Products of Luzin-type sets with combinatorial properties

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joint work with Grzegorz Wiśniewski

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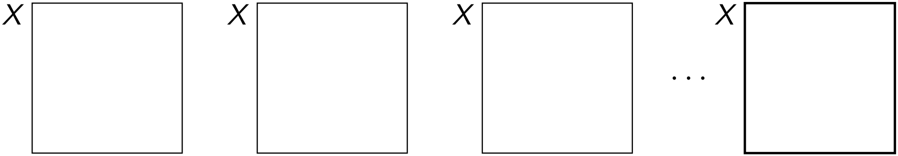

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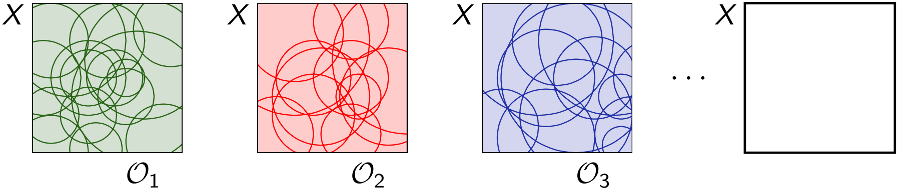

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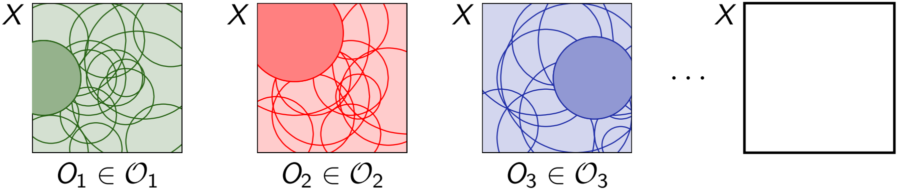

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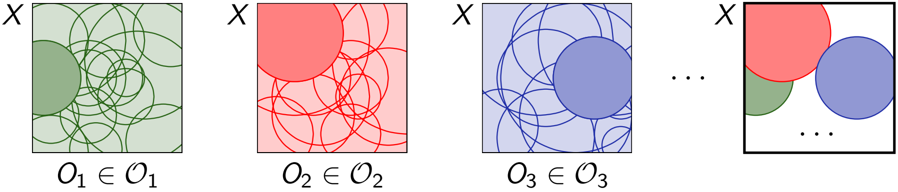

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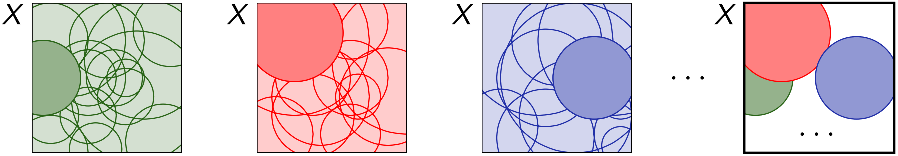

$O_1 \in \mathcal{O}_1$ $\quad$ $O_2 \in \mathcal{O}_2$ $\quad$ $O_3 \in \mathcal{O}_3$

countable $\longrightarrow$ $\mathsf{S}_1(\mathcal{O},\mathcal{O})$

Strong Measure Zero: for every sequence of positive numbers $\epsilon_1, \epsilon_2, \ldots$ there are intervals $I_1, I_2, \ldots$ such that $\mathrm{diam}(I_n) \leq \epsilon_n$ and $X \subseteq \bigcup_n I_n$

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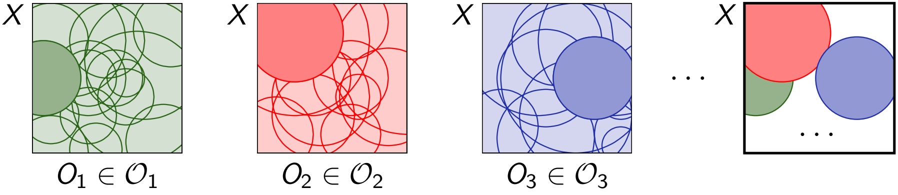

$$\text{countable} \longrightarrow \mathsf{S}_1(\mathcal{O},\mathcal{O}) \longrightarrow \mathsf{SMZ}$$

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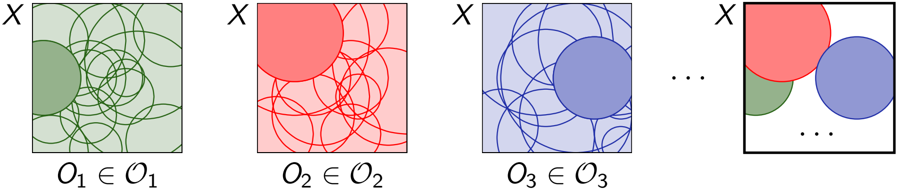

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Borel's conjecture: SMZ $=$ countable

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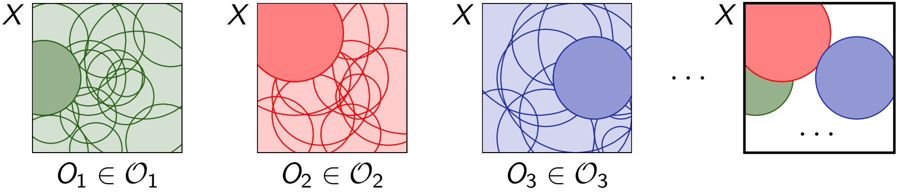

- $\mathbb{N}^{\mathbb{N}} \supseteq Y$ is guessable if $\exists_{g\in\mathbb{N}^{\mathbb{N}}} \forall_{y \in Y} \{\, n : g(n) = y(n) \,\}$ is infinite

Theorem (Recław)

X is $\mathsf{S}_1(\mathcal{O},\mathcal{O})$ $\Leftrightarrow$ every continuous image of X into $\mathbb{N}^{\mathbb{N}}$ is **guessable**

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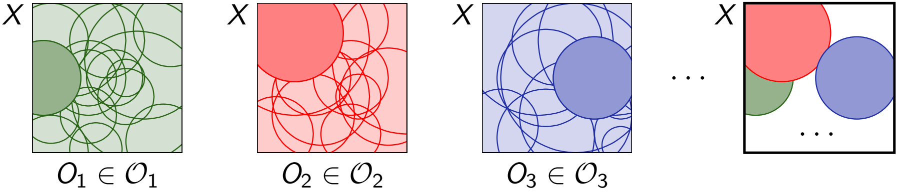

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Theorem (Recław)

X is $\mathsf{S}_1(\mathcal{O},\mathcal{O}) \Leftrightarrow$ every continuous image of X into $\mathbb{N}^{\mathbb{N}}$ is **guessable**

- $\mathrm{cov}(\mathcal{M})$: minimal cardinality of a nonguessable set
- $|X| < \mathrm{cov}(\mathcal{M}) \Rightarrow X$ is $\mathsf{S}_1(\mathcal{O},\mathcal{O})$

Luzin-type sets

Luzin set:

X is uncountable and $|X \cap M| = \aleph_0$ for every meager M

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- cov$(\mathcal{M})$: min cardinality of a family of meager sets that covers $\mathbb{R}$
- cof$(\mathcal{M})$: min cardinality of a cofinal family of meager sets in $\mathbb{R}$

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Theorem (Just, Miller, Scheepers, Szeptycki ($\mathrm{cov}(\mathcal{M}) = \mathfrak{c}$))

There is a $\mathrm{cov}(\mathcal{M})$-Luzin set L such that L^n is $\mathsf{S}_1(\mathcal{O}, \mathcal{O})$ for all n.

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Sakai: X^n is $\mathsf{S}_1(\mathcal{O}, \mathcal{O})$ for all $n \Leftrightarrow X$ is $\mathsf{S}_1(\Omega, \Omega)$

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Goal: find a new construction and improve the theorem

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- For every meager $M \subseteq [\mathbb{N}]^\infty$, there are a set $f \in [\mathbb{N}]^\infty$ and an interval partition $a \in [\mathbb{N}]^\infty$

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- In other words, pick x such that $d \leq_U x$ and for each $\beta < \alpha$

$$x \cap [a_\beta(n), a_\beta(n+1)) = f_\beta \cap [a_\beta(n), a_\beta(n+1)) \text{ for inf many } n$$

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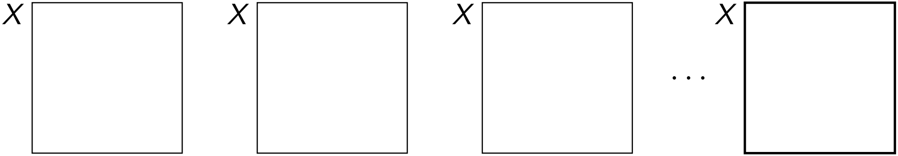

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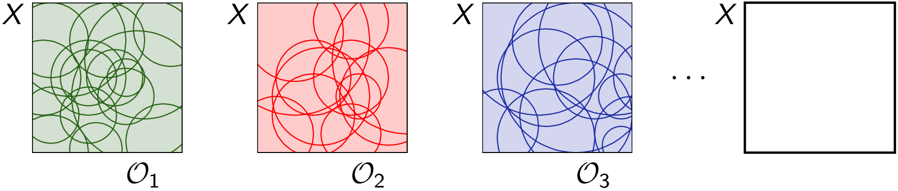

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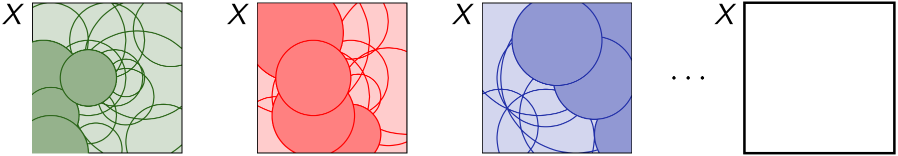

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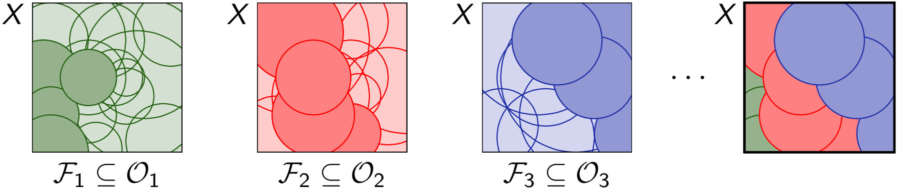

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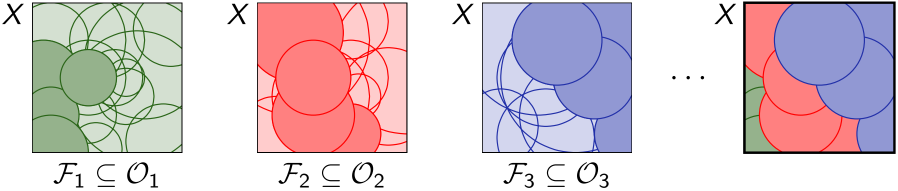

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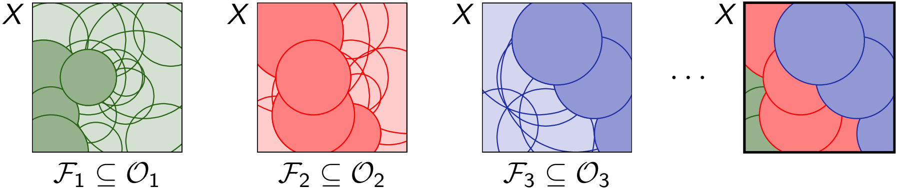

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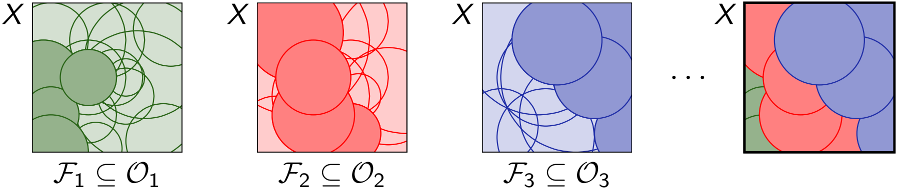

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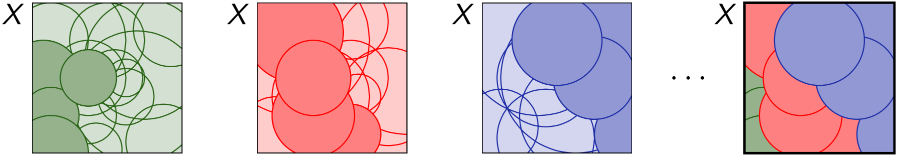

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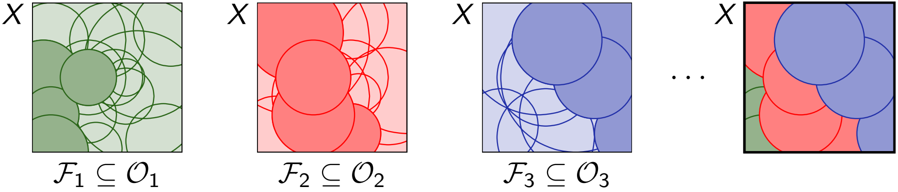

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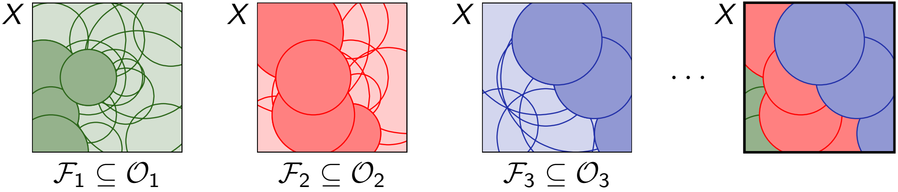

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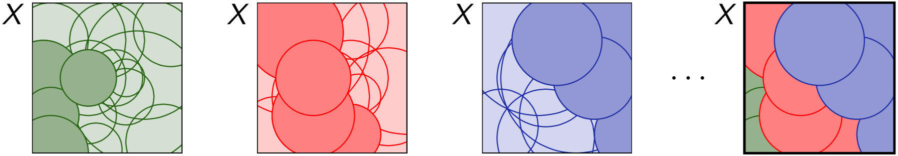

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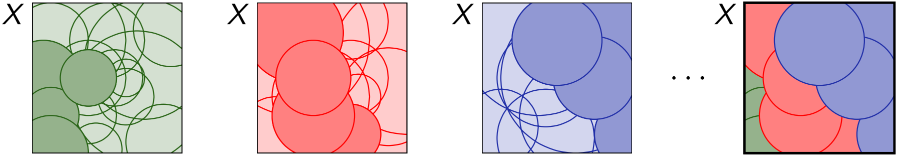
- $\mathfrak{d}$: minimal cardinality of a dominating set
- $|X| < \mathfrak{d} \Rightarrow X$ is $\mathsf{S}_{\mathrm{fin}}(\mathcal{O},\mathcal{O})$

Products of Luzin-type sets

$\mathsf{S}_{\mathrm{fin}}(\mathcal{O},\mathcal{O})$: for every sequence $\mathcal{O}_1, \mathcal{O}_2, \ldots$ of open covers of X there are finite $\mathcal{F}_1 \subseteq \mathcal{O}_1, \mathcal{F}_2 \subseteq \mathcal{O}_2, \ldots$ such that $\mathcal{F}_1 \cup \mathcal{F}_2 \cup \ldots$ covers X



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Theorem (Bartoszyński, Shelah, Tsaban ($\mathrm{cov}(\mathcal{M}) = \mathfrak{c}$))

There are $\mathrm{cov}(\mathcal{M})$*-Luzin sets* K, L *such that*

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Theorem (Sz, Wiśniewski ($\mathrm{cov}(\mathcal{M}) = \mathrm{cof}(\mathcal{M}) + \mathrm{cov}(\mathcal{M})$ is regular))

There are ultrafilters U, $\widetilde{U}$ and sets X, Y such that

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Assume that U is an ultrafilter and $\mathfrak{b}(U) = \mathrm{cov}(\mathcal{M})$.
There is a U-scale that is a $\mathrm{cov}(\mathcal{M})$-Luzin set.

Separation of properties

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$\mathsf{U}_{\mathrm{fin}}(\mathcal{O}, \Omega)$: for every sequence $\mathcal{O}_1, \mathcal{O}_2, \ldots$ of open covers of X there are finite $\mathcal{F}_1 \subseteq \mathcal{O}_1, \mathcal{F}_2 \subseteq \mathcal{O}_2, \ldots$ such that $\{\bigcup \mathcal{F}_n : n \in \mathbb{N}\}$ is an ω-cover

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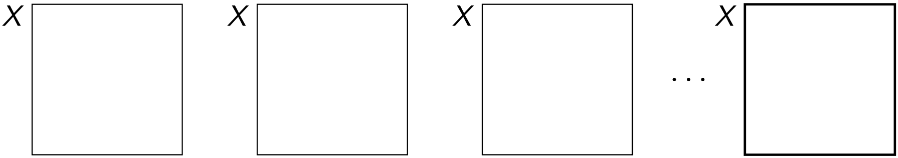

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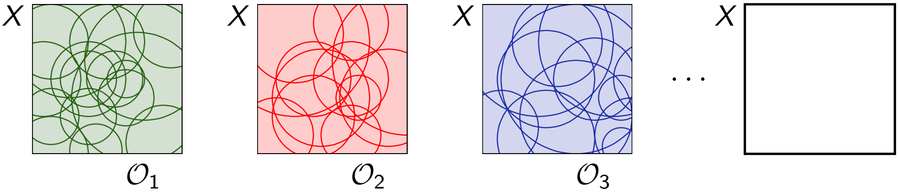

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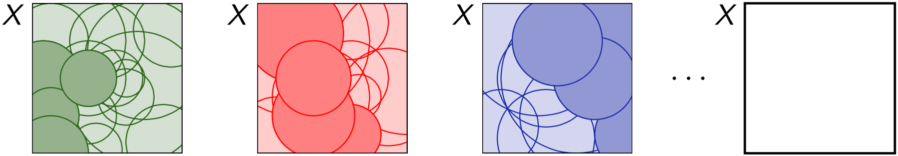

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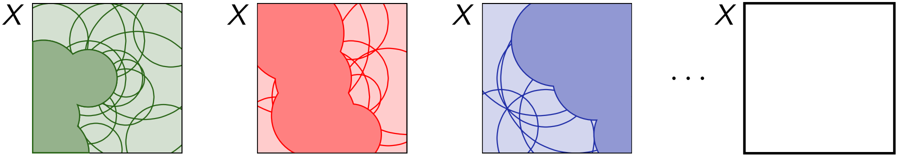

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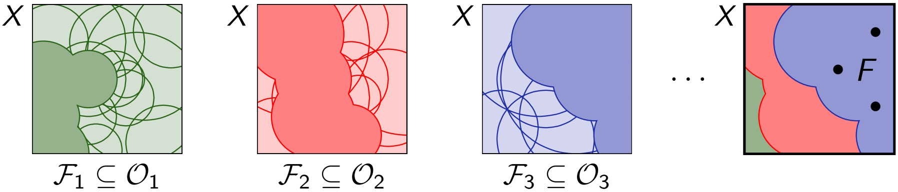

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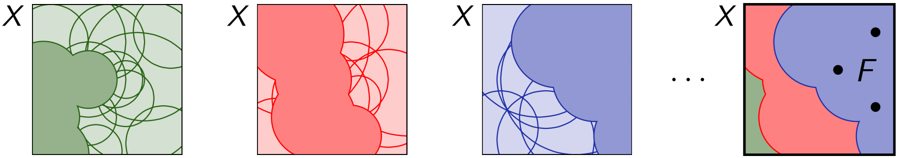

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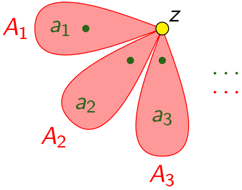

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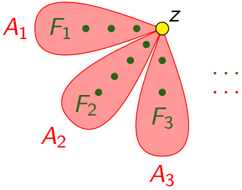

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Theorem (Sz, Tsaban, Zdomskyy ($\mathrm{cov}(\mathcal{M}) = \mathfrak{d} + \mathfrak{d}$ is regular))

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Theorem (Sz, Tsaban, Zdomskyy ($\mathrm{cov}(\mathcal{M}) = \mathfrak{d}$))

There is a set that is $\mathsf{S}_1(\mathcal{O}, \mathcal{O})$ *but no* $\mathsf{U}_{\mathrm{fin}}(\mathcal{O}, \Omega)$.