The continuity of Darboux functions between manifolds

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Definition

A function $f : X \to Y$ between topological spaces is called *Darboux* if for every connected set $C \subset X$ the image $f(C)$ is connected.

It is clear that each continuous function is Darboux.
The function

$$f(x) = \begin{cases} \sin(\frac{1}{x}) & \text{if } x \neq 0; \\ 0 & \text{if } x = 0. \end{cases}$$

is Darboux but not continuous.

Theorem (Darboux, 1875)

For any differentiable function $f : \mathbb{R} \to \mathbb{R}$ the derivative is Darboux (but not necessarily continuous).

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Tanaka-Pervin-Levine Theorem

A topological space X is caleld semilocally-connected if X has a base of the topology consisting of open sets $U \subset X$ whose complements have finitely many connected components.

The Euclidean spaces $\mathbb{R}^n$ are semilocally-connected.

Theorem (Tanaka, 1955; Pervin, Levine, 1958)

A bijective map $f : X \to Y$ between semilocally-connected T_1-spaces is a homeomorphism iff both f and f^{-1} are Darboux.

This theorem follows from a more general

Lemma

A map $f : X \to Y$ to a semilocally-connected T_1-space Y is continuous if f is Darboux and for any connected set $C \subset Y$ the preimage $f^{-1}(C)$ has finitely many connected components.

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$$x \in \overline{X \setminus f^{-1}(O_y)} = \overline{f^{-1}(Y \setminus O_y)} = \bigcup_{i=1}^{n} \bigcup_{j=1}^{n_i} \overline{C_{i,j}},$$

there exist i, j such that $x \in \overline{C_{i,j}}$. Then the subset $C := \{x\} \cup C_{i,j}$ is connected and so is its image $f(C) = \{f(x)\} \cup f(C_{i,j})$. On the other hand, the singleton $\{x\} = f(C) \cap O_y$ is clopen in $f(C)$, so $f(C)$ is not connected.

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Wong's Problem

Theorem (Tanaka, 1955; Pervin, Levine, 1958)

A bijective map $f : X \to Y$ between semilocally-connected T_1-spaces is a homeomorphism iff both f and f^{-1} are Darboux.

On the other hand, the Invariance of Domain Principle implies

Theorem

Each continuous bijection of $\mathbb{R}^n$ is a homeomorphism.

Problem (Wong, 2016)

Is each Darboux bijection of $\mathbb{R}^n$ a homeomorphism?

For $n = 1$ the answer is "yes"! For $n \geq 2$ the problem is still open!

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A Generalized Problem and a Partial Answer

To put Wong's Problem into a wider context,
let us ask a more

General Problem

Recognize pairs of topological spaces X, Y such that each Darboux injection (or bijection) $f : X \to Y$ is continuous (homeomorphism).

Partial Answer = Our Main Theorem

A Darboux injection $f : X \to Y$ between metrizable spaces is continuous if one of the following conditions is satisfied:

1. Y is a 1-manifold and X is compact and connected;
2. Y is a 2-manifold and X is a closed 2-manifold;
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A metrizable space X is called

- an *n-manifold* if each point $x \in X$ has a neighborhood, homeomorphic to an open subset of the closed half-space $\mathbb{R}^n_+ := \{(x_1, \dots, x_n) \in \mathbb{R}^n : x_1 \geq 0\}$;
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A topological space X is *simply-connected* if X is path-connected and has trivial fundamental group $\pi_1(X)$.

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A subset $S \subset X$ is called a *separator* of X if $X \setminus S$ is disconnected.

Lemma

Any separator S of a T_3-space X contains a closed separator of X.

Proof: If S has a non-empty interior, then we can choose a non-empty open set $U \subset X$ such that $\bar{U} \subset S$, and conclude that the boundary $\bar{U} \setminus U \subset S$ is a closed separator of X.
If S has empty interior, then $X \setminus S$ is dense in X. Since $X \setminus S$ is disjoint, there are open sets $U, V \subset X$ such that $X \setminus S \subset U \cup V$ and the $U \cap (X \setminus S)$ and $V \cap (X \setminus S)$ are disjoint and non-empty. Since $X \setminus S$ is dense, $U \cap (X \setminus S) \cap V = \emptyset$ implies $U \cap V = \emptyset$.
Then $L := X \setminus (U \cup V) \subset S$ is a required closed separator of X in S.

Key Lemma

If $f : X \to Y$ is a Darboux surjection, then for any separator $S \subset Y$ the preimage $f^{-1}(S)$ separates of X.
If X is regular, then $f^{-1}(S)$ contains a closed separator of X.

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Proof: If S has a non-empty interior, then we can choose a non-empty open set $U \subset X$ such that $\bar{U} \subset S$, and conclude that the boundary $\bar{U} \setminus U \subset S$ is a closed separator of X.
If S has empty interior, then $X \setminus S$ is dense in X. Since $X \setminus S$ is disjoint, there are open sets $U, V \subset X$ such that $X \setminus S \subset U \cup V$ and the $U \cap (X \setminus S)$ and $V \cap (X \setminus S)$ are disjoint and non-empty. Since $X \setminus S$ is dense, $U \cap (X \setminus S) \cap V = \emptyset$ implies $U \cap V = \emptyset$.
Then $L := X \setminus (U \cup V) \subset S$ is a required closed separator of X in S.

Key Lemma

If $f : X \to Y$ is a Darboux surjection, then for any separator $S \subset Y$ the preimage $f^{-1}(S)$ separates of X.
If X is regular, then $f^{-1}(S)$ contains a closed separator of X.

Separators and the Key Lemma

A subset $S \subset X$ is called a *separator* of X if $X \setminus S$ is disconnected.

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Varieties

In fact, Main Theorem holds not only for Darboux injections into n-manifolds but for generalizations of n-manifolds called n-varieties.

The definition of an n-variety is inductive.

Definition

A metrizable space Y is

(1) a *1-variety* if each point $y \in Y$ has a neighborhood homeomorphic to $\mathbb{R}$;

(n) an $(n+1)$-variety for some $n \in \mathbb{N}$ if

- each connected component of Y is semilocally-connected and
- for each convergent sequence $\{y_n\}_{n\in\omega} \subset Y$ and connected subset $C \subset Y$ containing more than one point, there exists a connected compact n-variety $S \subset Y$ such that $C \setminus S$ is disconnected and S contains infinitely many points y_n.

By induction we can show that each n-manifold of dimension $n \geq 2$ is an n-variety.

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General Main Theorem

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A Darboux injection $f : X \to Y$ between metrizable spaces is continuous if one of the following conditions is satisfied:

1. Y is a 1-variety and X is compact and connected;
2. Y is a 2-variety and X is a closed 2-manifold;
3. Y is a 3-variety and X is a simply-connected closed 3-manifold.

The case of dimension 1

Theorem 1

Any Darboux injection $f : X \to S^1$ from a compact connected space X to the circle S^1 is a topological embedding.

Proof.

We prove that:

1. Any Darboux injection $(0,1) \to \mathbb{R}$ is a topological embedding.
2. Any Darboux injection $[0,1] \to \mathbb{R}$ is a topological embedding.
3. Any Darboux injection $X \to \mathbb{R}$ from a path-connected space X is a topological embedding.
4. A connected compact metrizable space X admitting a Darboux injection $X \to S^1$ to a circle is locally connected.
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Multiacrs

A compact metrizable space X is called a *multiarc* if each connected component of X is homeomorphic to a segment $[a, b] \subset \mathbb{R}$ with $a \leq b$.

Using Poincaré Duality between Čech cohomologies of a closed subset A in a closed manifold and singular homologies of the complement $M \setminus A$ it is possible to prove two lemmas:

Lemma

For any multiarc A in a connected closed n-manifold of dimension $n \geq 2$ the complement $M \setminus A$ is connected.

Lemma

Let A be a multiarc in a closed n-manifold M of dimension $n \geq 3$. If for some coefficient group G and some positive $k < n$ the homology group $H_k(M; G)$ is trivial, then $H_k(M \setminus A; G) = 0$, too.

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Main Theorem in dimension 2

Theorem 2

Any Darboux injection $f : X \to Y$ of a connected closed 2-manifold X to a connected 2-variety is a homeomorphism.

Theorem 2g

Let X be a compact space that contains more than one point and cannot be separated by a multiarc. Any Darboux injection $f : X \to Y$ to a connected 2-variety Y is a homeomorphism.

Lemma 2

Let X, Y and f be from Theorem 2g. For any sequence $\{y_n\}_{n\in\omega} \subset Y$ that converges to a point $y \in Y$ there exists a topological circle $S \subset Y$ such that S contains infinitely many points y_n and $f{\restriction}f^{-1}(S) : f^{-1}(S) \to S$ is a homeomorphism.

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Proof of Lemma 2

Since Y is a 2-variety, there exists a topological circle $S \subset Y$ such that $f(X) \setminus S$ is disconnected and S contains infinitely many points y_n.

Since f is Darboux, $f^{-1}(S)$ is a separator of X and by Lemma, it contains a closed separator $L \subset X$.

By Theorem 1, for any connected complement C of L the restriction $f{\restriction}C : C \to S$ is a topological embedding. If $f(C) \neq S$, then $f(C)$ is either a singleton or an arc.

Since X cannot be separated by a multiarc, for some connected component C of L we have $f(C) = S$. Then $f^{-1}(S) = C$ and $f{\restriction}f^{-1}(S) : f^{-1}(S) \to S$ is a homeomorphism.

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Main theorem in dimension 3

Theorem 3

Any Darboux injection $f : X \to Y$ from a simply-connected closed 3-manifold X to a connected 3-variety is a homeomorphism.

Theorem 3g

Let X be a Peano continuum such that the complement $X \setminus A$ of any multiarc $A \subset X$ has trivial homology group $H_1(X \setminus A; G)$ for some coefficient group G. Then any Darboux injection $f : X \to Y$ to a connected 3-variety Y is a homeomorphism.

Lemma 3

Let X, Y f be as in Theorem 3g. For any sequence $\{y_n\}_{n\in\omega} \subset Y$ that converges to a point $y \in Y$ there exists a connected compact 2-variety $S \subset Y$ such that S contains infinitely many points y_n and $f{\restriction}f^{-1}(S) : f^{-1}(S) \to S$ is a homeomorphism.

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Using the Mayer-Vietoris exact sequence, we can prove that C is connected and moreovers, cannot be separated by a multiarc. So, we can apply Theorem 2 and conclude that the restriction $f{\restriction}C : C \to S$ is a homeomorphism. By injectivity of f, $C = f^{-1}(S)$ and hence $f{\restriction}f^{-1}(S) : f^{-1}(S) \to S$ is a homeomorphism.

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Open Problems

Problem 1 (Wong)

Is any Darboux bijection of $\mathbb{R}^n$ continuous?

Problem 2

Is any Darboux bijection of the plane continuous?

Problem 3

Is any Darboux bijection of the 3-dimensional torus continuous?

Problem 4

Is any Darboux bijection of the 4-dimensional sphere continuous?

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