On sets which can be moved away from sets of a certain family

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General properties of * operation for abelian (topological) groups.

$(X, +)$ - an abelian group, $A, B \subset X$,
$A + B := \{a + b : a \in A, b \in B\}$, $-A := \{-a : a \in A\}$,
$A + x := A + \{x\}$ for $x \in X$.

Definition

For $\mathcal{F} \subset P(X)$ let

$$\mathcal{F}^* := \{A \subset X : \forall_{F \in \mathcal{F}}\ A + F \neq X\}$$

or equivalently $\mathcal{F}^* := \{A \subset X : \forall_{F \in \mathcal{F}}\ \exists_{x \in X}\ (x - A) \cap F = \emptyset\}$.

$$\mathcal{F}_* := \{A \subset X : \forall_{F \in \mathcal{F}}\ F - A \neq X\} = \{A \subset X : \forall_{F \in \mathcal{F}}\ \exists_{x \in X}\ (A + x) \cap F = \emptyset\}$$

If we consider a family $\mathcal{F}$ which is reflection invariant then $\mathcal{F}_* = \mathcal{F}^*$.

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We write $\mathcal{F}^{**} := (\mathcal{F}^*)^*$ and $\mathcal{F}^{*(n+1)} := (\mathcal{F}^{*(n)})^*$ for $n \geqslant 2$.

Proposition

Let $\mathcal{F}, \mathcal{G}$ be arbitrary nonempty families of subsets of X. Then

- $\mathcal{G} \subset \mathcal{F}^* \Leftrightarrow \mathcal{F} \subset \mathcal{G}^*$,
- $\mathcal{F} \subset \mathcal{F}^{**}$,
- $\mathcal{G} \subset \mathcal{F} \Rightarrow \mathcal{F}^* \subset \mathcal{G}^*$,
- $\mathcal{F}^*$ is closed under taking subsets and it is translation invariant,
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$$\mathcal{F}^* := \{A \subset X : \forall_{F \in \mathcal{F}}\ A + F \neq X\}$$

Theorem

For any $\mathcal{F} \subset P(X)$ the following conditions are equivalent:

- $\forall_{A \notin \mathcal{F}}\ (\mathcal{F} \cup \{A\})^* \neq \mathcal{F}^*$,
- $\mathcal{F}^{**} = \mathcal{F}$.

Corollary

For any $\mathcal{F} \subset P(X)$ the family $\mathcal{F}^{**}$ is a maximal element (with respect to inclusion) of the set $\{\mathcal{G} \subset P(X) : \mathcal{G}^* = \mathcal{F}^*\}$.

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$\mathcal{F}in^*$ is the union of all proper, translation invariant ideals of subsets of X.

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$Count^*$ is the union of all proper, translation invariant σ-ideals of subsets of X.

For $X = \mathbb{R}$ the unit interval $[0,1]$ is an example of a set belonging to $\mathcal{F}in^*$ but not to $Count^*$.

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For $X = \mathbb{R}$ the unit interval $[0, 1]$ is an example of a set belonging to $\mathcal{F}in^*$ but not to $\mathcal{C}ount^*$.

On the real line

$\mathcal{K}$ - the σ-ideal of all meager sets,
$\mathcal{SMZ}$ - the σ-ideal of all strong measure zero sets.

A set $E \subset \mathbb{R}$ is of strong measure zero if for each sequence of positive reals $\{\epsilon_n\}_{n\in\mathbb{N}}$ there exists a sequence of intervals $\{I_n\}_{n\in\mathbb{N}}$ such that

$$E \subset \bigcup_{n\in\mathbb{N}} I_n \text{ and } m(I_n) \leqslant \epsilon_n \text{ for } n \in \mathbb{N}.$$

Galvin F., Mycielski J., Solovay R. M., *Strong measure zero sets*, Notices Am. Math. Soc. 26(3), 1979, Abstract A-280

$$\mathcal{K}^* = \mathcal{SMZ}.$$

It works also for a σ-compact metrizable group.

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The Borel Conjecture (BC): $\mathcal{SMZ} = \mathcal{C}ount.$

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Under CH: $\mathcal{SMZ} \setminus \mathcal{C}ount \neq \emptyset.$

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In Laver's model: $\mathcal{K}^* = \mathcal{C}ount.$

In ZFC: $\mathcal{K} \subset \mathcal{SMZ}^* \subset \mathcal{C}ount^*.$

$$\mathcal{K} \neq \mathcal{C}ount^*.$$

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We say that an ideal $\mathcal{J} \subset P(X)$ is continuum generated if there exists a family $\mathcal{F} \subset \mathcal{J}$ of subets of X with $card\mathcal{F} = \mathfrak{c}$ such that for every set $I \in \mathcal{J}$ there exists a set $F \in \mathcal{F}$ covering I ($I \subset F$).

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Assume CH. If $\mathcal{J} \subset P(\mathbb{R})$ is a continuum generated, translation and reflection invariant proper σ-ideal, and $A \notin \mathcal{J}$, then

$$(\mathcal{J} \cup \{A\})^* \subsetneq \mathcal{J}^*.$$

The proof here repeats reasoning from

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$\mathcal{A} \downarrow := \{B \subset \mathbb{R} : \exists_{A \in \mathcal{A}}\ A \supset B\}$.
It is easy to observe that if $\mathcal{A}, \mathcal{B}$ are σ-ideals, then $(\mathcal{A} + \mathcal{B}) \downarrow$ is also a σ-ideal.

Define

$$\mathcal{O} := (\mathcal{K} + \mathcal{SMZ}) \downarrow .$$

$\mathcal{O}$ is a proper σ-ideal invariant under translations.

$$\mathcal{C}ount \subset \mathcal{O}^*.$$

$$\mathcal{K} \subset \mathcal{O} \quad \Rightarrow \quad \mathcal{O}^* \subset \mathcal{K}^* = \mathcal{SMZ}.$$

$$\mathcal{SMZ} \subset \mathcal{O} \quad \Rightarrow \quad \mathcal{O}^* \subset \mathcal{SMZ}^* = \mathcal{K} \quad \text{(under CH)}.$$

Therefore, assuming CH,

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It is easy to observe that if $\mathcal{A}, \mathcal{B}$ are σ-ideals, then $(\mathcal{A} + \mathcal{B}) \downarrow$ is also a σ-ideal.

Define

$$\mathcal{O} := (\mathcal{K} + \mathcal{SMZ}) \downarrow .$$

$\mathcal{O}$ is a proper σ-ideal invariant under translations.

$$\mathcal{C}ount \subset \mathcal{O}^*.$$

$$\mathcal{K} \subset \mathcal{O} \quad \Rightarrow \quad \mathcal{O}^* \subset \mathcal{K}^* = \mathcal{SMZ}.$$

$$\mathcal{SMZ} \subset \mathcal{O} \quad \Rightarrow \quad \mathcal{O}^* \subset \mathcal{SMZ}^* = \mathcal{K} \quad \text{(under CH)}.$$

Therefore, assuming CH,

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Let $\mathcal{F}$ be a family of subsets of $\mathbb{R}$. We say that a set X, $X \subset \mathbb{R}$ is $\mathcal{F}$- additive if for every set $F \in \mathcal{F}$ the set $X + F$ belongs to $\mathcal{F}$.

Theorem

The following conditions are equivalent:

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- P is $\mathcal{SMZ}$-additive

and the condition

- P is $\mathcal{K}$-additive (meager-additive)

implies the previous two.
Under CH all three conditions are equivalent.

O. Zindulka, *Strong measure zero and meager-additive sets through the prism of fractal measures* - submitted

Corollary

$\mathcal{O}^*$ is a σ-ideal.

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T. J. Carlson, *Strong measure zero and strongly meager sets*, Proc. Amer. Math. Soc., 118(2), 1993

$$\mathcal{SM} := \mathcal{N}^*,$$

where $\mathcal{N}$ is the σ-ideal of Lebesgue null sets.

Corollary

Assume CH. If $\mathcal{J} \subset P(\mathbb{R})$ is a continuum generated, translation and reflection invariant proper σ-ideal, then

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Assume CH.

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The last corollary gives the positive answer to the question 1369 from W. Seredyński, *Some operations related to translations*, Colloq. Math. 57, 1989.

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Thank You for Your kind attention.