Weak versions of compactness and their productivity

Joint meeting ....

Mirna Džamonja, in joint work with David Buhagiar of the University of Malta)

University of East Anglia, UK,
Associated member, IHPST, Université Panthéon- Sorbonne, Paris 1

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It turns out that this property is connected with large cardinals and that this is the case exactly when κ is a strongly compact cardinal, as we shall review.

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The simply stated question if a weakly compact cardinal is sufficient, has been **open** since that time. In results subject to verification we believe that the notion of a square compact cardinal is strictly stronger than that of the weak compactness.

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[Why? For each $f\in 2^\lambda$ define $U_f=\{g\in 2^\kappa : f\subseteq g\}$, so an open set in the $(<\kappa)$-box topology of 2^κ.

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Let $\Sigma = \mathcal{P}(X)$ and consider $\mathcal{P}(\Sigma)$, identified with 2^Σ, in the $(< \kappa)$-box product topology. Define the closed sets:

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$\{\mathcal{F}_0\cup\{S\}$ generates a $(<\kappa)$-complete filter$\}_S$. This family has the $(<\kappa)$-intersection property.

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We shall use the last statement to extract a relevant property. But let us first go through the counter-examples to the productivity, as that property is indicated there too.

Counterexample, to get "square compact $\implies$ weak compact"

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An example of a $(< \kappa)$-Sikorski space is 2^θ in the $(< \kappa)$-box topology, for $\theta \leq \kappa$, a space that is relevant to the strong compactness result above.

The Tube Lemma

A proof of Tychonoff theorem that does not use extension of filters to ultrafilters but just transfinite induction on i where we consider $\Pi_{i<i^*} X_i$, goes through the so called Tube Lemma, which lets us handle the successor stages.

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Tube Lemma Suppose that X is a compact space and Y is any space. If $\mathcal{O}$ is an open cover of Y with no finite subcover, then there is $x \in X$ such that $\{x\} \times Y$ is not covered by any family of finitely many sets of $\mathcal{O}$.

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Theorem 1 If κ is square compact then the class of $(<\kappa)$-compact $(<\kappa)$ Sikorski spaces is closed under Tychonoff products.

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Theorem 2 If κ is square compact and the class of $(<\kappa)$-compact $(<\kappa)$ Sikorski spaces is closed under $(<\lambda)$-box products for all $\lambda<\kappa$, then it is closed under $(<\kappa)$-box products.

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Through work with elementary embeddings and forcing it is known that these notions are strictly increasing in strength, with θ.

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The same proof gives a connection with weakly compact cardinals, through their filter characterisation (due to Kunen): κ is a weakly compact cardinal if for any κ many subsets of κ, there is a $(<\kappa)$-closed filter on κ measuring them all. But somewhat surprisingly, we also get an equivalence of weak compactness to a seemingly weaker version of square compactness.

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We believe that this is not the case, as we shall now proceed to indicate.

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The conclusion would be that square compactness is strictly stronger than weak compactness.