C-K fractional derivative of order α then, by applying L'Hospital rule, we get

$$\lim_{\rho\to 0^+} {}^{C}D_{a^+}^{\alpha,\rho} f(t) = \lim_{\rho\to 0^+} \frac{\rho^\alpha}{\Gamma(1-\alpha)} \int_a^t \frac{f'(s)}{(t^\rho - s^\rho)^\alpha} ds$$
$$= \frac{1}{\Gamma(1-\alpha)} \int_a^t \lim_{\rho\to 0^+} \frac{\rho^\alpha f'(s)}{(t^\rho - s^\rho)^\alpha} ds = \frac{1}{\Gamma(1-\alpha)} \int_a^t \frac{f'(s)}{(\ln t - \ln s)^\alpha} ds,$$

which is the Caputo-Hadamard (CH) fractional derivative ${}^{CH}D_{a^+}^{\alpha} f(t)$.

Main results - Generalisation of the results obtained in [12]

In [12], the authors studied extremum principle for the following nonlinear time-fractional diffusion equation involving the CH derivative

$$\left({}^{CH}D_{1^+}^{\alpha}u\right)(t,x) = \nu\Delta_x u(t,x) + F(t,x,u), \quad (t,x) \in (1, T] \times \Omega, \tag{25}$$

where $\nu > 0$ and $\alpha \in (0,1)$, under the following boundary conditions

$$\left.\begin{array}{ll} u(1,x) = \varphi(x), & x \in \overline{\Omega}, \\ u(t,x) = \gamma(t,x), & (t,x) \in [1, T] \times \partial\Omega, \end{array}\right\} \tag{26}$$

being $\varphi \in \mathcal{C}(\overline{\Omega})$, $\gamma \in \mathcal{C}([1, T] \times \partial\Omega)$ and $F \in \mathcal{C}([1, T] \times \overline{\Omega} \times \mathbb{R})$.

Main results - Generalisation of the results obtained in [12]

By arguments similar to those used in this paper and taking into account Proposition 3.1 of [12], we can prove the extremum principle for the following more general equation than (25)

$$\left({}^{CH}D^{\alpha}_{1^+}u\right)(t,x) = div(p(x)\nabla u(t,x)) + F(t,x,u), \quad (t,x) \in (1,T] \times \Omega,$$ S14

where $\alpha \in (0,1)$, $p \in \mathcal{C}^{(1)}(\overline{\Omega})$, $p(x) > 0$ for $x \in \Omega$ and $F \in \mathcal{C}([1,T] \times \overline{\Omega} \times \mathbb{R})$ under the boundary conditions (26).

Thank you for your attention!

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