The Baire theorem, an analogue of the Banach fixed point theorem

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Baire space

Definition

A topological space X is a *Baire space* if the intersection of countably many dense open subsets of X is a dense subset of X.

Equivalently, the countable union of closed sets with empty interiors has empty interior.

Theorem (Baire)

Every complete-metrisable topological space or Hausdorff compact space is Baire space.

T_1 - Baire space

Theorem 1

If X is a T_1 second countable compact space, TFAE

- X is a Baire space,
- every nonempty open subset of X contains a closed subset with nonempty interior.

Proof of the Theorem 1 ($\rightarrow$ direction)

Lemma

If X is a T_1 second countable compact space, then each closed subset of X is a countable intersection of open sets.

Namely, by Lemma every open set is a union of countably many closed sets and one of them must have nonempty interior because X is a Baire space.

$\square$

Proof of the Theorem 1 ($\leftarrow$ direction).

Firstly, assume that for every open subset U of X there exists a nonempty open set V s.t. $cl(V) \subseteq U$.

- Let $\mathcal{F} = \{F_n : n \in \omega\}$ be a family of closed subsets with $int(F_n) = \emptyset$.
- $\emptyset \neq W \subseteq X$ - open subset. We show that $W \setminus \bigcup \mathcal{F} \neq \emptyset$,
- define family $\{V_n : n \in \omega\}$ of nonempty open sets in X s.t.:
  - $V_0 \subseteq cl(V_0) \subseteq W \cap F_0^c$,
  - $V_{n+1} \subseteq cl(V_{n+1}) \subseteq V_n \cap F_{n+1}^c$ for each $n \in \omega$.
- then
$$\bigcap_{n=0}^{\infty} cl(V_n) \cap \bigcup \mathcal{F} = \emptyset,$$
- As X is compact, $W \cap \bigcap_{n=1}^{\infty} cl(V_n) \neq \emptyset$. Hence
$$W \nsubseteq \bigcup \mathcal{F}.$$

Example 1

Set $\tau = \{U :\in P(\omega) : \ \omega \setminus U \in [\omega]^{<\omega}\}$.
Then (ω, τ) is a T_1 second-countable compact space which is not a Baire space.
Only ω is a closed with nonempty interior set in (ω, τ).

Remark

Example 1 shows a difference between the T_1 and T_2 cases, because every T_2 compact space is a Baire space.

Example 2, A condition in Theorem 1 does not imply T_2

Let $X = [-1, 1]$ and base of topology $\mathscr{B} = \mathscr{B}_{(-1,1)} \cup \mathscr{B}_{-1} \cup \mathscr{B}_1$:

$$\mathscr{B}_{(-1,1)} = \{(-1, 1) \cap (a, b) : \ a, b \in \mathbb{R}\},$$

$$\mathscr{B}_{-1} = \{[-1, 1) \setminus [p, q] : \ p, q \in \mathbb{Q} \cap (-1, 1) \wedge p < q\}$$

$$\mathscr{B}_1 = \{(-1, 1] \setminus [p, q] : \ p, q \in \mathbb{Q} \cap (-1, 1) \wedge p < q\}$$

Then

- X is a compact T_1 but not T_2, second countable Baire space,
- if $[p, q] \subseteq U \subseteq (-1, 1)$ for rationals $-1 < p < q < 1$ then

  $$[p, q] = ([-1, 1]\setminus[p, q])^c = \big(([-1, 1)\setminus[p, q])\cup((-1, 1]\setminus[p, q])\big)^c,$$

  and thus $[p, q]$ is closed in X. But (p, q) is a open set in X.
- let $U \in \mathscr{B}_{-1}$ then $[p, q] \subseteq U$ for some rationals $-1 < p < q < 1$. Then U contains open (p, q) in X and $[p, q]$ is closed in X. If $U \in \mathscr{B}_1$ then we argue as above.

In Theorem 1 we cannot drop the second countabilty

Example 3

Let $X = [0,1]$; a base of a toplogy on X: $\mathscr{B} = \mathscr{B}_{[0,1)} \cup \mathscr{B}_1$ where

$$\mathscr{B}_{[0,1)} = \{[0,1) \cap (a,b) :\ a, b \in \mathbb{R}\}$$

$$\mathscr{B}_1 = \{U \in \mathscr{P}([0,1]) :\ 1 \in U \wedge [0,1] \setminus U \text{ is finite }]\}$$

Then we have

- X is compact and T_1,
- X is a Baire space,
- if $U \subseteq [0,1)$ is open then each closed set $F \subseteq U$ is finite (because $1 \in F^c$). Then $int(F) = \emptyset$.

Theorem (Banach fixed-point theorem, 1920)

Every Lipschitz contraction on complete metric space has unique fixed point.

Here $f : X \to X$ is a Lipschitz contraction iff existst $c \in [0, 1)$ s.t. for every $x, y \in X$

$$d(f(x), f(y)) \leq c \cdot d(x, y).$$

Topological contraction

Definition

Let X be a T_1-topological space and $f : X \to X$. We say that f is a topological contraction on X iff for every distinct $x, y \in X$ there exists $n \in \omega$ s.t.

$$f^n[X] \subseteq \{x\}^c \text{ or } f^n[X] \subseteq \{y\}^c.$$

For the compact metric spaces we have

Theorem (Lebesgue number)

For every compact metric space, X and any open cover $\mathcal{U}$ there exists $\epsilon > 0$ s.t.

$$\forall x \in X \exists U \in \mathcal{U}\ B(x, \epsilon) \subseteq U.$$

Fact

Every Lipschitz contraction on a compact metric space is a topological contraction.

Fixed point theorem for compact T_1 spaces

Theorem 2

Let X be T_1 topological space and $f : X \to X$ be a closed topological contraction on X. Then there exsists an unique $x \in X$ s.t. $x = f(x)$.

Corollary

Every Lipschitz contraction on compact metric space has unique fixed point.

Example 4

Let (ω, τ) be T_1 topological space where

$$\tau = \{\emptyset\} \cup \{A \in \mathscr{P}(\omega) : A^c \text{ is finite } \}.$$

Then $\omega \ni n \mapsto f(n) = n + 1 \in \omega$ is a continuous, topological contraction without any fixed point, (f is not closed map !!!).

Proof of the Theorem 2

- For each $n \in \omega$, $f^n[X]$ is a closed subset of X with $f^{n+1}[X] \subseteq f^n[X]$,
- because X is compact

$$F = \bigcap \{f^n[X] : n \in \omega\} \neq \emptyset.$$

- If $x, y \in F$ are two distinct points then $\{\{x\}^c, \{y\}^c\}$ is an open cover of T_1-space X and then there exists $n \in \omega$ s.t.

$$F \subseteq f^n[X] \subseteq \{x\}^c \ \vee \ F \subseteq f^n[X] \subseteq \{y\}^c,$$

  which is impossible.
- If $F = \{x\}$ then for every $n \in \omega$ $x \in f^n[X]$ so $f(x) \in f^{n+1}[X] \subseteq f^n[X]$. Then $f(x) \in F$, hence $x = f(x)$.
- for each $y \in X$ if $y = f(y)$ then $y \in F$. Hence $y = x$.

Theorem 3

Let X be a T_1-topological space and $f : X \to X$ be a closed map. Then f is a topological contraction iff for every open cover $\mathcal{U}$ of X there are $n \in \omega$ and $U \in \mathcal{U}$ s.t. $f^n[X] \subseteq U$.

Proof.

Let $\mathcal{U}$ be an open cover of X.

- By fixed point theorem there is $x \in X$ s.t. $x = f(x)$.
- then $x \in U$ for some $U \in \mathcal{U}$
- for some $n \in \omega$ $f^n[X] \subseteq U$. If not then for each $n \in \omega$ $f^n[X] \cap U^c \neq \emptyset$,
- there is y s.t. $y \in F := \bigcap\{f^n[X] : n \in \omega\} \cap U^c \neq \emptyset$,
- $F \subseteq f^n[X] \subseteq \{x\}^c$ or $F \subseteq f^n[X] \subseteq \{y\}^c$ for some $n \in \omega$, contradiction.

The other direction is obvious. $\square$

Lipschitz contraction is continuous but topological not neccessary.

Example 5

Let $X = \{1/n : n \in \mathbb{N}\} \cup \{0, 2, 3\}$ be endowed with the usual Euclidean metric from the real line. Let for $x \in X$:

$$f(x) := \begin{cases} 2 & \text{if} \quad x = 1/n, \\ 3 & \text{if} \quad x = 0, 2, 3. \end{cases}$$

The mapping f is a closed topological contraction because $f^2[X] = \{3\}$; it is closed because $f[X] = \{2, 3\}$; and it is not continuous because

$$f\left(\lim_n \frac{1}{n}\right) = f(0) = 3 \neq 2 = \lim_n f\left(\frac{1}{n}\right).$$

(Of course, the fixed point here is 3).

IFS - iterated function systems

Let X be a T_1 compact space, $m \in \omega$ then

$$\mathcal{F} = \{f_i : i < m\} \in [X^X]^{<\omega} \text{ is an IFS.}$$

$\mathcal{F}$ is a contractive IFS if

- each $f \in \mathcal{F}$ is closed,
- for every open cover $\mathcal{U}$ of X there is $n \in \omega$ s.t.

$$\forall s \in \{0, \ldots, m-1\}^n \; \exists U \in \mathcal{U} \; f_s[X] \subseteq U,$$

where $f_s = f_{s(n-1)} \circ \ldots \circ f_{s(0)}$ and $\circ$ is a composition.

Lebesgue number Lemma implies

Fact

Every Lipschitz contractive IFS on compact metric space is contractive as above.

Hutchinson operator

Set 2^X hyperspace of all closed subsets of X with Vietoris topology.
Let $\mathcal{F} = \{f_i : i < m\}$ be an IFS on a T_1 space X.
The *Hutchinson operator* $F : 2^X \to 2^X$ *induced by* $\mathcal{F}$ is given by

$$2^X \ni K \mapsto F(K) = \bigcup_{i<m} f_i[K] \in 2^X.$$

Every fixed point of the Hutchinson operator is called attractor.

Theorem 4

Let X be a T_1 compact space. Let $\mathcal{F}$ be an IFS on X. Then the Hutchinson operator induced by $\mathcal{F}$ has a fixed point.

Proof.

Let F be the Hutchinson operator of IFS $\mathcal{F}$. Let $F^0(X) = X$.

for $\alpha + 1$: $F^{\alpha+1}(X) = F(F^{\alpha}(X))$

for a limit λ: $F^{\lambda}(X) = \bigcap_{\alpha<\lambda} F^{\alpha}(X)$.

Then for all $\alpha \in On$

- $F^{\alpha}(X)$ are closed and nonempty (compactness of X),
- if $\alpha < \beta$ then $F^{\beta}(X) \subseteq F^{\alpha}(X)$ (by $A \subseteq B \to F(A) \subseteq F(B)$).

Thus it must stabilize at some ordinal α

$$F^{\alpha}(X) = F^{\alpha+1}(X) = \dots.$$

Thus $F^{\alpha}(X)$ is a fixed point of F. $\square$

Example 6

$$\text{Let } X = \left\{ \frac{1}{n} : n \in \mathbb{N} \right\} \cup \{0, -1\}$$

be considered with the usual Euclidean topology. Let $\mathcal{F}$ consist of only one mapping f defined as follows:

$$f(0) = 0, \; f(-1) = 0,$$

$$f(1) = 1/2, \, f(1/2) = -1,$$

$$f(1/3) = 1/4, \, f(1/4) = 1/5, \, f(1/5) = -1,$$

$$f(1/6) = 1/7, \, f(1/7) = 1/8, \, f(1/8) = 1/10, f(1/9) = -1$$

$$\ldots,$$

If F is the Hutchinson operator $\{f\}$ IFS, then $F^n(K) = f^n[K]$, and then

$$F^{\omega}(X) = \bigcap_{n=1}^{\infty} f^n[X] = \{0, -1\} \text{ but } F^{\omega+1}(X) = F^{\omega+2}(X) = \ldots = \{0\}.$$

Two fixed points

Example 7

Let X be any T_1 compact topological space and $|X| \geq 2$.
Fix $x_0 \in X$ and $f \equiv x_0$ be a constant mapping.
Define an IFS as

$$\mathcal{F} = \{\mathrm{id}_X, f\},$$

where id_X is the identity mapping on X.
The Hutchinson operator F induced by this IFS has two fixed points: $\{x_0\}$ and X.

Theorem 5

Let X be a T_1 compact space. Let $\mathcal{F}$ be a contractive IFS on X. Then the Hutchinson operator induced by $\mathcal{F}$ is a topological contraction on 2^X.

Applying the Fix Point Theorem 2

Corollary

If X is a T_1 compact space then every contractive IFS for which its Hutchinson operator is closed in 2^X has a unique attractor.

But by Theorem 4

Corollary

If X is a T_1 compact space then every contractive IFS has a unique attractor.

Example 8

We consider the space $X = \{0, 1, 2\}$ with the discrete topology. Let $f_1(2) = 1$, $f_1(1) = 0$, $f_1(0) = 0$, and $f_2(1) = 2$, $f_2(2) = 0$, $f_2(0) = 0$. Both f_1 and f_2 are contractions because $f_1^2[X] = f_2^2[X] = \{0\}$. Let $\mathcal{F} = \{f_1, f_2\}$ and let F be the Hutchinson operator induced by the IFS $\mathcal{F}$. We have

$$F(X) = f_1[X] \cup f_2[X] = \{0, 1\} \cup \{0, 2\} = X$$

and

$$F(\{0\}) = f_1[\{0\}] \cup f_2[\{0\}] = \{0\}.$$

Thus F has more than one fixed point and by last Corrolary the IFS $\mathcal{F}$ is not contractive and by Theorem 2, F is not a topological contraction.

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Thank You