Group actions on Polish space

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Motivation

Question (Y. Kuznetsova)

For every null subset $A \subseteq \mathbb{R}$ there exists a subset $S \subseteq \mathbb{R}$ such that $A + S$ is nonmeasurable.

Theorem (Z. Kostana)

For every meager subset $A \subseteq \mathbb{R}$ there exists a subset $S \subseteq \mathbb{R}$ such that $A + S$ does not have the Baire property.

Cardinal coefficients

Definition

Let $I \subset \mathscr{P}(X)$ be a σ-ideal on a Polish space X. Assume that I has a Borel base and contains all singletons.

$$cov(I) = min\{|\mathscr{A}| : \mathscr{A} \subset I \wedge \bigcup \mathscr{A} = X\},$$

$$cov_h(I) = min\{|\mathscr{A}| : (\mathscr{A} \subset I) \wedge (\exists B \in \mathscr{B}(X) \setminus I)\ (B \subseteq \bigcup \mathscr{A})\},$$

$$cof(I) = min\{|\mathscr{A}| : \mathscr{A} \subset I \wedge \mathscr{A} \text{ is a Borel base of } I\},$$

$$non(I) = min\{|A| : A \subseteq X \wedge A \notin I\}.$$

Let X - Polish space, I σ-ideal with Borel base containing all singletons. Define

$$\mathscr{B}_I^+(X) = Bor(X) \setminus I$$

and

$$cof(\mathscr{B}_I^+(X)) = min\{|\mathscr{A}| : \mathscr{A} \subset \mathscr{B}_I^+(X) \wedge \mathscr{A} \text{ is a base of } \mathscr{B}_I^+(X)\}.$$

We have $cof(\mathscr{B}_I^+(X)) \leq \mathfrak{c}$.
Moreover, $cof(\mathscr{N}) = cof(\mathscr{B}_{\mathscr{N}}^+)(X)$ and $cof(\mathscr{M}) = cof(\mathscr{B}_{\mathscr{M}}^+)(X)$, (due to Cichoń, Kamburelis and Pawlikowski)

Definition (Polish ideal space)

We say that a pair (X, I) is a Polish ideal space iff

- X is an uncountable Polish space,
- $I \subset \mathscr{P}(X)$ is a σ-ideal containing singletons and having a Borel base.

Examples:

- $(X, \mathscr{M})$ - Polish ideal space, where X is a Polish space, $\mathscr{M}$ is σ-ideal of meager sets,
- $(X, \mathscr{N})$ - Polish ideal space, where $X = \mathbb{R}, [0,1]$ and $\mathscr{N}$ σ-ideal of null subsets with respect to Lebesgue measure,
- $(2^\omega, \mathscr{N})$ where $\mathscr{N}$ - σ-ideal of null subsets with respect to Haar measure.

The cardinal coefficients connected to $\mathscr{M}$ and $\mathscr{N}$ do not depend on X.
Moreover $cov(\mathscr{M}) = cov_h(\mathscr{M})$, $cov(\mathscr{N}) = cov_h(\mathscr{N})$.

Definition (Polish ideal group)

We say that a triple $(G, \cdot, J)$ is a Polish ideal group if (G, J) is a Polish ideal space and $(G, \cdot)$ is a Polish group.

If X is a compact Polish space then $\mathscr{H}(X)$ with compact-open topology is a Polish group.

Definition

Let (X, I) be a Polish ideal space. We say that $C \subseteq X$ is completely I-nonmeasurable in X iff

$$(\forall B \in \mathscr{B}_I^+(X))\,(B \cap C \neq \emptyset \wedge B \cap C^c \neq \emptyset).$$

- $A \subseteq 2^\omega$ is completely *ctbl*-nonmeasurable iff A is Bernstein set,
- $A \subseteq 2^\omega$ is completely $\mathscr{N}$-nonmesurable iff $\lambda_*(A) = 0$ and $\lambda^*(A) = 1$.

Theorem

Let (X, I) be a Polish ideal space and $(G, \cdot, J)$ be an uncountable Polish ideal group acting on X. Assume that there are bases $\mathcal{B}_G \subset \mathscr{B}_J^+(G)$ and $\mathcal{B}_X \subset \mathscr{B}_I^+(X)$ with

$$|\mathcal{B}_G| = |\mathcal{B}_X| \leq |\{Gb : \ b \in B\}|.$$

Then there exists a completely J-nonmeasurable subgroup $H \leq G$ and a pairwise disjoint family $\{A_\alpha : \alpha < cof(\mathscr{B}_I^+(X))\} \subset \mathscr{P}(X)$ such that:

1. $(\forall \alpha < cof(\mathscr{B}_I^+(X)))\ A_\alpha, HA_\alpha$ *are completely I-nonmeasurable in X,*
2. $(\forall \alpha, \beta)\alpha < \beta < cof(\mathscr{B}_I^+(X)) \to HA_\alpha \cap HA_\beta = \emptyset.$

Corollary

$$G = \{T_X : X \in \mathscr{P}(\{n \in \omega : n \equiv 0 \mod 2\})\} \leq Iso(2^\omega),$$

where for any $x \in 2^\omega$ and $n \in \omega$

$$T_X(x)(n) = \begin{cases} x(n) & \text{when } n \notin X \\ 1 - x(n) & \text{when } n \in X. \end{cases}$$

Then there is a subgroup H of G and family $\{A_\alpha \subset 2^\omega : \alpha < cof(\mathscr{M})\}$ such that

- *HA_α are completely $\mathscr{M}$-nonmeasurable in the Cantor space 2^ω for any $\alpha < cof(\mathscr{M})$,*
- *$\{HA_\alpha : \alpha < cof(\mathscr{M})\}$ forms a pairwise disjoint family of subsets of the Cantor space.*

Theorem

Let (X, I) be a Polish ideal space. Assume that $(G, \cdot, J)$ forms a Polish ideal group acting on X. If for some (every) $x \in X$ $Gx = X$ and

$$(\exists \lambda < 2^{\omega})(\forall x, y \in X)\, x \neq y \rightarrow |G_{x,y}| \leq \lambda$$

where $G_{x,y} = \{g \in G : \ y = gx\}$ then there exists a subgroup $H \leq G$ and a subset $A \subset X$ such that A and HA are completely I-nonmeasurable sets in X and H is completely J-nonmeasurable in G.

Corollary

Let $(G, \cdot, J)$ be a Polish ideal group. Then there exist $H < G$ and $A \subseteq G$ such that H, A, HA are completely J-nonmeasurable.

Theorem

Let $(G, \cdot, J)$ be a Polish ideal group which acts on a Polish ideal space (X, I). Let us assume that

1. $cov_h(J) = cof(\mathscr{B}_J^+(G)) = cof(\mathscr{B}_I^+(X))$,
2. *there exists $G' \subseteq G$ with $G \setminus G' \in J$ such that for any $n \in \omega$, $s \in \mathbb{Z}^n$ and for every $g \in G'$, $a \in G'^n$ the following condition holds:*
$$S_{a,s,g} = \{h \in G : \prod_{i \in n} a_i \cdot h^{s_i} = g\} \in J,$$
3. *there exists $X' \subseteq X$ such that $X \setminus X' \in I$ such that for any $n \in \omega$, $s \in \mathbb{Z}^n$, for every $a \in G'^n$ and every $x, y \in X'$ the following condition holds:*
$$T_{a,s.x,y} = \{h \in G : \left(\prod_{i \in n} a_i \cdot h^{s_i}\right)x = y\} \in J.$$

Then there is a completely J-nonmeasurable subgroup $H \leq G$ and a completely I-nonmeasurable subset $A \subseteq X$ such that HA is completely I-nonmeasurable in the space X.

Proposition

Let $(G,\cdot)$ be an uncountable Polish group. Fix $n \in \omega$, $s \in \mathbb{Z}^n$. Then there exists comeager $G' \subseteq G$ such that for every $g \in G'$, $a \in G'^n$ the following set

$$\{h \in G : \prod_{i \in n} a_i \cdot h^{s_i} = g\}$$

is meager.

Proposition

Let $(G,\cdot)$ *be an uncountable Polish group acting on an uncountable Polish space* X*. Fix* $n \in \omega$, $s \in \mathbb{Z}^n$*. Then there exists comeager* $G' \subseteq G$ *and comeager* $X' \subseteq X$ *such that for every* $a \in G'^n$ *and every* $x, y \in X'$ *the following set*

$$T_{a,s.x,y} = \{h \in G : \ \big(\prod_{i \in n} a_i \cdot h^{s_i}\big)x = y\}$$

is meager.

Corollary

Assume that

- $cov(\mathscr{M}) = cof(\mathscr{M})$,
- X *be a compact Polish space without isolated points,*
- $\mathscr{H}(X)$ *space of all homeomorhisms of* X.

Then there are $H \leq \mathscr{H}(X)$ *and* $A \subseteq X$ *such that*

- H *is completely* $\mathscr{M}$*-nonmeasurable subgroup in* $\mathscr{H}(X)$,
- A *and* HA *are completely* $\mathscr{M}$*-nonmeasurable in* X.

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Thank You