Partial Differential Equations with Applications in Industry

Problem Set 4: Laplace and Poisson's Equations

1. (*Gravitational field and potential*) Let $\mathbf{g}(\mathbf{x}) = \mathbf{x}|\mathbf{x}|^{-3}$ and $u(\mathbf{x}) = |\mathbf{x}|^{-1}$. Show that

$$\nabla \cdot \mathbf{g} = 0, \quad \mathbf{g} = -\nabla u \quad \text{for } \mathbf{x} \neq 0.$$

Deduce that $\Delta u = 0$ for $\mathbf{x} \neq 0$.

2. (*Laplacian in curvilinear coordinates*) Find the form of Laplacian in polar and spherical coordinates. If you are a daredevil, find the form of the Laplacian in a *general* curvilinear coordinate system.

3. Solve the Laplace equation on a square

$$u_{xx} + u_{yy} = 0, \quad (x, y) \in (0, 1)^2,$$

with the following conditions

$$\text{a)} \begin{cases} u(x,0) = f(x), & x \in [0,1]; \\ u(x,1) = 0, & x \in [0,1]; \\ u(0,y) = 0, & y \in [0,1]; \\ u(1,y) = 0, & y \in [0,1], \end{cases} \qquad \text{b)} \begin{cases} u_y(x,0) = g(x), & x \in [0,1]; \\ u(x,1) = 0, & x \in [0,1]; \\ u(0,y) = 0, & y \in [0,1]; \\ u(1,y) = 0, & y \in [0,1]. \end{cases}$$

How to proceed this problem when nonzero BCs are given on all the sides of the square?

4. Solve the Poisson equation on a square

$$\begin{cases} u_{xx} + u_{yy} = f, & (x,y) \in (0,1)^2, \\ u(x,y) = 0, & (x,y) \in \partial[0,1]^2. \end{cases}$$

Here, operator ∂ denotes the boundary of a set. *Hint.* You can use the same method as we did in finding a solution of the nonhomogeneous heat equation (*eigenfuction method*).

5. (*Poisson's kernel*) Show that

$$\frac{1}{2} + \sum_{n=1}^{\infty} \cos\left(n(\phi - \theta)\right) r^n = \frac{1}{2} \frac{1 - r^2}{1 + r^2 - 2r\cos\left(\phi - \theta\right)}.$$

Hint. One way of summing the above series is to use Euler's formula for expressing trigonometric functions in terms of the exponentials.

6. Use a separation of variables to solve the Laplace equation on the *exterior* of an unit circle, i.e.

$$\begin{cases} \frac{1}{r}\left(r u_r\right)_r + \frac{1}{r^2} u_{\theta\theta} = 0, & r > 1, \ -\pi \le \theta \le \pi, \\ u(1,\theta) = f(\theta), & -\pi \le \theta \le \pi. \end{cases}$$

7. (*Isotherms*)

   (a) Solve the Laplace's equation on a unit disc with the following boundary condition

$$u(1,\theta) = \begin{cases} u_0, & -\pi \le \theta < 0; \\ 0, & 0 \le \theta < \pi. \end{cases}, \quad -\pi \le \theta \le \pi.$$

(b) Next, use the fact that

$$\sum_{n=1}^{\infty} \frac{\sin(n\theta)}{n} r^n = \arctan\left(\frac{r \sin\theta}{1 - r\cos\theta}\right), \quad 0 < r < 1, \quad \theta \in [-\pi, \pi],$$

to express the solution of the above problem in a closed form, i.e. without the series.

(c) Find isotherms of u, that is curves $r = r(\theta)$ for which $u(r, \theta) = T$ for fixed $T > 0$. Sketch them.

8. Solve Laplace equation on a quarter of a circle, i.e.

$$\begin{cases} \frac{1}{r}(ru_r)_r + \frac{1}{r^2}u_{\theta\theta} = 0, & 0 < r < 1,\ 0 \le \theta \le \frac{\pi}{2}, \\ u_r(1, \theta) = f(\theta), & 0 \le \theta \le \frac{\pi}{2}, \\ u(r, 0) = 0, \quad u(r, \frac{\pi}{2}) = 0, & 0 \le r \le 1. \end{cases}$$

9. Find a solution of the Laplace equation on an annulus

$$\begin{cases} \frac{1}{r}(ru_r)_r + \frac{1}{r^2}u_{\theta\theta} = 0, & a < r < b,\ -\pi \le \theta \le \pi, \\ u(a, \theta) = 0, \quad u(b, \theta) = f(\theta), & -\pi \le \theta \le \pi. \end{cases}$$

How to deal with both nonzero BCs?

10. (*Semi-infinite strip*) Solve the Laplace equation on a semi-infinite strip

$$\begin{aligned} &\Delta u = 0, \quad x \in \mathbb{R}_+, \quad y \in (0, 1), \\ &u(0, y) = f(y), \quad u(x, 0) = u(x, 1) = 0. \end{aligned}$$

11. (*Solvability condition*) Find the solvability condition for the Poisson equation on D with Neumann BC

$$\Delta u = f, \quad \nabla u \cdot \mathbf{n}|_{\partial D} = g,$$

where $\mathbf{n}$ is the normal vector to D.
Hint. Use the divergence theorem when integrating Laplacian.

12. (*Green function for a square*)

(a) Write the solution of the Problem 4 as an integral over $[0, 1]^2$ of a product of function f and some kernel G.

(b) Show that G is indeed the Green function for the problem, i.e. $\Delta G(\mathbf{x}, \mathbf{x}_0) = \delta(\mathbf{x} - \mathbf{x}_0)$ and is zero on the $\partial[0, 1]^2$.

13. (*Two-dimensional full-space Green function*) Write down the formula for a solution of two-dimensional Poisson equation for any bounded region D with inhomogeneous Dirichlet BC. Next, find the Green function for the $D = \mathbb{R}^2$ space by mimicking derivation from the lecture.

14. (*Symmetry of Green function*) Using the Green formula with suitably chosen u and v show that $G(\mathbf{x}, \mathbf{x}_0) = G(\mathbf{x}_0, \mathbf{x})$.

15. (*A potential for a globular cluster*) In astrophysics one is interested in describing the mass distributions of various configurations of stellar objects. One of them is a gravitationally

bound spherical collection of stars - *globular cluster*. The mass density of such system can be given by the *Plummer model*

$$\rho(r) = \frac{3M}{4\pi b^3}\left(1+\frac{r^2}{b^2}\right)^{-\frac{5}{2}}.$$

Here, $r = |\mathbf{x}|$ is the distance from the origin. Assuming that u is a spherically symmetric function find the solution of the Poisson's equation

$$\Delta u = 4\pi G\rho, \quad u = u(r).$$

Compare the found solution with the potential for the point mass.
Hint. Use the Laplacian in spherical coordinates from Prob. 2 or look it up in the tables.

Łukasz Płociniczak