The role of ideals in topological selection principles

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Basic terms

Definition

The family $\mathcal{I} \subseteq \mathcal{P}(\omega)$ is called **ideal**, if it has properties:

(I1) $A \in \mathcal{I},\ B \subseteq A \ \rightarrow\ B \in \mathcal{I}$,

(I2) $A, B \in \mathcal{I} \ \rightarrow\ A \cup B \in \mathcal{I}$,

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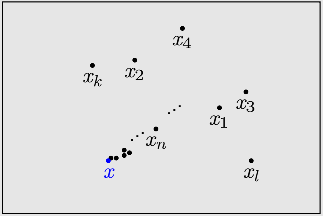
A sequence $\langle x_n :\ n \in \omega\rangle$ elements of a topological space X is $\mathcal{I}$-**convergent** to $x \in X$ if the set $\{n \in \omega :\ x_n \notin U\} \in \mathcal{I}$ for each neighborhood U of x, (written $x_n \xrightarrow{\mathcal{I}} x$).



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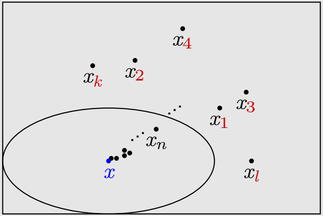

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- $\mathrm{C}_p(X)$ denotes the set of all continuous functions on X.
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- $f_n \xrightarrow{\mathcal{I}} f \ \Leftrightarrow \ \{n \in \omega : |f_n(x) - f(x)| \geq \varepsilon\} \in \mathcal{I}$ for each $x \in X$ and for each $\varepsilon > 0$.

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- Let $\mathbf{0}$ denote constant zero-value function on X.

$$\Omega_{\mathbf{0}} = \left\{ A \in {}^{\omega}(\mathrm{C}_p(X) \setminus \{\mathbf{0}\}) : \ \mathbf{0} \in \overline{\{y : \ (\exists n \in \omega) \ A(n) = y\}} \right\}.$$

$$\mathcal{I}\text{-}\Gamma_{\mathbf{0}} = \{A \in {}^{\omega}(\mathrm{C}_p(X) \setminus \{\mathbf{0}\}) : \ A \text{ is } \mathcal{I}\text{-convergent to } \mathbf{0}\}.$$

- We use $\Gamma_{\mathbf{0}}$ instead of $\mathrm{Fin}\text{-}\Gamma_{\mathbf{0}}$.

Selection principles

Let $\mathcal{P}$ and $\mathcal{R}$ be families of sets.

- X has $\begin{bmatrix}\mathcal{P}\\\mathcal{R}\end{bmatrix}$ or X is a $[\mathcal{P},\mathcal{R}]$-space if for every $\langle p_n : n \in \omega\rangle \in \mathcal{P}$ there is $\langle n_m : m \in \omega\rangle$ such that $\langle p_{n_m} : m \in \omega\rangle \in \mathcal{R}$.

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  - If $\mathcal{P}$ and $\mathcal{R}$ denote convergences then X is a $[\mathcal{P}_p, \mathcal{R}_p]$-space if for every $\langle p_n : n \in \omega \rangle$ such that $p_n \xrightarrow{\mathcal{P}} p$ there is $\langle n_m : m \in \omega \rangle$ such that $p_{n_m} \xrightarrow{\mathcal{R}} p$.

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- X is an $\mathrm{S}_1(\mathcal{P},\mathcal{R})$-space if for a sequence $\langle \mathcal{U}_n : n\in\omega\rangle$ of elements of $\mathcal{P}$ we can select a set $U_n\in\mathcal{U}_n$ for each $n\in\omega$ such that $\langle U_n : n\in\omega\rangle$ is a member of $\mathcal{R}$.

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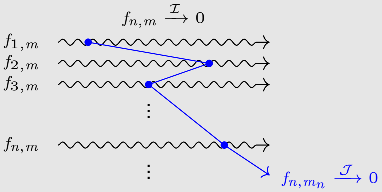
$\mathrm{S}_1(\mathcal{I}\text{-}\Gamma_{\mathbf{0}},\mathcal{J}\text{-}\Gamma_{\mathbf{0}})$ can be imagined in the following way



$\mathrm{S}_1(\mathcal{I}\text{-}\Gamma_{\mathbf{0}}, \mathcal{J}\text{-}\Gamma_{\mathbf{0}})$

classical convergence $\Rightarrow$ $\mathcal{I}$-convergence

$\mathrm{S}_1(\mathcal{I}\text{-}\Gamma_{\mathbf{o}}, \mathcal{J}\text{-}\Gamma_{\mathbf{o}})$

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Observation

(1) If $\mathrm{C}_p(X)$ is an $\mathrm{S}_1(\mathcal{I}\text{-}\Gamma_{\mathbf{o}}, \mathcal{J}\text{-}\Gamma_{\mathbf{o}})$-space then $\mathrm{C}_p(X)$ is an $\mathrm{S}_1(\Gamma_{\mathbf{o}}, \mathcal{J}\text{-}\Gamma_{\mathbf{o}})$-space.

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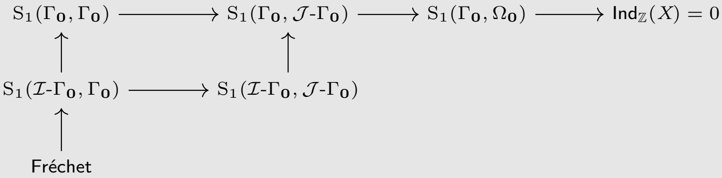

Diagram. Selection principles for functions.

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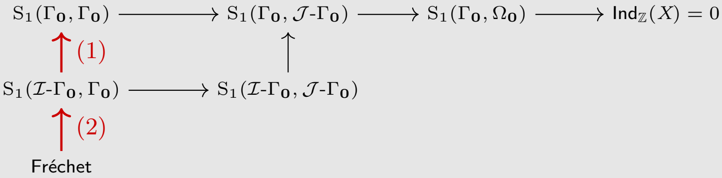

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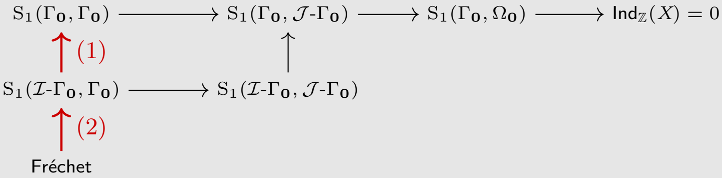

Diagram. Selection principles for functions.

- $\mathrm{S}_1(\Omega_{\mathbf{0}}, \Gamma_{\mathbf{0}})$-space $\Leftrightarrow [\Omega_{\mathbf{0}}, \Gamma_{\mathbf{0}}]$-space $\Leftrightarrow$ Fréchet space

(1) The relation between $\mathrm{S}_1(\mathcal{I}\text{-}\Gamma_{\mathbf{0}}, \Gamma_{\mathbf{0}})$-space and $\mathrm{S}_1(\Gamma_{\mathbf{0}}, \Gamma_{\mathbf{0}})$.

Proposition (V. Š., J. Šupina)

Let X be a topological space. Then $\mathrm{C}_p(X)$ is an $\mathrm{S}_1(\mathcal{I}\text{-}\Gamma_{\mathbf{0}}, \Gamma_{\mathbf{0}})$-space if and only if $\mathrm{C}_p(X)$ has $\left[\begin{smallmatrix}\mathcal{I}\text{-}\Gamma_{\mathbf{0}}\\ \Gamma_{\mathbf{0}}\end{smallmatrix}\right]$ and $\mathrm{S}_1(\Gamma_{\mathbf{0}}, \Gamma_{\mathbf{0}})$.

- In general, $\mathrm{S}_1(\mathcal{I}\text{-}\Gamma_{\mathbf{0}}, \mathcal{J}\text{-}\Gamma_{\mathbf{0}}) \Rightarrow \left[\begin{smallmatrix}\mathcal{I}\text{-}\Gamma_{\mathbf{0}}\\ \mathcal{J}\text{-}\Gamma_{\mathbf{0}}\end{smallmatrix}\right]$ for arbitrary ideals $\mathcal{I}$, $\mathcal{J}$.

(2) The relation between $\mathcal{I}$-$\Gamma_{\mathbf{0}}$ and $\Omega_{\mathbf{0}}$.

Lemma

For any countable family of functions $\mathcal{E}$ on X such that $\mathbf{0} \in \overline{\mathcal{E} \setminus \{\mathbf{0}\}}$ and its bijective enumeration $\langle f_n : n \in \omega \rangle$ there is an ideal $\mathcal{I}$ such that $f_n \xrightarrow{\mathcal{I}} \mathbf{0}$.

Theorem (V. Š., J. Šupina)

Let X be a Tychonoff topological space. The following statements are equivalent.

(a) *X is an $\mathrm{S}_1(\Omega_{\mathbf{0}}, \Gamma_{\mathbf{0}})$-space.*

(b) *$\mathrm{C}_p(X)$ is an $\mathrm{S}_1(\mathcal{I}\text{-}\Gamma_{\mathbf{0}}, \Gamma_{\mathbf{0}})$-space for every ideal $\mathcal{I}$.*

(c) *$\mathrm{C}_p(X)$ has $\begin{bmatrix} \mathcal{I}\text{-}\Gamma_{\mathbf{0}} \\ \Gamma_{\mathbf{0}} \end{bmatrix}$ for every ideal $\mathcal{I}$.*

$\begin{bmatrix}\mathcal{P}\\ \mathcal{R}\end{bmatrix}$ and $\mathrm{S}_1(\mathcal{P}, \mathcal{R})$

$\mathcal{P}$, $\mathcal{R}$ as covers.

- Ω denotes the family of all open ω-covers of X.
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- $\mathcal{I}$-Γ denotes the family of all open $\mathcal{I}$-γ-covers of X.
  - the set $\{n \in \omega : \ x \notin U_n\} \in \mathcal{I}$ for each $x \in X$,
  - $\mathrm{Fin}\text{-}\Gamma = \Gamma$.

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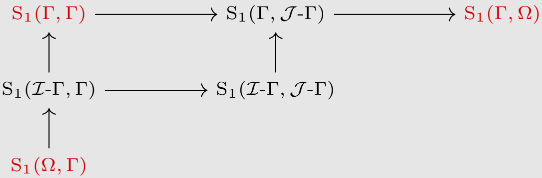

Diagram. Covering selection principles.

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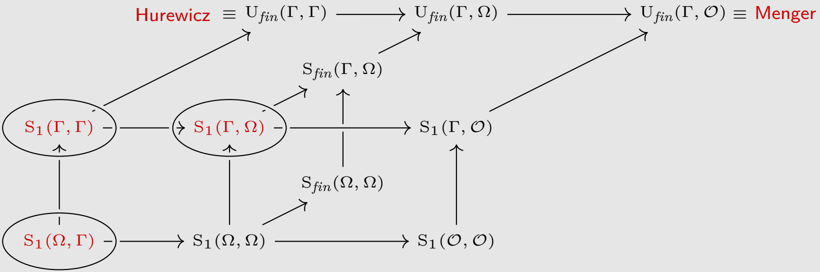

Diagram. Scheepers' diagram (1996).

$\begin{bmatrix}\mathcal{P}\\\mathcal{R}\end{bmatrix}$ and $S_1(\mathcal{P},\mathcal{R})$

- We say that a sequence $\langle f_n : \ n\in\omega\rangle$ is **monotone sequence** if for any $n\in\omega$ and $x\in X$ we have $f_n(x)\ge f_{n+1}(x)$.
- $\Gamma_{\mathbf{0}}^m = \{A\in {}^{\omega}(\mathrm{C}_p(X)\setminus\{\mathbf{0}\}) : \ A \text{ is monotone and convergent to } \mathbf{0}\}$.

$\left[{\mathcal{P} \atop \mathcal{R}}\right]$ and $\mathrm{S}_1(\mathcal{P}, \mathcal{R})$

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- We say that a sequence $\langle f_n : \ n \in \omega \rangle$ is $\mathcal{I}$-**monotone sequence** if $\{n : \ f_n \not\leq f_m\} \in \mathcal{I}$ for every $m \in \omega$.
- $\mathcal{I}\text{-}\Gamma_{\mathbf{0}}^m = \{A \in {}^\omega(\mathrm{C}_p(X) \setminus \{\mathbf{0}\}) : \ A \text{ is } \mathcal{I}\text{-monotone and } \mathcal{I}\text{-convergent to } \mathbf{0}\}$.

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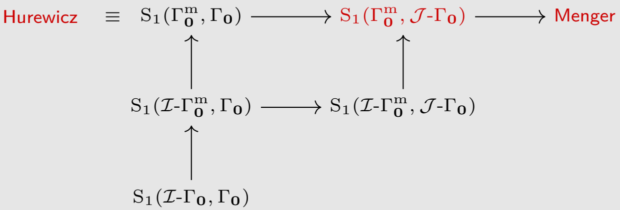

Diagram. Monotonic selection principles for functions.

$\begin{bmatrix}\mathcal{P}\\\mathcal{R}\end{bmatrix}$ and $\mathrm{S}_1(\mathcal{P},\mathcal{R})$

- M. Scheepers (1997):

$$\text{Hurewicz} \equiv \mathrm{S}_1(\Gamma_{\mathbf{0}}^{\mathrm{m}}, \Gamma_{\mathbf{0}})$$

- P. Szewczak, B. Tsaban (2017):

$$\text{Hurewicz} \Rightarrow \mathcal{J}\text{-Hurewicz} \Rightarrow \text{Menger.}$$

Proposition (V. Š, J. Šupina)

If X is a perfectly normal topological space then the following are equivalent. Moreover, if X is arbitrary topological space then $(\mathrm{a}) \equiv (\mathrm{b})$.

(a) $\mathrm{C}_p(X)$ *has* $\begin{bmatrix}\Gamma_{\mathbf{0}}^{\mathrm{m}}\\ s\mathcal{J}\mathrm{QN}_{\mathbf{0}}\end{bmatrix}$.

(b) $\mathrm{C}_p(X)$ *has the property* $\mathrm{S}_1(\Gamma_{\mathbf{0}}^{\mathrm{m}}, \mathcal{J}\text{-}\Gamma_{\mathbf{0}})$.

(c) X *possesses a* $\mathcal{J}$*-Hurewicz property.*

- L. Bukovský, P. Das and J. Šupina (2017): the ideal version of Scheepers' result.

$$\mathrm{S}_1(\mathcal{I}\text{-}\Gamma, \mathcal{J}\text{-}\Gamma) \to S_1(\mathcal{I}\text{-}\Gamma^{sh}, \mathcal{J}\text{-}\Gamma) \Leftrightarrow \mathrm{S}_1(\mathcal{I}\text{-}\Gamma_{\mathbf{0}}, \mathcal{J}\text{-}\Gamma_{\mathbf{0}}) \to \mathrm{S}_1(\mathcal{I}\text{-}\Gamma_{\mathbf{0}}^{\mathrm{m}}, \mathcal{J}\text{-}\Gamma_{\mathbf{0}}).$$

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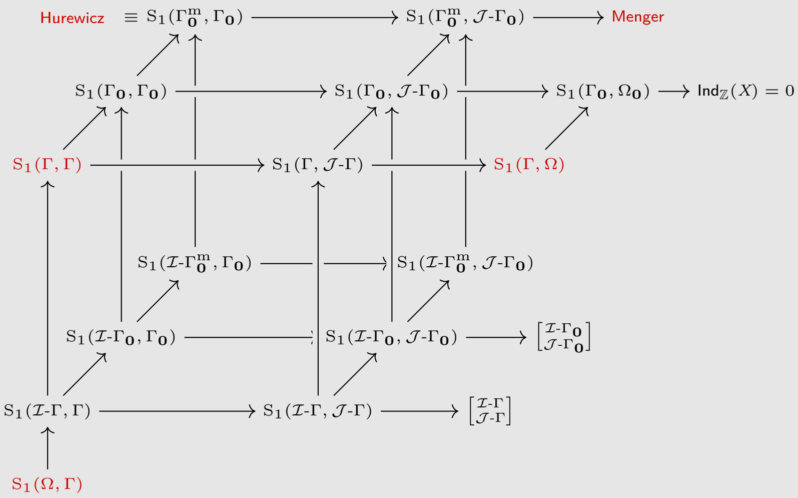

Diagram. The overall relations of investigated properties.

Cardinal invariants

- $\mathrm{non}(\mathrm{S}_1(\mathcal{I}\text{-}\Gamma, \mathcal{J}\text{-}\Gamma)\text{-space})$ denotes the minimal cardinality of a perfectly normal space which is not an $\mathrm{S}_1(\mathcal{I}\text{-}\Gamma, \mathcal{J}\text{-}\Gamma)$-space.

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- M. Scheepers (1996):
  - $\mathrm{non}(\mathrm{S}_1(\Gamma, \Gamma)) = \mathfrak{b}$,
  - $\mathrm{non}(\mathrm{S}_1(\Gamma, \Omega)) = \mathfrak{d}$,
  - $\mathrm{non}(\mathrm{S}_1(\Omega, \Gamma)) = \mathfrak{p}$.

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- M. Hrušák, F. Hernández

$$\mathsf{cov}^*(\mathcal{I}) = \min\{|\mathcal{A}| : \ \mathcal{A} \subseteq \mathcal{I} \wedge (\forall S \in [\omega]^\omega)(\exists A \in \mathcal{A})\ |S \cap A| = \omega\}.$$

  - $\mathfrak{p} = \min\{\kappa : \ (\exists \text{ an ideal } \mathcal{I})\ \mathsf{cov}^*(\mathcal{I}) = \kappa\}$.
- M. Repický (2018)

$$\mathfrak{k}_{\mathcal{I},\mathcal{J}} = \min\{|\mathcal{A}| : \ \mathcal{A} \subseteq \mathcal{I} \wedge \mathcal{A} \not\leq_K \mathcal{J}\}.$$

  - if $\mathcal{I}$ is tall then $\mathfrak{k}_{\mathcal{I},\mathrm{Fin}} = \mathsf{cov}^*(\mathcal{I})$.

Let $\mathcal{A} \subseteq \mathcal{P}(\omega)$.

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- J. Šupina's results (2016):
  - $\lambda(\mathrm{Fin}, \mathcal{J}) = \mathfrak{b}_{\mathcal{J}}$,
  - if $\mathcal{I}_1 \leq_K \mathcal{I}_2$ and $\mathcal{J}_1 \leq_{KB} \mathcal{J}_2$ then $\lambda(\mathcal{I}_2, \mathcal{J}_1) \leq \lambda(\mathcal{I}_1, \mathcal{J}_2)$,
  - $\mathrm{non}(\mathrm{S}_1(\mathcal{I}\text{-}\Gamma, \mathcal{J}\text{-}\Gamma) = \lambda(\mathcal{I}, \mathcal{J}))$.

Theorem (V. Š., J. Šupina)

(1) *If* $\mathcal{I} \not\leq_K \mathcal{J}$ *then* $\lambda(\mathcal{I}, \mathcal{J}) \leq \min\{\mathfrak{k}_{\mathcal{I},\mathcal{J}}, \mathfrak{b}_{\mathcal{J}}\}$.

(2) *If* $\mathcal{I} \not\leq_K \mathcal{J}$ *and* $\mathcal{J} \leq_K \mathcal{I}$ *then* $\lambda(\mathcal{I}, \mathcal{J}) = \min\{\mathfrak{k}_{\mathcal{I},\mathcal{J}}, \lambda(\mathcal{J}, \mathcal{J})\}$.

(3) *If* $\mathcal{I}$ *is tall then* $\lambda(\mathcal{I}, \mathrm{Fin}) = \min\{\mathtt{cov}^*(\mathcal{I}), \mathfrak{b}\}$.

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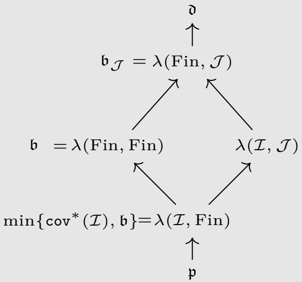

Diagram. Cardinal $\lambda(\mathcal{I},\mathcal{J})$.

Critical cardinality

- Let D being a discrete topological space.

$$|D| < \lambda(\mathcal{I},\mathcal{J}) \Leftrightarrow \mathrm{C}_p(D) \text{ has } \left[{\mathcal{I}\text{-}\Gamma_{\mathbf{0}} \atop \mathrm{s}\mathcal{J}\mathrm{QN}_{\mathbf{0}}}\right] \Leftrightarrow \mathrm{C}_p(D) \text{ has } \mathrm{S}_1(\mathcal{I}\text{-}\Gamma_{\mathbf{0}}, \mathcal{J}\text{-}\Gamma_{\mathbf{0}})$$
$$\Leftrightarrow \mathrm{C}_p(D) \text{ has } \mathrm{S}_1(\mathcal{I}\text{-}\Gamma^{\mathrm{m}}_{\mathbf{0}}, \mathcal{J}\text{-}\Gamma_{\mathbf{0}}).$$

- A. Kwela–M. Repický (2018)

$$|D| < \mathsf{cov}^*(\mathcal{I}) \Leftrightarrow \mathrm{C}_p(D) \text{ has } \left[{\mathcal{I}\mathrm{QN}_{\mathbf{0}} \atop \mathrm{QN}_{\mathbf{0}}}\right] \Leftrightarrow \mathrm{C}_p(D) \text{ has } \left[{\mathcal{I}\text{-}\Gamma_{\mathbf{0}} \atop \Gamma_{\mathbf{0}}}\right] \Leftrightarrow D \text{ has the property } \left[{\mathcal{I}\text{-}\Gamma \atop \Gamma}\right].$$

- Let D being a discrete topological space.

$$|D| < \lambda(\mathcal{I},\mathcal{J}) \Leftrightarrow \mathrm{C}_p(D) \text{ has } \left[\begin{smallmatrix}\mathcal{I}\text{-}\Gamma_{\mathbf{0}}\\ \mathrm{s}\mathcal{J}\mathrm{QN}_{\mathbf{0}}\end{smallmatrix}\right] \Leftrightarrow \mathrm{C}_p(D) \text{ has } \mathrm{S}_1(\mathcal{I}\text{-}\Gamma_{\mathbf{0}},\mathcal{J}\text{-}\Gamma_{\mathbf{0}})$$
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$$|D| < \mathsf{cov}^*(\mathcal{I}) \Leftrightarrow \mathrm{C}_p(D) \text{ has } \left[\begin{smallmatrix}\mathcal{I}\mathrm{QN}_{\mathbf{0}}\\ \mathrm{QN}_{\mathbf{0}}\end{smallmatrix}\right] \Leftrightarrow \mathrm{C}_p(D) \text{ has } \left[\begin{smallmatrix}\mathcal{I}\text{-}\Gamma_{\mathbf{0}}\\ \Gamma_{\mathbf{0}}\end{smallmatrix}\right] \Leftrightarrow D \text{ has the property } \left[\begin{smallmatrix}\mathcal{I}\text{-}\Gamma\\ \Gamma\end{smallmatrix}\right].$$

Corollary (V. Š., J. Šupina)

- *Let $\mathcal{I},\mathcal{J} \subseteq \mathcal{P}(\omega)$ be ideals.*

(1) $\mathrm{non}(\mathrm{S}_1(\mathcal{I}\text{-}\Gamma_{\mathbf{0}},\mathcal{J}\text{-}\Gamma_{\mathbf{0}})) = \mathrm{non}(\mathrm{S}_1(\mathcal{I}\text{-}\Gamma^{\mathrm{m}}_{\mathbf{0}},\mathcal{J}\text{-}\Gamma_{\mathbf{0}})) = \mathrm{non}(\left[\begin{smallmatrix}\mathcal{I}\text{-}\Gamma_{\mathbf{0}}\\ s\mathcal{J}\mathrm{QN}_{\mathbf{0}}\end{smallmatrix}\right]) = \lambda(\mathcal{I},\mathcal{J})$.

(2) $\mathrm{non}(\mathrm{S}_1(\Gamma_{\mathbf{0}},\mathcal{J}\text{-}\Gamma_{\mathbf{0}})) = \mathrm{non}(\mathrm{S}_1(\Gamma^{\mathrm{m}}_{\mathbf{0}},\mathcal{J}\text{-}\Gamma_{\mathbf{0}})) = \mathrm{non}(\left[\begin{smallmatrix}\Gamma_{\mathbf{0}}\\ s\mathcal{J}\mathrm{QN}_{\mathbf{0}}\end{smallmatrix}\right]) = \mathfrak{b}_{\mathcal{J}}$.

- *If $\mathcal{I}$ is tall then*

(3) $\mathrm{non}(\mathrm{S}_1(\mathcal{I}\text{-}\Gamma,\Gamma)) = \mathrm{non}(\mathrm{S}_1(\mathcal{I}\text{-}\Gamma_{\mathbf{0}},\Gamma_{\mathbf{0}})) = \mathrm{non}(\mathrm{S}_1(\mathcal{I}\text{-}\Gamma^{m}_{\mathbf{0}},\Gamma_{\mathbf{0}})) = \mathrm{non}(\left[\begin{smallmatrix}\mathcal{I}\text{-}\Gamma_{\mathbf{0}}\\ \mathrm{QN}_{\mathbf{0}}\end{smallmatrix}\right]) = \min\{\mathsf{cov}^*(\mathcal{I}),\mathfrak{b}\}$.

(4) **(A. Kwela–M. Repický)** $\mathrm{non}(\left[\begin{smallmatrix}\mathcal{I}\mathrm{QN}_{\mathbf{0}}\\ \mathrm{QN}_{\mathbf{0}}\end{smallmatrix}\right]) = \mathrm{non}(\left[\begin{smallmatrix}\mathcal{I}\text{-}\Gamma_{\mathbf{0}}\\ \Gamma_{\mathbf{0}}\end{smallmatrix}\right]) = \mathrm{non}(\left[\begin{smallmatrix}\mathcal{I}\text{-}\Gamma\\ \Gamma\end{smallmatrix}\right]) = \mathsf{cov}^*(\mathcal{I})$.

Critical cardinality

Proposition

(1) Let $X \subseteq {}^{\omega}\omega$. If $\mathrm{C}_p(X)$ has the property $\mathrm{S}_1(\mathcal{I}\text{-}\Gamma^{\mathrm{m}}_{\mathbf{0}}, \mathcal{J}\text{-}\Gamma_{\mathbf{0}})$, then X is bounded in $({}^{\omega}\omega, \leq^{\mathcal{J}})$.

(2) Let $\mathcal{I}$ be a tall ideal. If $\mathcal{A} \subseteq \mathcal{I}$ has $\left[\begin{smallmatrix}\mathcal{I}\text{-}\Gamma \\ \Gamma\end{smallmatrix}\right]$ or $\mathrm{C}_p(\mathcal{A})$ has $\left[\begin{smallmatrix}\mathcal{I}\text{-}\Gamma_{\mathbf{0}} \\ \Gamma_{\mathbf{0}}\end{smallmatrix}\right]$ or $\left[\begin{smallmatrix}\mathcal{I}\mathrm{QN}_{\mathbf{0}} \\ \mathrm{QN}_{\mathbf{0}}\end{smallmatrix}\right]$ then $\mathcal{A}$ has a pseudounion.

(3) Let $\mathcal{I}$ be a tall ideal. If $\mathcal{A} \subseteq \mathcal{I} \cap [\omega]^{\omega}$ and $\mathrm{C}_p(\mathcal{A})$ is an $\mathrm{S}_1(\mathcal{I}\text{-}\Gamma_{\mathbf{0}}, \Gamma_{\mathbf{0}})$ then $\mathcal{A}$ has a pseudounion and the family of increasing enumerations of its elements is bounded in $({}^{\omega}\omega, \leq^{*})$.

if	there is a set X of reals of cardinality μ such that:
$\mathfrak{b}_{\mathcal{J}} \leq \mu \leq \mathfrak{c}$	$\mathrm{C}_p(X)$ does not have the property $\mathrm{S}_1(\mathcal{I}\text{-}\Gamma^{\mathrm{m}}_{\mathbf{0}}, \mathcal{J}\text{-}\Gamma_{\mathbf{0}})$ $\mathrm{C}_p(X)$ does not have the property $\mathrm{S}_1(\mathcal{I}\text{-}\Gamma_{\mathbf{0}}, \mathcal{J}\text{-}\Gamma_{\mathbf{0}})$
$\mathtt{cov}^{*}(\mathcal{I}) \leq \mu \leq \mathfrak{c}$	$\mathrm{C}_p(X)$ does not have the property $\left[\begin{smallmatrix}\mathcal{I}\text{-}\Gamma_{\mathbf{0}} \\ \Gamma_{\mathbf{0}}\end{smallmatrix}\right]$ $\mathrm{C}_p(X)$ does not have the property $\left[\begin{smallmatrix}\mathcal{I}\mathrm{QN}_{\mathbf{0}} \\ \mathrm{QN}_{\mathbf{0}}\end{smallmatrix}\right]$ X does not have $\left[\begin{smallmatrix}\mathcal{I}\text{-}\Gamma \\ \Gamma\end{smallmatrix}\right]$
$\min\{\mathtt{cov}^{*}(\mathcal{I}), \mathfrak{b}\} \leq \mu \leq \mathfrak{c}$	$\mathrm{C}_p(X)$ does not have the property $\mathrm{S}_1(\mathcal{I}\text{-}\Gamma_{\mathbf{0}}, \Gamma_{\mathbf{0}})$ X does not have $\mathrm{S}_1(\mathcal{I}\text{-}\Gamma, \Gamma)$.

- Consistency

*	$\mathfrak{b} = \mathfrak{c}$	$\Rightarrow$	$\mathrm{non}(\mathrm{S}_1(\mathcal{I}\text{-}\Gamma, \Gamma)) = \mathsf{cov}^*(\mathcal{I})$ for every tall ideal $\mathcal{I}$
*	$\mathfrak{b} < \mathsf{cov}^*(\mathcal{I})$	$\Rightarrow$	$\mathrm{non}(\mathrm{S}_1(\mathcal{I}\text{-}\Gamma, \Gamma)) < \mathsf{cov}^*(\mathcal{I})$ for every tall ideal $\mathcal{I}$
	$\mathfrak{p} = \mathfrak{b}$	$\Rightarrow$	$\mathrm{non}(\mathrm{S}_1(\mathcal{I}\text{-}\Gamma, \Gamma)) = \mathfrak{b}$
*	$\mathsf{cov}^*(\mathcal{I}) < \mathfrak{b}$	$\Rightarrow$	$\mathrm{non}(\mathrm{S}_1(\mathcal{I}\text{-}\Gamma, \Gamma)) < \mathfrak{b}$
	$\mathfrak{b}_{\mathcal{J}} < \mathfrak{d}$	$\Rightarrow$	$\mathrm{non}(\mathrm{S}_1(\mathcal{I}\text{-}\Gamma, \mathcal{J}\text{-}\Gamma)) < \mathfrak{d}$

Table: $\mathrm{non}(\mathrm{S}_1(\mathcal{I}\text{-}\Gamma, \Gamma))$ regarding to cardinal consistency.

* We can reformulate for $\mathrm{S}_1(\mathcal{I}\text{-}\Gamma_{\mathbf{0}}, \mathcal{J}\text{-}\Gamma_{\mathbf{0}})$-space and its monotone version.

Critical cardinality

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	$\mathfrak{p} = \mathfrak{b}$	$\Rightarrow$	$\mathrm{non}(\mathrm{S}_1(\mathcal{I}\text{-}\Gamma, \Gamma)) = \mathfrak{b}$
*	$\mathrm{cov}^*(\mathcal{I}) < \mathfrak{b}$	$\Rightarrow$	$\mathrm{non}(\mathrm{S}_1(\mathcal{I}\text{-}\Gamma, \Gamma)) < \mathfrak{b}$
	$\mathfrak{b}_{\mathcal{J}} < \mathfrak{d}$	$\Rightarrow$	$\mathrm{non}(\mathrm{S}_1(\mathcal{I}\text{-}\Gamma, \mathcal{J}\text{-}\Gamma)) < \mathfrak{d}$

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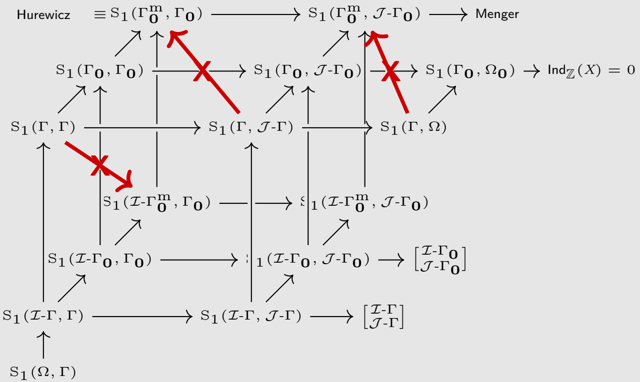
condition	X is	$\mathrm{C}_p(X)$ is not
$\mathfrak{p} < \mathfrak{b}$	$\mathrm{S}_1(\Gamma, \Gamma)$-space	$\mathrm{S}_1(\mathcal{U}\text{-}\Gamma_{\mathbf{0}}^m, \Gamma_{\mathbf{0}})$-space
$\mathrm{cov}^*(\mathcal{I}) < \mathfrak{b}$	$\mathrm{S}_1(\Gamma, \Gamma)$-space	$\mathrm{S}_1(\mathcal{I}\text{-}\Gamma_{\mathbf{0}}^m, \Gamma_{\mathbf{0}})$-space
$\mathfrak{b} < \mathfrak{b}_{\mathcal{U}}$	$\mathrm{S}_1(\Gamma, \mathcal{U}\text{-}\Gamma)$-space	$\mathrm{S}_1(\Gamma_{\mathbf{0}}^m, \Gamma_{\mathbf{0}})$-space
$\mathfrak{b}_{\mathcal{J}} < \mathfrak{d}$	$\mathrm{S}_1(\Gamma, \Omega)$-space	$\mathrm{S}_1(\Gamma_{\mathbf{0}}^m, \mathcal{J}\text{-}\Gamma_{\mathbf{0}})$-space
$\mathfrak{b} < \mathrm{cov}^*(\mathcal{I})$	$[\mathcal{I}\text{-}\Gamma, \Gamma]$-space	$\mathrm{S}_1(\mathcal{I}\text{-}\Gamma_{\mathbf{0}}^m, \Gamma_{\mathbf{0}})$-space

Critical cardinality

condition	X is	$C_p(X)$ is not
$\mathfrak{p} < \mathfrak{b}$	$S_1(\Gamma, \Gamma)$-space	$S_1(\mathcal{U}\text{-}\Gamma^m_{\mathbf{0}}, \Gamma_{\mathbf{0}})$-space
$\mathrm{cov}^*(\mathcal{I}) < \mathfrak{b}$	$S_1(\Gamma, \Gamma)$-space	$S_1(\mathcal{I}\text{-}\Gamma^m_{\mathbf{0}}, \Gamma_{\mathbf{0}})$-space
$\mathfrak{b} < \mathfrak{b}_{\mathcal{U}}$	$S_1(\Gamma, \mathcal{U}\text{-}\Gamma)$-space	$S_1(\Gamma^m_{\mathbf{0}}, \Gamma_{\mathbf{0}})$-space
$\mathfrak{b}_{\mathcal{J}} < \mathfrak{d}$	$S_1(\Gamma, \Omega)$-space	$S_1(\Gamma^m_{\mathbf{0}}, \mathcal{J}\text{-}\Gamma_{\mathbf{0}})$-space
$\mathfrak{b} < \mathrm{cov}^*(\mathcal{I})$	$[\mathcal{I}\text{-}\Gamma, \Gamma]$-space	$S_1(\mathcal{I}\text{-}\Gamma^m_{\mathbf{0}}, \Gamma_{\mathbf{0}})$-space



Thank you for your attention

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