Mycielski among trees
Nonstandard proofs of Mycielski and Egglestone like Theorems

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Kosice, 9-th September 2019

Theorem (Jan Mycielski - $2D$ case)

If $I = \mathscr{M}, \mathscr{N}$ then for every $B \subseteq [0,1]^2$ such that $B^c \in I$ then there is perfect set $P \subseteq [0,1]$ such that

$$P \times P \subseteq B \cup \Delta,$$

where $\Delta = \{(x,x) : \; x \in [0,1]\}$.

Slalom tree

Let $T \subseteq \omega^{<\omega}$ be a tree. Then T is a slalom tree iff

$$(\forall m \in \omega)(\forall s \in \omega^m)(\exists k \in \omega)(\forall x \in [T])(\forall i \in m)x(k+i) = s(i).$$

Theorem

If $G \subseteq \omega^\omega \times \omega^\omega$ is dense G_δ then there is a slalom tree $T \subseteq \omega^{<\omega}$ such that $[T] \times [T] \subseteq G \cup \Delta$.

Strategy of proof

- fix V as the any ZFC transitive universe,
- prove the required theorem φ in some generic extension $V[G]$,
- check the complexity of proved formula φ,
- if our formula φ is Σ^1_2 or $\prod^1_2$ or simpler then use Schoenfield Theorem,
- and we are getting $V \models \varphi$.

Absolutness

Let $M \subseteq N$ - transitive models of ZF theory, $\varphi \in \mathscr{L}(\epsilon)$ set theory formula with n free variables. Then φ is absolute between M, N if for every parameters $a_0, \ldots, a_{n-1} \in M$

$$M \models \varphi(a_0, \ldots, a_{n-1}) \text{ iff } N \models \varphi(a_0, \ldots, a_{n-1}).$$

Σ^1_2 sentence

X canonical Polish space if is a countable product of $2^\omega, \omega^\omega, \mathbb{R}, [0,1]$ and $Perf(\mathbb{R})$ as space of perfect sets.

φ is Σ^1_2 sentence if for some canonical spaces X, Y and Borel set $B \subseteq X \times Y$ φ is

$$(\exists x \in X)(\forall y \in Y)\ (x, y) \in B.$$

(X, Y, b) are parameters where $b \in \omega^\omega$ is a Borel code for B.

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Schoenfield Absolutness Theorem

Theorem

Let $M \subseteq N$ be a standard transitive models of ZFC and $\omega_1^N \subseteq M$. Let φ be a Σ_2^1 formula with parameters from M then

$$M \models \varphi \text{ iff } N \models \varphi.$$

If N is a generic extension of M then $Ord^M = Ord^N$ and $\omega_1^N \subseteq M$.

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If N is a generic extension of M then $Ord^M = Ord^N$ and $\omega_1^N \subseteq M$.

Let $T \subseteq \omega^\omega$ - tree then define

$$\text{leaves}(T) = \{\sigma \in T : \neg(\exists \tau \in T)\ \sigma \subseteq \tau \wedge \tau \neq \sigma\}$$

For any $\sigma \in T$ define

$$rank_T(\sigma) = \sup\{rank_T(\tau) + 1 : \tau \in T \wedge \sigma \subsetneq \tau\}$$

$ht(T) = rank_T(\emptyset)$.
$T \subseteq \omega^{<\omega}$ is a nice cutted tree if

$$(\exists n \in \omega) ht(T) = n \wedge leaves(T) \subseteq \omega^n.$$

Forcing notion

Define $(\mathscr{C}, \leq)$ as follows

- $\mathscr{C} = \{p \subseteq \omega^{<\omega} : p$ is nice cutted tree and is finite$\}$,
- $(\forall p, q \in \mathscr{C})$ $(p \leq q$ iff $q \subseteq p \wedge p \cap \omega^{ht(q)} = leaves(q))$, ($p$ is stronger than q).

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Theorem in one Cohen real extension

Theorem

After adding one Cohen real there is a perfect slalom tree T such that $[T] \times [T] \subseteq W \cup \Delta$ for every dense G_δ set $W \subseteq \omega^\omega \times \omega^\omega$ from the ground model.

Sketch of Proof

Lemma

For every open dense set $U \subseteq \omega^\omega \times \omega^\omega$ *a finite open sets* $(V_k : \ k \in n)$ *in* ω^ω : *there is sequence* $\{ \ \sigma_k : k \in n\}$ *of sequences such that for any* $k, l \in n$

- $[\sigma_k] \subseteq V_k$,
- $|\sigma_k| = |\sigma_l|$,
- $k \neq l \to [\sigma_k] \times [\sigma_l] \cup [\sigma_l] \times [\sigma_k] \subseteq U$.

Claim

If $G \subseteq \mathscr{C}$ is generic filter over V and $T_G = \bigcup G$ then

1. T_G *is slalom perfect tree,*
2. *for every dense open set* $U \subseteq \omega^\omega \times \omega^\omega$ *coded in ground model and any* $n \in \omega$ *the set* $D_{n,U}$

   $$\{p \in \mathscr{C} : (\forall s, t \in leaves(p))(n \leq |t|, |s| \wedge t \neq s) \rightarrow [t] \times [s] \subseteq U\}$$

   is dense in $(\mathscr{C}, \leq)$.
3. *Fix* $\dot{x} \in V^{\mathscr{C}}$ *and* $p, q \in G$. *Assume that*

   $$p \Vdash \dot{x} \in [T_G] \wedge q \Vdash \dot{x} \restriction n \subseteq s$$

   for $n \leq ht(q)$ *and* $s \in q$. *Then there is* $r \in G$ *and* $m \geq n$ *such that* $r \leq p, q$ *and* $r \Vdash \dot{x} \restriction m \in leaves(q)$.

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continuation of the proof

By first condition T_G is slalom perfect tree.
Let $\dot{x}, \dot{y} \in V^{\mathscr{C}}$ and $p \in G$ such
$p \Vdash \dot{x}, \dot{y} \in [\dot{T}_G] \wedge \dot{x} \restriction n_{x,y} \neq y \restriction \dot{n}_{x,y}$.
By the Claim 2) there is $q \in G$ such that $q \leq p$ and for any
$s, t \in leaves(q)$ if $t \neq s \rightarrow [t] \times [s] \subseteq U$.
Then by the Claim 3) there is $r \in G$ stronger than q and there is
$m > n_{x,y}$ such that

$$r \Vdash \dot{x} \restriction m \in leaves(q) \wedge \dot{y} \restriction m \in leaves(q)$$

Then we have for $r \in G$

$$r \Vdash (\dot{x}, \dot{y}) \in [\dot{x} \restriction m] \times [\dot{y} \restriction m] \subseteq U.$$

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Theorem

For every $G \in G_\delta$ dense subset of $\omega^\omega \times \omega^\omega$ there exists slalom perfect set $P \subseteq \omega^\omega$ such that $P \times P \subseteq G \cup \Delta$.

Proof.

Let assume

- V - ZFC ground model,
- $W \in V$ dense G_δ in $\omega^\omega \times \omega^\omega$,
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Then by previous Theorem, in $V[G]$ there is a generic tree T_G such that $[T_G] \times [T_G] \subseteq W \cup \Delta$.But

$$\varphi = (\exists P \in Perf(\omega^\omega))(\forall x, y \in P)(x \neq y \to (x, y) \in W)$$

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Egglestone like Theorem

Theorem

Let $I = \mathscr{M}, \mathscr{N}$ - σ-ideal on $\mathbb{R}$ and $G \subseteq \mathbb{R}^2$ be a Borel set such that $G^c \in I(\mathbb{R}^2)$ Then there are $B, P \subseteq \mathbb{R}$ such that P - perfect and $B^c \in I$ such that $P \times B \subseteq G$.

Sz. Żeberski, Nonstandard proofs of Egglestone like theorems, Proceedings of the Ninth Topological Symposium, 2001, 353-357.

Proof

Let V' - extension of ZFC model V and $V' \models \aleph_2 < add(I) \leq \mathfrak{c}$.
Let $G \in V$ and $b \in \omega^\omega$ a borel code for G. Set $G^* = \#b^{V'}$
In V' define $Z = \{x \in \mathbb{R} : \; {G_x^*}^c \in I\}$.
By Fubini (or Kuratowski - Ulam) theorem $Z^c \in I$ so $|Z| = \mathfrak{c} > \aleph_2$.
Choose $T \subseteq Z$ with $T = \aleph_2$. Then $(\bigcap_{t \in T} G_t^*)^c \in I$.
Let $B \in Bor(\mathbb{R})$ s.t. $B^c \in I$ and $B \subseteq \bigcap_{t \in T} G_t^*$. Observe that

$$A = \{x \in \mathbb{R} : \; B \subseteq G_x^*\} \text{ is coanalytic.}$$

and $T \subseteq A$ then A contain perfect P.

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which is Σ^1_2.

In category case the set B can be dense G_δ which can by written as arithmetical formula which is absolute between V and V' (analogoulsy in measure case).

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