Continuous images of Bernstein set

Jacek Cichoń, Michał Morayne and Robert Rałowski
Wrocław University of Science and Technology

Winter School in Abstract Analysis
Section: Set Theory & Topology
Hejnice, 28-th January 2019

Question (Alexandr Osipov)

"It is true that for every Bernstein set in real line there are countably many continuous functions for which the union of images of Bernstein set by the family functions is whole real line? "

Very simple observation

Example

Let $[0,1] = A_0 \cup A_1$, A_0, A_1 - Bernstein sets in $[0,1]$,
Set $f : [0,2] \to [0,2]$ by $f(x) = 2x(2-x)$ and $A = A_0 \cup 2 - A_1$.
But $f(x) = f(2-x)$, $f[A] = [0,2]$

Very simple observation

Example

Let $[0,1] = A_0 \cup A_1$, A_0, A_1 - Bernstein sets in $[0,1]$,
Set $f : [0,2] \to [0,2]$ by $f(x) = 2x(2-x)$ and $A = A_0 \cup 2 - A_1$.
But $f(x) = f(2-x)$, $f[A] = [0,2]$

Bernstein vs. Vitali

Theorem

There is a Vitali set which is not Bernstein set.

Proof.

Let $Q = \{(x, y) \in \mathbb{R}^2 : \ x - y \in \mathbb{Q}\} \in \mathscr{M}(\mathbb{R}^2)$.
By Mycielski Theorem is $P \in Perf(\mathbb{R})$ such that $P \times P \subset Q^c \setminus \Delta$
Find $V \subseteq \mathbb{R}$ s.t. $P \subseteq V$ - Vitali set. □

Theorem (Beriashvili)

There exists set which is a Bernstein and Vitali set.

M. Beriashvili. *On some paradoxical subsets of the real line*, Georgian International Journal of Science and Technology 6 (4) (2014).

Bernstein vs. Vitali

Theorem
There is a Vitali set which is not Bernstein set.

Proof.
Let $Q = \{(x, y) \in \mathbb{R}^2 : \ x - y \in \mathbb{Q}\} \in \mathscr{M}(\mathbb{R}^2)$.
By Mycielski Theorem is $P \in Perf(\mathbb{R})$ such that $P \times P \subset Q^c \setminus \Delta$
Find $V \subseteq \mathbb{R}$ s.t. $P \subseteq V$ - Vitali set. $\square$

Theorem (Beriashvili)
There exists set which is a Bernstein and Vitali set.

M. Beriashvili. *On some paradoxical subsets of the real line*, Georgian International Journal of Science and Technology 6 (4) (2014).

Bernstein vs. Vitali

Theorem

There is a Vitali set which is not Bernstein set.

Proof.

Let $Q = \{(x, y) \in \mathbb{R}^2 : \ x - y \in \mathbb{Q}\} \in \mathscr{M}(\mathbb{R}^2)$.
By Mycielski Theorem is $P \in Perf(\mathbb{R})$ such that $P \times P \subset Q^c \setminus \Delta$
Find $V \subseteq \mathbb{R}$ s.t. $P \subseteq V$ - Vitali set. □

Theorem (Beriashvili)

There exists set which is a Bernstein and Vitali set.

M. Beriashvili. *On some paradoxical subsets of the real line*, Georgian International Journal of Science and Technology 6 (4) (2014).

Bernstein vs. Vitali

Theorem

There is a Vitali set which is not Bernstein set.

Proof.

Let $Q = \{(x, y) \in \mathbb{R}^2 : \ x - y \in \mathbb{Q}\} \in \mathscr{M}(\mathbb{R}^2)$.
By Mycielski Theorem is $P \in Perf(\mathbb{R})$ such that $P \times P \subset Q^c \setminus \Delta$
Find $V \subseteq \mathbb{R}$ s.t. $P \subseteq V$ - Vitali set. □

Theorem (Beriashvili)

There exists set which is a Bernstein and Vitali set.

M. Beriashvili. *On some paradoxical subsets of the real line*, Georgian International Journal of Science and Technology 6 (4) (2014).

Bernstein vs. Vitali

Theorem

There is a Vitali set which is not Bernstein set.

Proof.

Let $Q = \{(x, y) \in \mathbb{R}^2 : \ x - y \in \mathbb{Q}\} \in \mathscr{M}(\mathbb{R}^2)$.
By Mycielski Theorem is $P \in Perf(\mathbb{R})$ such that $P \times P \subset Q^c \setminus \Delta$
Find $V \subseteq \mathbb{R}$ s.t. $P \subseteq V$ - Vitali set. □

Theorem (Beriashvili)

There exists set which is a Bernstein and Vitali set.

M. Beriashvili. *On some paradoxical subsets of the real line*, Georgian International Journal of Science and Technology 6 (4) (2014).

Proposition

There exists a Bernstein set $B \subseteq \mathbb{R}$ and countable many continuous functions $f_n : B \to \mathbb{R}$ such that

$$\bigcup_{n\in\omega} f_n[B] = \mathbb{R}.$$

Proof.

Consider a Bernstein set which is Vitali set also and $f_n(x) = q_n + x$ for $\mathbb{Q} = \{q_n : \; n \in \omega\}$ $\square$

Proposition

There exists a Bernstein set $B \subseteq \mathbb{R}$ and countable many continuous functions $f_n : B \to \mathbb{R}$ such that

$$\bigcup_{n\in\omega} f_n[B] = \mathbb{R}.$$

Proof.

Consider a Bernstein set which is Vitali set also and $f_n(x) = q_n + x$ for $\mathbb{Q} = \{q_n : \ n \in \omega\}$ □

Proposition

There exists a Bernstein set $B \subseteq \mathbb{R}$ and two continuous functions $f_0, f_1 : B \to \mathbb{R}$ such that

$$f_0[B] \cup f_1[B] = \mathbb{R}.$$

Proof.

Let A Bernstein and Vitali set and $f_0(x) = x$, $f_1(x) = x + 1$.
$Q \cap \mathbb{Q} \cap [0, 1]$, $B_0 = A + Q + 2n\mathbb{Z}$, $B_1 = A + Q + (2n + 1)\mathbb{Z}$

- $A \subseteq B_0$, $A + 1 \subseteq B_1$,
- $B_0 \cap B_1 = \emptyset$, $B_0 \cup B_1 = \mathbb{R}$,
- $B_1 = f_1[B_0]$.

$\mathbb{R} = f_0[B] \cup f_1[B_0]$. □

Proposition

There exists a Bernstein set $B \subseteq \mathbb{R}$ and two continuous functions $f_0, f_1 : B \to \mathbb{R}$ such that

$$f_0[B] \cup f_1[B] = \mathbb{R}.$$

Proof.

Let A Bernstein and Vitali set and $f_0(x) = x$, $f_1(x) = x + 1$.
$Q \cap \mathbb{Q} \cap [0, 1]$, $B_0 = A + Q + 2n\mathbb{Z}$, $B_1 = A + Q + (2n + 1)\mathbb{Z}$

- $A \subseteq B_0$, $A + 1 \subseteq B_1$,
- $B_0 \cap B_1 = \emptyset$, $B_0 \cup B_1 = \mathbb{R}$,
- $B_1 = f_1[B_0]$.

$\mathbb{R} = f_0[B] \cup f_1[B_0]$. □

Proposition

There exists a Bernstein set $B \subseteq \mathbb{R}$ and two continuous functions $f_0, f_1 : B \to \mathbb{R}$ such that

$$f_0[B] \cup f_1[B] = \mathbb{R}.$$

Proof.

Let A Bernstein and Vitali set and $f_0(x) = x$, $f_1(x) = x + 1$.
$Q \cap \mathbb{Q} \cap [0,1]$, $B_0 = A + Q + 2n\mathbb{Z}$, $B_1 = A + Q + (2n+1)\mathbb{Z}$

- $A \subseteq B_0$, $A + 1 \subseteq B_1$,
- $B_0 \cap B_1 = \emptyset$, $B_0 \cup B_1 = \mathbb{R}$,
- $B_1 = f_1[B_0]$.

$\mathbb{R} = f_0[B] \cup f_1[B_0]$. □

Proposition

There exists a Bernstein set $B \subseteq \mathbb{R}$ and two continuous functions $f_0, f_1 : B \to \mathbb{R}$ such that

$$f_0[B] \cup f_1[B] = \mathbb{R}.$$

Proof.

Let A Bernstein and Vitali set and $f_0(x) = x$, $f_1(x) = x + 1$.
$Q \cap \mathbb{Q} \cap [0, 1]$, $B_0 = A + Q + 2n\mathbb{Z}$, $B_1 = A + Q + (2n + 1)\mathbb{Z}$

- $A \subseteq B_0$, $A + 1 \subseteq B_1$,
- $B_0 \cap B_1 = \emptyset$, $B_0 \cup B_1 = \mathbb{R}$,
- $B_1 = f_1[B_0]$.

$\mathbb{R} = f_0[B] \cup f_1[B_0]$. □

Productable spaces

Topological space X is productible if there are

- a topological space Y of size $\mathfrak{c}$,
- a continuous surjection $f : X \to X \times Y$.

$[0, 1]$, $\mathbb{R}$, 2^ω and ω^ω - productible spaces.

Productable spaces

Topological space X is productible if there are

- a topological space Y of size $\mathfrak{c}$,
- a continuous surjection $f : X \to X \times Y$.

$[0, 1]$, $\mathbb{R}$, 2^ω and ω^ω - productible spaces.

Peano Continuum

X is Peano continuum if is continuous image of $0, 1]$.

Theorem (Hahn-Mazurkiewicz)

Let X - topological Haussdorf space. The X is a Peano continuum if

- *X is second countable,*
- *X is compact,*
- *X is connected and locally connected.*

Fact

Every Peano continuum is productible.

Proof.

Let $f : [0,1] \to X$-continuous and onto X,
$g : X \to [0,\infty)$ as $g(x) = d(f(0),x)$.
Then $g[X] = [a,b]$ for some $a, b \in \mathbb{R}$.
Let

- $h : [a,b] \to [0,1]$ - continuous bijection and
- $T : [0,1] \to [0,1]^2$ - Peano map,
- $[0,1]^2 \ni (x,y) \mapsto \varphi(x,y) = (f(x), f(y)) \in X \times X$ continuous surjection.

Then $\varphi \circ T \circ h \circ g : X \to X \times X$ is required. □

Fact

Every Peano continuum is productible.

Proof.

Let $f : [0, 1] \to X$-continuous and onto X,
$g : X \to [0, \infty)$ as $g(x) = d(f(0), x)$.
Then $g[X] = [a, b]$ for some $a, b \in \mathbb{R}$.
Let

- $h : [a, b] \to [0, 1]$ - continuous bijection and
- $T : [0, 1] \to [0, 1]^2$ - Peano map,
- $[0, 1]^2 \ni (x, y) \mapsto \varphi(x, y) = (f(x), f(y)) \in X \times X$ continuous surjection.

Then $\varphi \circ T \circ h \circ g : X \to X \times X$ is required. □

Fact

Every Peano continuum is productible.

Proof.

Let $f : [0, 1] \to X$-continuous and onto X,
$g : X \to [0, \infty)$ as $g(x) = d(f(0), x)$.
Then $g[X] = [a, b]$ for some $a, b \in \mathbb{R}$.
Let

- $h : [a, b] \to [0, 1]$ - continuous bijection and
- $T : [0, 1] \to [0, 1]^2$ - Peano map,
- $[0, 1]^2 \ni (x, y) \mapsto \varphi(x, y) = (f(x), f(y)) \in X \times X$ continuous surjection.

Then $\varphi \circ T \circ h \circ g : X \to X \times X$ is required. $\square$

Fact

Every Peano continuum is productible.

Proof.

Let $f : [0,1] \to X$-continuous and onto X,
$g : X \to [0,\infty)$ as $g(x) = d(f(0), x)$.
Then $g[X] = [a,b]$ for some $a, b \in \mathbb{R}$.
Let

- $h : [a,b] \to [0,1]$ - continuous bijection and
- $T : [0,1] \to [0,1]^2$ - Peano map,
- $[0,1]^2 \ni (x,y) \mapsto \varphi(x,y) = (f(x), f(y)) \in X \times X$ continuous surjection.

Then $\varphi \circ T \circ h \circ g : X \to X \times X$ is required. □

Fact

Every Peano continuum is productible.

Proof.

Let $f : [0,1] \to X$-continuous and onto X,
$g : X \to [0,\infty)$ as $g(x) = d(f(0), x)$.
Then $g[X] = [a,b]$ for some $a, b \in \mathbb{R}$.
Let

- $h : [a,b] \to [0,1]$ - continuous bijection and
- $T : [0,1] \to [0,1]^2$ - Peano map,
- $[0,1]^2 \ni (x,y) \mapsto \varphi(x,y) = (f(x), f(y)) \in X \times X$ continuous surjection.

Then $\varphi \circ T \circ h \circ g : X \to X \times X$ is required. □

Fiberable space

Topological space X is fiberable if there is $f \in C(X)$ such that

$$(\forall y \in X)\ (|f^{-1}[\{y\}]| = \mathfrak{c}).$$

f - fiberable map.

Proposition

Every productible space is fiberable.

Fiberable space

Topological space X is fiberable if there is $f \in C(X)$ such that

$$(\forall y \in X)\ (|f^{-1}[\{y\}]| = \mathfrak{c}).$$

f - fiberable map.

Proposition

Every productible space is fiberable.

Main Theorem

Theorem

Let X - fiberable Polish space, Then there exists $f \in C(X)$ such that for every Bernstein set $B \subset X$, $f[B] = X$.

Proof.

Let B-Bernstein set, $f : X \to X$ - fiberable map and $y \in X$ - any point.
Then $f^{-1}[\{y\}]$ is a perfect subset of X.
Find $x \in B \cap f^{-1}[\{y\}]$ and then $f(x) = y$. Then $f[B] = X$. □

Main Theorem

Theorem

Let X - fiberable Polish space, Then there exists $f \in C(X)$ such that for every Bernstein set $B \subset X$, $f[B] = X$.

Proof.

Let B-Bernstein set, $f : X \to X$ - fiberable map and $y \in X$ - any point.

Then $f^{-1}[\{y\}]$ is a perfect subset of X.

Find $x \in B \cap f^{-1}[\{y\}]$ and then $f(x) = y$. Then $f[B] = X$. □

Main Theorem

Theorem

Let X - fiberable Polish space, Then there exists $f \in C(X)$ such that for every Bernstein set $B \subset X$, $f[B] = X$.

Proof.

Let B-Bernstein set, $f : X \to X$ - fiberable map and $y \in X$ - any point.
Then $f^{-1}[\{y\}]$ is a perfect subset of X.
Find $x \in B \cap f^{-1}[\{y\}]$ and then $f(x) = y$. Then $f[B] = X$. □

Main Theorem

Theorem

Let X - fiberable Polish space, Then there exists $f \in C(X)$ such that for every Bernstein set $B \subset X$, $f[B] = X$.

Proof.

Let B-Bernstein set, $f : X \to X$ - fiberable map and $y \in X$ - any point.
Then $f^{-1}[\{y\}]$ is a perfect subset of X.
Find $x \in B \cap f^{-1}[\{y\}]$ and then $f(x) = y$. Then $f[B] = X$. □

Answer for Osipov question is YES!

Question (Alexandr Osipov)

"It is true that for every Bernstein set in real line there are countably many continuous functions for which the union of images of Bernstein set by the family functions is whole real line? "

Theorem

There exists a continuous function $f : \mathbb{R} \to \mathbb{R}$ such that for every Bernstein set $B \subseteq \mathbb{R}$, $f[B] = \mathbb{R}$.

Moreover

Theorem

For every $A \in \Sigma^1_1(\mathbb{R})$ there exists a continuous function $f : \mathbb{R} \to \mathbb{R}$, such that for every Bernstein set $B \subseteq \mathbb{R}$, $f[B] = A$.

Answer for Osipov question is YES!

Question (Alexandr Osipov)

"It is true that for every Bernstein set in real line there are countably many continuous functions for which the union of images of Bernstein set by the family functions is whole real line? "

Theorem

There exists a continuous function $f : \mathbb{R} \to \mathbb{R}$ such that for every Bernstein set $B \subseteq \mathbb{R}$, $f[B] = \mathbb{R}$.

Moreover

Theorem

For every $A \in \Sigma^1_1(\mathbb{R})$ there exists a continuous function $f : \mathbb{R} \to \mathbb{R}$, such that for every Bernstein set $B \subseteq \mathbb{R}$, $f[B] = A$.

Answer for Osipov question is YES!

Question (Alexandr Osipov)

"It is true that for every Bernstein set in real line there are countably many continuous functions for which the union of images of Bernstein set by the family functions is whole real line? "

Theorem

There exists a continuous function $f : \mathbb{R} \to \mathbb{R}$ such that for every Bernstein set $B \subseteq \mathbb{R}$, $f[B] = \mathbb{R}$.

Moreover

Theorem

For every $A \in \Sigma^1_1(\mathbb{R})$ there exists a continuous function $f : \mathbb{R} \to \mathbb{R}$, such that for every Bernstein set $B \subseteq \mathbb{R}$, $f[B] = A$.

Theorem

Every proper coanalytic set (i.e. which is not analytic) cannot be continuous image of any Bernstein set.

Proof.

Assume that some $C \in \Pi^1_1 \setminus \Sigma^1_1$ and Bernstein set X, $f : X \to C$ is continuous and onto map.Then there exists a Borel function $g : \mathbb{R} \to \mathbb{R}$ and dense G_δ set $G \subseteq \mathbb{R}$ such that

- $X \subseteq G$,
- $f \subseteq g$,
- g is contiunuous on G.

Then $g[G] \in \Sigma^1_1$ and $C = f[X] = g[X] \subseteq g[G]$.
Then $g[G] \setminus C$ is countable analytic and then Borel set
if not then $g[G] \setminus C$ would be contain some perfect set P but $g^{-1}[P]$ contain some perfect set thus $X \cap g^{-1}[P] \neq \emptyset$ what is impossible.
Then $C = g[G] \setminus (g[G] \setminus C)$ is also analytic set, contradiction. □

Theorem

Every proper coanalytic set (i.e. which is not analytic) cannot be continuous image of any Bernstein set.

Proof.

Assume that some $C \in \Pi^1_1 \setminus \Sigma^1_1$ and Bernstein set X, $f : X \to C$ is continuous and onto map. Then there exists a Borel function $g : \mathbb{R} \to \mathbb{R}$ and dense G_δ set $G \subseteq \mathbb{R}$ such that

- $X \subseteq G$,
- $f \subseteq g$,
- g is contiunuous on G.

Then $g[G] \in \Sigma^1_1$ and $C = f[X] = g[X] \subseteq g[G]$.
Then $g[G] \setminus C$ is countable analytic and then Borel set
if not then $g[G] \setminus C$ would be contain some perfect set P but $g^{-1}[P]$ contain some perfect set thus $X \cap g^{-1}[P] \neq \emptyset$ what is impossible.
Then $C = g[G] \setminus (g[G] \setminus C)$ is also analytic set, contradiction. □

Theorem

Every proper coanalytic set (i.e. which is not analytic) cannot be continuous image of any Bernstein set.

Proof.

Assume that some $C \in \Pi^1_1 \setminus \Sigma^1_1$ and Bernstein set X, $f : X \to C$ is continuous and onto map. Then there exists a Borel function $g : \mathbb{R} \to \mathbb{R}$ and dense G_δ set $G \subseteq \mathbb{R}$ such that

- $X \subseteq G$,
- $f \subseteq g$,
- g is contiunuous on G.

Then $g[G] \in \Sigma^1_1$ and $C = f[X] = g[X] \subseteq g[G]$.
Then $g[G] \setminus C$ is countable analytic and then Borel set
if not then $g[G] \setminus C$ would be contain some perfect set P but $g^{-1}[P]$ contain some perfect set thus $X \cap g^{-1}[P] \neq \emptyset$ what is impossible.
Then $C = g[G] \setminus (g[G] \setminus C)$ is also analytic set, contradiction. □

Theorem

Every proper coanalytic set (i.e. which is not analytic) cannot be continuous image of any Bernstein set.

Proof.

Assume that some $C \in \Pi^1_1 \setminus \Sigma^1_1$ and Bernstein set X, $f : X \to C$ is continuous and onto map. Then there exists a Borel function $g : \mathbb{R} \to \mathbb{R}$ and dense G_δ set $G \subseteq \mathbb{R}$ such that

- $X \subseteq G$,
- $f \subseteq g$,
- g is contiunuous on G.

Then $g[G] \in \Sigma^1_1$ and $C = f[X] = g[X] \subseteq g[G]$.
Then $g[G] \setminus C$ is countable analytic and then Borel set
if not then $g[G] \setminus C$ would be contain some perfect set P but $g^{-1}[P]$ contain some perfect set thus $X \cap g^{-1}[P] \neq \emptyset$ what is impossible.
Then $C = g[G] \setminus (g[G] \setminus C)$ is also analytic set, contradiction. □

Theorem

Every proper coanalytic set (i.e. which is not analytic) cannot be continuous image of any Bernstein set.

Proof.

Assume that some $C \in \Pi^1_1 \setminus \Sigma^1_1$ and Bernstein set X, $f : X \to C$ is continuous and onto map. Then there exists a Borel function $g : \mathbb{R} \to \mathbb{R}$ and dense G_δ set $G \subseteq \mathbb{R}$ such that

- $X \subseteq G$,
- $f \subseteq g$,
- g is contiunuous on G.

Then $g[G] \in \Sigma^1_1$ and $C = f[X] = g[X] \subseteq g[G]$.
Then $g[G] \setminus C$ is countable analytic and then Borel set
if not then $g[G] \setminus C$ would be contain some perfect set P but $g^{-1}[P]$ contain some perfect set thus $X \cap g^{-1}[P] \neq \emptyset$ what is impossible.
Then $C = g[G] \setminus (g[G] \setminus C)$ is also analytic set, contradiction. □

Theorem

Every proper coanalytic set (i.e. which is not analytic) cannot be continuous image of any Bernstein set.

Proof.

Assume that some $C \in \Pi^1_1 \setminus \Sigma^1_1$ and Bernstein set X, $f : X \to C$ is continuous and onto map.Then there exists a Borel function $g : \mathbb{R} \to \mathbb{R}$ and dense G_δ set $G \subseteq \mathbb{R}$ such that

- $X \subseteq G$,
- $f \subseteq g$,
- g is contiunuous on G.

Then $g[G] \in \Sigma^1_1$ and $C = f[X] = g[X] \subseteq g[G]$.
Then $g[G] \setminus C$ is countable analytic and then Borel set
if not then $g[G] \setminus C$ would be contain some perfect set P but $g^{-1}[P]$ contain some perfect set thus $X \cap g^{-1}[P] \neq \emptyset$ what is impossible.
Then $C = g[G] \setminus (g[G] \setminus C)$ is also analytic set, contradiction. □

Theorem

Every proper coanalytic set (i.e. which is not analytic) cannot be continuous image of any Bernstein set.

Proof.

Assume that some $C \in \Pi^1_1 \setminus \Sigma^1_1$ and Bernstein set X, $f : X \to C$ is continuous and onto map.Then there exists a Borel function $g : \mathbb{R} \to \mathbb{R}$ and dense G_δ set $G \subseteq \mathbb{R}$ such that

- $X \subseteq G$,
- $f \subseteq g$,
- g is contiunuous on G.

Then $g[G] \in \Sigma^1_1$ and $C = f[X] = g[X] \subseteq g[G]$.
Then $g[G] \setminus C$ is countable analytic and then Borel set
if not then $g[G] \setminus C$ would be contain some perfect set P but $g^{-1}[P]$ contain some perfect set thus $X \cap g^{-1}[P] \neq \emptyset$ what is impossible.
Then $C = g[G] \setminus (g[G] \setminus C)$ is also analytic set, contradiction. □

Theorem

Every proper coanalytic set (i.e. which is not analytic) cannot be continuous image of any Bernstein set.

Proof.

Assume that some $C \in \Pi^1_1 \setminus \Sigma^1_1$ and Bernstein set X, $f : X \to C$ is continuous and onto map. Then there exists a Borel function $g : \mathbb{R} \to \mathbb{R}$ and dense G_δ set $G \subseteq \mathbb{R}$ such that

- $X \subseteq G$,
- $f \subseteq g$,
- g is contiunuous on G.

Then $g[G] \in \Sigma^1_1$ and $C = f[X] = g[X] \subseteq g[G]$.
Then $g[G] \setminus C$ is countable analytic and then Borel set
if not then $g[G] \setminus C$ would be contain some perfect set P but $g^{-1}[P]$ contain some perfect set thus $X \cap g^{-1}[P] \neq \emptyset$ what is impossible.
Then $C = g[G] \setminus (g[G] \setminus C)$ is also analytic set, contradiction. □

Banach condition T_2

A real function f is T_2 if

$$\{y \in \mathbb{R} : \ |f^{-1}[\{y\}]| = \mathfrak{c}\} \in \mathcal{N}.$$

Every Lipschitz or defferentiable function fulfill T_2 condition.

Theorem

There is a Bernstein set $B \subseteq \mathbb{R}$ such that for every ctbl family $\mathcal{F} \subseteq T_c \cap C(\mathbb{R})$ we have

$$\bigcup \{f[B] : \ f \in \mathcal{F}\} \neq \mathbb{R}.$$

Banach condition T_2

A real function f is T_2 if

$$\{y \in \mathbb{R} : \ |f^{-1}[\{y\}]| = \mathfrak{c}\} \in \mathscr{N}.$$

Every Lipschitz or defferentiable function fulfill T_2 condition.

Theorem

There is a Bernstein set $B \subseteq \mathbb{R}$ such that for every ctbl family $\mathcal{F} \subseteq T_c \cap C(\mathbb{R})$ we have

$$\bigcup \{f[B] : \ f \in \mathcal{F}\} \neq \mathbb{R}.$$

Banach condition T_2

A real function f is T_2 if

$$\{y \in \mathbb{R} : \ |f^{-1}[\{y\}]| = \mathfrak{c}\} \in \mathscr{N}.$$

Every Lipschitz or defferentiable function fulfill T_2 condition.

Theorem

There is a Bernstein set $B \subseteq \mathbb{R}$ such that for every ctbl family $\mathcal{F} \subseteq T_c \cap C(\mathbb{R})$ we have

$$\bigcup\{f[B] : \ f \in \mathcal{F}\} \neq \mathbb{R}.$$

Completely nonmeasurable set and its images

Definition

Assume X is Polish space $I \subseteq P(X)$ is σ-ideal with Borel base. A set $A \subseteq X$ is completely I-nonmeasurable if

$$(\forall B \in Bor \setminus I)\ (A \cap B \neq \emptyset \wedge A^c \cap B \neq \emptyset)$$

A is completely $ctbl$-nonmeasurble iff A is Bernstein set.
A is completely $\mathscr{M}$-nonmeasurable then A has no Baire property in each nonempty open set.
A is completely $\mathscr{N}$-nonmeasurable then A in not measurable in every positive measure Borel set.

Completely nonmeasurable set and its images

Definition

Assume X is Polish space $I \subseteq P(X)$ is σ-ideal with Borel base. A set $A \subseteq X$ is completely I-nonmeasurable if

$$(\forall B \in Bor \setminus I)\ (A \cap B \neq \emptyset \wedge A^c \cap B \neq \emptyset)$$

A is completely *ctbl*-nonmeasurble iff A is Bernstein set.
A is completely $\mathscr{M}$-nonmeasurable then A has no Baire property in each nonempty open set.
A is completely $\mathscr{N}$-nonmeasurable then A in not measurable in every positive measure Borel set.

Relative negative answer for completely nonmeasurable sets

Theorem

It is a relatively consistent with ZFC theory that there are two subsets $A, B \subseteq \mathbb{R}$ of the cardinality equal to $\mathfrak{c}$ such that:

1. *B is completely $\mathscr{M}$-nonmeasurable in $\mathbb{R}$,*
2. *A is strongly null,*
3. *for every family $\mathcal{F} \subset \mathbb{R}^B$ of countinuous functions on B such that $|\mathcal{F}| < \mathfrak{c}$, we have $A \setminus \bigcup\{f[B] : f \in \mathcal{F}\} \neq \emptyset$.*

Relative negative answer for completely nonmeasurable sets

Theorem

It is a relatively consistent with ZFC theory that there are two subsets $A, B \subseteq \mathbb{R}$ of the cardinality equal to $\mathfrak{c}$ such that:

1. *B is completely $\mathscr{M}$-nonmeasurable in $\mathbb{R}$,*
2. *A is strongly null,*
3. *for every family $\mathcal{F} \subset \mathbb{R}^B$ of countinuous functions on B such that $|\mathcal{F}| < \mathfrak{c}$, we have $A \setminus \bigcup\{f[B] : f \in \mathcal{F}\} \neq \emptyset$.*

Relative negative answer for completely nonmeasurable sets

Theorem

It is a relatively consistent with ZFC theory that there are two subsets $A, B \subseteq \mathbb{R}$ of the cardinality equal to $\mathfrak{c}$ such that:

1. *B is completely $\mathscr{M}$-nonmeasurable in $\mathbb{R}$,*
2. *A is strongly null,*
3. *for every family $\mathcal{F} \subset \mathbb{R}^B$ of countinuous functions on B such that $|\mathcal{F}| < \mathfrak{c}$, we have $A \setminus \bigcup\{f[B] : f \in \mathcal{F}\} \neq \emptyset$.*

Relative negative answer for completely nonmeasurable sets

Theorem

It is a relatively consistent with ZFC theory that there are two subsets $A, B \subseteq \mathbb{R}$ of the cardinality equal to $\mathfrak{c}$ such that:

1. *B is completely $\mathscr{M}$-nonmeasurable in $\mathbb{R}$,*
2. *A is strongly null,*
3. *for every family $\mathcal{F} \subset \mathbb{R}^B$ of countinuous functions on B such that $|\mathcal{F}| < \mathfrak{c}$, we have $A \setminus \bigcup\{f[B] : f \in \mathcal{F}\} \neq \emptyset$.*

Thank You

Thank You