On a Cardinal Invariant Related to the Haar Measure Problem

Gianluca Paolini
(joint work with Saharon Shelah)

Einstein Institute of Mathematics
Hebrew University of Jerusalem

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The Haar Measure

Our (locally) compact topological groups are Hausdorff.

Fact

Every locally compact topological group admits a unique left-invariant probability measure. This measure is called the Haar measure. If the group is compact, then this measure is both left-invariant and right-invariant (a.k.a. translation-invariant).

This measure was introduced by Haar in 1933, though its special case for Lie groups had been introduced by Hurwitz in 1897 under the name "invariant integral". Haar measures are used in many parts of analysis, number theory, group theory, representation theory, statistics, probability theory, and ergodic theory.

Compact Groups and non-Haar-Measurable Subsets

In this talk we will focus on compact groups.

Fact

A compact group G is either finite or uncountable. Furthermore, if G is metrizable, then it is either finite or continuum-sized.

Fact

Every infinite compact group has a non-Haar-measurable subset.

There is a wide literature on non-Haar-measurable subsets of infinite compact groups, we focus here on another theme.

The Haar Measure Problem

Problem (Haar Measure Problem)

Does every infinite compact group have a non-Haar-measurable subgroup?

The Haar Measure Problem dates back at least to 1963, when Hewitt and Ross gave a positive answer in the abelian case[1]. It was formulated explicitly a paper of Saeki and Stromberg (1985[2]). The problem remains open despite substantial efforts.

[1] E. Hewitt and K. Ross. *Abstract Harmonic Analysis*. Vol. I. Springer-Verlag, Berlin-Gttingen-Heidelberg, 1963.

[2] S. Saeki and K. Stromberg. *Measurable Subgroups and Nonmeasurable Characters*. Math. Scand. **57** (1985), 359-374.

State of the Art I

Hernández, Hofmann, and Morris (2016[3]) proved that if all subgroups of an infinite compact group are measurable, then the group must be profinite and metrizable (definition delayed).

[3] S. Hernáandez, K. Hofmann, and S. Morris. *Nonmeasurable Subgroups of Compact Groups*. J. Group Theory **19** (2016) 179-189.

State of the Art II

Building on that, Brian and Mislove (2016[4]) proved that a positive answer to the Haar Measure Problem is consistent with ZFC.

[4] W. Brian and M. Mislove. *Every Infinite Compact Group can have a non-Measurable Subgroup*. Topology Appl. **210** (2016), 144-146.

State of the Art III

Przeździecki, Szewczak, and Tsaban (2018[5]) introduced a certain cardinal invariant of the continuum $\mathfrak{fm}(G)$, depending on an infinite metrizable profinite group G, and proved that if $\mathrm{non}(\mathcal{N}) \leqslant \mathfrak{fm}(G)$ then G has a non-Haar measurable subgroup.

[5] A. J. Przeździecki, P. Szewczak, and B. Tsaban. *The Haar Measure Problem*. Proc. Amer. Math. Soc., to appear.

A Conjecture of Przeździecki, Szewczak, and Tsaban

Conjecture (Przeździecki, Szewczak, and Tsaban)

Let G be an infinite metrizable profinite group. Then:

$$\mathrm{non}(\mathcal{N}) \leqslant \mathfrak{fm}(G).$$

Clearly a positive answer to the conjecture of Przeździecki, Szewczak, and Tsaban would settle the Haar measure problem...

Our Theorem

Unfortunately, as Shelah says, mathematics is a harsh mistress:

Theorem (P. and Shelah)

It is consistent with ZFC that there exists an infinite metrizable profinite group G_* such that:

$$\mathrm{non}(\mathcal{N}) > \mathfrak{fm}(G_*).$$

Metrizable Profinite Grouos

Definition

A metrizable profinite group G is a profinite group of the form $\varprojlim^{\bar{\varphi}}_{i<\omega} G_i$, for $\bar{\varphi} = (\varphi_i : i < \omega)$ and $\varphi_i \in \mathrm{Hom}(G_{i+1}, G_i)$, i.e. G is an inverse $\bar{\varphi}$-limit of an $(\omega, <)$-inverse system of finite groups. When the homorphisms φ_i are clear from the context, we might forget to mention $\bar{\varphi}$ and simply write $\varprojlim_{i<\omega} G_i$.

Notation

Let $1 < n < \omega$, $A \subseteq G^n$ and $g \in G$. We let:

$$A_g = \{(h_1, ..., h_{n-1}) \in G^{n-1} : (h_1, ..., h_{n-1}, g) \in A\}.$$

Fubini-Markov sets

Definition

Let G be a metrizable profinite group.

(1) *We say that $X \subseteq G^n$ is an elementary algebraic set if there is a group word $w(\bar{x}, \bar{z})$ and a sequence of parameters $\bar{c} \in G^{|\bar{z}|}$ such that:*

$$X = \{\bar{a} \in G^{|\bar{x}|} : G \models w(\bar{a}, \bar{c}) = e\}.$$

(2) *We say that $X \subseteq G^n$ is an elementary algebraic null set if X is an elementary algebraic set which is null with respect to the Haar measure μ on G.*

(3) *We say that $X \subseteq G$ is Fubini-Markov if either of the following happens:*
   - (a) *X is an elementary algebraic null set;*
   - (b) *there is $1 < n < \omega$ and an elementary algebraic null set $A \subseteq G^n$ such that:*

$$X = \{g \in G : \mu(A_g) > 0\}.$$

The Cardinal Invariant $\mathfrak{fm}(G)$

Notice that by Fubini's Theorem every Fubini-Markov set is null.

Definition

Let G be an infinite metrizable profinite group. The cardinal invariant $\mathfrak{fm}(G)$ is the smallest size of a collection of Fubini-Markov sets whose union has (Haar) measure 1.

Definition

We denote by $\mathcal{N}$ the ideal of null sets in the Cantor space 2^ω, and by $\mathrm{non}(\mathcal{N})$ the minimal cardinality of a non-null subset of 2^ω.

The Construction of G_*

For every $i < \omega$, we construct a certain finite group G_i^* a then take as our metrizable profinite group G_* the group $\prod_{i<\omega} G_i^*$.

We denote by $[g, h] = g^{-1}h^{-1}gh$, i.e. the commutator of g and h.

Proposition

There are k_i and m_i (satisfying certain numerical constraints) such that: if $x_\ell \in G_i^*$, for $\ell < k_i$, then for some $c_1, c_2 \in G_i^*$ we have:

(a) $G_i^* \models [[x_\ell, c_1], c_2] = e$;

(b) $\{y \in G_i^* : G_i^* \models [[[x_\ell, c_1], c_2], y] = e\} = G_i^*$;

(c)

$$|\{(x, y) \in G_i^* \times G_i^* : G_i^* \models [[[x, c_1], c_2], y] = e\}| \leqslant |G_i^* \times G_i^*| / m_i.$$

The First Sufficient Condition

The sequences $(|G_i^*| : i < \omega)$, $(k_i : i < \omega)$, and $(m_i : i < \omega)$ are carefully choosen so that using the proposition above we can prove:

Lemma

A sufficient condition for $\mathfrak{fm}(G_*) \leqslant \lambda$ is:

$(\star)_1$ *there is $\mathcal{F} \subseteq \prod_{i<\omega} [G_i^*]^{k_i}$ of cardinality $\leqslant \lambda$ such that:*

$$(\forall \eta \in \prod_{i<\omega} G_i^*)(\exists \nu \in \mathcal{F})[\eta(i) \in \nu(i)].$$

The Second Sufficient Condition

Below $(m_i^* : i < \omega)$ and $(m_i^{**} : i < \omega)$ are certain sequences defined in function of the previously considered ones, i.e. $(|G_i^*| : i < \omega)$, $(k_i : i < \omega)$, and $(m_i : i < \omega)$.

Lemma

A sufficient condition for $\mathrm{non}(\mathcal{N}) > \lambda$ *is:*

$(\star)_2$ *for every* $Y \subseteq \prod_{i<\omega} m_i^{**}$ *of cardinality* $\leqslant \lambda$ *there is* ν *such that:*

(a) $\nu \in \prod_{i<\omega} [m_i^{**}]^{m_i^*}$;

(b) *if* $\eta \in Y$*, then, for infinitely many* $i < \omega$, $\eta(i) \in \nu(i)$.

Epilogue: The Forcing

The following forcing is taken from [KrSh:872][6].

Theorem

Assume that $\mathbf{V} \models CH$. *Then for some* $\aleph_2$*-c.c. proper (in fact even cardinal preserving) forcing* $\mathbb{P}$ *we have that in* $\mathbf{V}[\mathbb{P}]$ *both of the conditions below are satisfied:*

(a) *the statement* $(\star)_1$*, for* $\lambda = \aleph_1$*;*

(b) *the statement* $(\star)_2$*, for* $\lambda = \aleph_1$*.*

Hence, in the forcing extension $\mathbf{V}[\mathbb{P}]$ we have that:

$$\mathfrak{fm}(G_*) = \aleph_1 < \mathrm{non}(\mathcal{N}).$$

[6] J. Kellner and S. Shelah. *Decisive Creatures and Large Continuum*. J. Symbolic Logic **74** (2009), no. 1, 73-104.

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