# $T_1$ and $T_2$ - Productable Compact Spaces

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Inspirations in Real Analysis
Bedlewo, April 2022

## Question 1 (Alexandr Osipov)

*"It is true that for every Bernstein set in real line there are countably many continuous functions for which the union of images of Bernstein set by the family functions is whole real line? "*

# Answer is yes

## Theorem 1

*There exists continuous real function such that image of every Bernstein set is a whole real line.*

## Proof.

Let $h : [0, 1] \to [0, 1]^2$ be Peano function such that $h(0) = (0, 0)$ and $h(1) = (1, 1)$ and set $g : \mathbb{R} \to \mathbb{R}^2$

$$\mathbb{R} \ni x \mapsto g(x) = ([x], [x]) + h(x - [x]) \in \mathbb{R}^2.$$

We have

$$g[\mathbb{R}] = \bigcup_{n \in \mathbb{Z}} ((n, n) + [0, 1] \times [0, 1]).$$

Then $f = \pi_2 \circ g : \mathbb{R} \to \mathbb{R}$ is a required function ( here $\pi_2 : \mathbb{R}^2 \to \mathbb{R}$ is projection on second axis).

$\square$

# Productable or fiberable map

## Definition 2

Let $X$ be a topological space, then

- $X$ is fiberable space if there exists a continuous map $f : X \to X$ s. that
$$(\forall x \in X)\ (|f^{-1}[\{x\}]| = \mathfrak{c}),$$
Here $f$ is called a fiberable map.
- $X$ is productible space if there are a topological space $T$ of size equal to $\mathfrak{c}$ and $g : X \to X \times T$ which is continuous and "onto".

Every productible space is fiberable – easy.
Every Cook continuum is not fiberable one.
$[0,1]^n, [0,1]^\omega$, $2^\omega$, $\omega^\omega$ are productible.

## Theorem 3

*If $X$ is fiberable Polish space then there is continuous $f : X \to X$ s.t. for every Bernstein subset $B \subseteq X$ $f[B] = X$.*

## Proposition 1

*Let $X$ be a topological space, then $X$ is fiberable iff $X$ is productible.*

## Proof.

It is enough to poof *if* direction. Let $X$ be a fiberable space and $T = \mathfrak{c}$ with coarst topology $\tau = \{\emptyset, T\}$ and

$$f : X \to X \text{ fiberable map.}$$

For every $x \in X$ enumerate $f^{-1}[\{x\}] = \{u_\xi^x : \xi < \mathfrak{c}\}$.
$\{f^{-1}[\{x\}] : x \in X\}$ is a partition of $X$.
Define $g : X \to X \times \mathfrak{c}$ as follows, for every $x \in X$ and $\xi \in \mathfrak{c}$

$$g(u_\xi^x, \xi) = (f(u_\xi^x), \xi) = (x, \xi) \in X \times \mathfrak{c}$$

For any open $U \subseteq X$ we have

$$g^{-1}[U \times \mathfrak{c}] = f^{-1}[U] \text{ is open in } X.$$

$\square$

### Theorem 4

*There exists a fiberable continuum $X \subseteq \mathbb{R}^3$ such that for no topological space $T$ whose every point has a nontrivial neighbourhood there exists a continuous surjection from $X$ onto $X \times T$ .*

*J. Cichoń, M. Morayne, R. Rałowski, A fiberable continuum which is not nontrivially productable, Topology and its Applications, vol. 304 (2021),*

### Theorem 5

*There exists a fiberable metric continuum $X$ such that for no nontrivial topological space $T$ (i.e. whose topology consists of more than two open sets, the whole space and the empty set) there exists a surjection from $X$ onto $X \times T$.*

*J. Cichoń, M. Morayne, R. Rałowski, A fiberable continuum which is not nontrivially productable II, Topology and its Applications, vol. 302 (2021),*

### Theorem 6 (Morayne–Ryll-Nardzewski)

*Every $T_1$ compact second countable topology can be enriched to Polish one*

*M. Morayne, Cz. Ryll-Nardzewski, Refinements of $T_1$, compact and second countable topologies, Topology and its Applications 56 (1994), 159–164*

### Theorem 7

*If $T$ Polish space, $X$ compact metric space $g : X \to T$ be such that $g^{-1}[\{t\}]$ is closed in $X$ for every point $t \in T$ then the map $h : T \to 2^X\}$ defined*

$$T \in t \mapsto h(t) = g^{-1}[\{t\}] \in 2^X$$

*is Baire measurable ($2^X$ - hyperspace with Vietoris topology).*

### Proof.

For fixed open nonempty $U \subseteq X$ if

$$R_U = \{F \in 2^X : \ F \cap U \neq \emptyset\}, \ \ S_U = \{F \in 2^X : \ F \subseteq U\}$$

then $h^{-1}[R_U] = g[U] \in \Sigma^1_1, \quad h^{-1}[S_U] = g[U^c]^c \in \Pi^1_1.$ □

# Main Result

**Theorem 8**

*Let assume that*

- $X$ *is compact metric space,*
- $T$ *is* $T_1$ *compact second countable space,*
- $f : X \to X \times T$ *is continuous surjection.*

*Then there is a continuous surjection of* $X$ *onto* $X \times [0, 1]$ *(where in interval is Euclidean topology).*

# Proof of Main Theorem

Let $f = (f_1, f_2) : X \to X \times T$ - continuous surjection. Then

- $g = \pi_T \circ f : X \to T$ is also continuous so for any $t \in T$
$$X_t = g^{-1}[\{t\}] \text{ is closed subset of } X,$$
- by Theorem 1 $T$ can be enriched to Polish space topology,
- define $T \ni t \mapsto h(t) = g^{-1}[\{t\}] \in 2^X$,
- by Theorem 2, $h : T \to 2^X$ is Baire measurable and $1-1$,
- by Lusin Theorem $h \restriction G$ is continuous on some dense $G_\delta$ subset of $T$.
- $\mathscr{C} \subseteq h[G] \subseteq 2^X$ contains a copy of the Cantor set,
- there is continuous surjection $\psi : \mathscr{C} \to [0,1]$.

...

### Claim 1

*If $\mathscr{H} \subseteq \mathscr{C}$ is closed in $2^X$ then $\bigcup \mathscr{H} \subseteq X$ is closed in $X$.*

Define $\varphi : \bigcup \mathscr{C} \to [0,1]$, let $X_t \in \mathscr{C}$, $x \in X_t$, and $s \in [0,1]$,

$$\varphi(x) = s \longleftrightarrow \Psi(X_t) = s.$$

### Claim 2

*$\varphi$ is continuous.*

Proof. Let $V \subseteq [0,1]$ open set, $x \in \bigcup \mathscr{C}$ s.t. $\varphi(x) \in V$. There are open subsets $U_0, \ldots, U_n \subseteq X$ and $X_t \in \mathscr{C}$ s.t. $x \in X_t$ and

$$X_t \in \{F \in 2^X : F \subseteq U_0\} \cap \bigcap_{i=1}^{n} \{F \in 2^X : F \cap U_i \neq \emptyset\} \in \Psi^{-1}[V].$$

...

Observe that

$$\{F \in 2^X : \neg(F \subseteq U_0)\}, \quad \bigcup_{i=1}^{n} \{F \in 2^X : F \cap U_i = \emptyset\}$$

are closed in $2^X$. Then $x \notin Z$ and $Z \subseteq X$ is closed in $X$, where

$$Z = \bigcup \mathscr{C} \cap \left(\{F \in 2^X : \neg(F \subseteq U_0)\} \cup \bigcup_{i=1}^{n} \{F \in 2^X : F \cap U_i = \emptyset\}\right)$$

Find open neighborhood $W$ of $x$ s.t. $W \cap Z = \emptyset$.
If $y \in W \cap \bigcup \mathscr{C}$ then there is $r \in T$ s.t. $X_r \in \mathscr{C}$ and $y \in X_r$ then

$$X_r \in \{F \in 2^X : F \subseteq U_0\} \cap \bigcap_{i=1}^{n} \{F \in 2^X : F \cap U_i \neq \emptyset\} \in \Psi^{-1}[V]$$

So $\Psi(X_r) \in V$ and then $\varphi(y) \in V$ also. But $y \in W \cap \bigcup \mathscr{C}$ is arbitrary then $\varphi$ is continuous in $x \in \bigcup \mathscr{C}$. $\Box$

$\bigcup \mathscr{C}$ is closed in $X$ then there is a continuous exstension $\varphi^*$ of $\varphi$ onto whole space $X$. Then $F = (f_1, \varphi^*) : X \to X \times [0,1]$ is continuous also.

$F$ is "onto" $X \times [0,1]$: let $x \in X$ and $s \in [0,1]$ then

- there is $t \in T$ s.t. $X_t \in \mathscr{C}$ and $\Psi(X_t) = s$,
- because $f[X_t] = X \times \{t\}$ then $f_1[X_t] = X$,
- there is $u \in X_t$ s.t. $f_1(u) = x$ and then $(f_1(u), \varphi^*(u)) = (f_1(u), \varphi(u)) = (x, s)$ because $u \in X_t$ and $\varphi(u) = \Psi(X_t) = s$.

$\blacksquare$

Thank You for your attention