Research statment

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1 Background

My research interests lie mainly in geometric group theory, large scale geometry, and nonlinear analysis. This means that I am trying to understand geometric and algebraic properties by finding nonlinear embeddings into "nice" spaces and using the properties of the ambient space to attack a problem. This kind of analytic approach has proven to have a lot of applications to geometry (growth of groups, Novikov conjecture, Gromov's positive scalar curvature conjecture, etc) as well as theoretical computer science (Sparsest Cut problem, Nearest Neighbor Search, Compressive sensing, etc).

Together with Baudier and Swift in [BSS] we have used diamond graph-like objects to show that for every $q \in (2, \infty)$, there exists a doubling subset of ℓ_q that does not admit any bi-Lipschitz embedding into $\mathbb{R}^d$ for any $d \in \mathbb{N}$. Our approach allows us to recover quantitative results about possible distortion bounds for embeddings into p-uniformly convex Banach spaces as well as embeddability obstructions into non-positively curved spaces. Another result that I would like to mention comes from [S2] where we showed the nonexistence of equivariant coarse embeddings of L_p into ℓ_p. Equivariant coarse embeddings are special types of coarse embeddings, which have been studied only very recently and one can hope that they will be a good tool for proving general nonembeddability results. Below I will explain some of the results I have obtained in the past, as well as the necessary background to understand how they fit within the field and some possible future generalizations.

It is important to mention that during my Ph.D. I have worked on several other projects, including whether Grigorchuk group has Finite Decomposition Property and calculating Dynamic Asymptotic Dimension of Mapping Class Group. Similarly, in my Master's thesis, I have studied systolic complexes which are simplicial analogs of $CAT(0)$ cube complexes. This more "geometric" side of geometric group theory, which is close to the study of low dimensional manifolds and algebraic topology, is very interesting to me and I would love to work on more projects in this direction. On the other hand, I am also interested in possible applications of the analytic tools I use, in particular to compressive sensing and neural nets. Overall I am a candidate with a broad background and am looking forward to further expanding my mathematical horizons and working on different ideas.

2 Introduction

Banach spaces are one of the cornerstones of functional analysis and modern mathematics in general. The development of the linear theory and its applications to differential equations, geometry, complexity theory, and other fields, were among some of the most important advances in mathematics of the 20th century. My own research focuses on the non-linear theory of Banach spaces and of its possible applications.

One particular area of research that is interested in the nonlinear properties of Banach spaces is large scale geometry. It studies spaces viewed from far away and is interested in global geometric features rather than the local ones. From this point of view, integers are the same as the real line since they look alike if you zoom out sufficiently. Similarly, a disk (or any other bounded geometric space) is indistinguishable from a single point. This is in stark contrast to a lot of classical notions in analysis that focus on the local properties of maps and spaces. Historically these ideas first appeared in the proof of Mostow's rigidity theorem [Mo] and its strengthening by Margulis [M]. They have later entered the field of geometric group theory for good with the work of Švarc, Milnor, and Wolf on the growth of groups. A celebrated result by Gromov [Gr2], which states that a finitely generated group has polynomial growth if and only if it is virtually nilpotent, relied heavily on this machinery. The definition of the quasi-isometric equivalence (i.e. a map that preserves the large scale geometry, the concept that was further generalized into coarse equivalence) was finally formulated by Gromov in [Gr1] and used in great success in different areas of mathematics ranging from differential geometry to data analysis. Many concepts have been studied extensively in group theory, for example, the existence of the Banach-Tarski paradox [BT] that had led von Neumann to introduce the concept of amenability in [VN], turned out to be a large scale invariant. One can find a good overview of the field in [NY].

A result by Yu from [Yu] motivated a lot of research in this field because it connected large scale geometry to K-theory. Namely, Yu proved that if a discrete metric space X with bounded geometry admits a coarse embedding into Hilbert space, then the coarse Baum-Connes conjecture holds for X, i.e. the coarse index map is an isomorphism. Note that if X happens to be a finitely generated group G whose classifying space BG has the homotopy type of a finite CW complex and the metric structure we consider comes from a word-length metric, one can deduce that the strong Novikov conjecture holds for G, i.e. the index map from $K_*(BG)$ to $K_*(C_r^*G)$ is injective, where C_r^*G is the reduced C^* algebra of G. Note that by the index theory, the strong Novikov conjecture implies the Novikov conjecture i.e. that higher signatures (certain numerical invariants in the Pontryagin classes of smooth manifolds) are homotopy invariant. This result was later generalized by Kasparov and Yu in [KY] where they proved that if X is a bounded geometry metric space, which admits a coarse embedding into a uniformly convex Banach space, then the coarse Novikov conjecture holds for X; that is, the coarse index map is injective. Those results inspired people to study not only coarse embeddings into Banach spaces but also various coarse embeddings between said spaces.

3 Bi-Lipschitz embeddings

Assouad proved in [A] an astonishing result that for every separable metric space (X, d) with doubling constant $M > 0$ we can construct a bi-Lipschitz embedding with distortion D of $(X, d^{1-\varepsilon})$ into $\mathbb{R}^N$, where $\varepsilon \in (0, 1)$, $N(\varepsilon, M) \in \mathbb{N}$ and $D(\varepsilon, M) \in (0, \infty)$. Since this result says that a huge class of metric spaces can be bi-Lipschitzly embedded into a finite dimensional vector space, as long as we change the original metric slightly, it motivated a question for what classes of spaces this can be done with the original metric.

It turns out that for a metric space (X, d) to admit a bi-Lipschitz embedding into a finite-dimensional vector space, it has to be doubling and embeddable into Hilbert space in a bi-Lipschitz way. This motivated Lang and Plaut to ask if every doubling subset of Hilbert space admits a bi-Lipschitz embedding into some $\mathbb{R}^N$? This remains a hard, open question that motivates a lot of research in nonlinear analysis.

In [BSS] we presented a new proof that for every $q \in (2, \infty)$, there exists a doubling subset of ℓ_q that does not admit any bi-Lipschitz embedding into $\mathbb{R}^d$ for any $d \in \mathbb{N}$. This result was proven before by Lafforgue and Naor in [LN] and independently by Bartal, Gottlieb, and Neiman in [BGN]. Note that the question for $q \in (1, 2]$ remains open.

The approach undertaken by Lafforgue and Naor is based on the subtle geometric properties of Heisenberg groups and they obtain the result in full generality as stated above. On the other hand proof by Bartal, Gottlieb, and Neiman use a very concrete construction of doubling subsets, but their subsets depend on the distortion and the dimension of the target space. Consequently, the nonembeddability is proven for one finite dimensional vector space at a time.

The proof that we present in [BSS] has the advantage of the simplicity of the construction in BGN with the generality of the approach in [LN]. It also allows us to recover optimal bounds for the distortion D of ℓ_q into a p-uniformly convex Banach space, namely that $D = \Omega\Big((\log n)^{\frac{1}{p}-\frac{1}{q}}\Big)$ if $p \in [2, q)$. Note that this is an improvement in comparison to both papers discussed before. We use Laakso-like objects inside of ℓ_q (inspired by [BGN]) and rely on a self-improvement argument (developed by Johnson and Schechtman in [JS]) to bound the distortion. Our approach also has the advantage of being easily translatable to a metric setting and we are able to show that the distortion D of ℓ_q^k in $CAT(0)$ metric spaces X (or more generally a space with roundness 2) is bounded from below by $\Omega\Big(k^{\frac{1}{2}-\frac{1}{q}}\Big)$ for $q > 2$.

Ribe's result in [R] that uniformly homeomorphic Banach spaces have uniformly linearly isomorphic finite-dimensional subspaces has inspired Bourgain and Lindenstrauss to propose a program of trying to classify local properties of Banach spaces using a purely metric language. Bourgain started what is now called Ribe's program (see [N] for an overview) by showing in [Bou] that a Banach space is superreflexive if and only if the family of binary trees of finite height does not equi-bi-Lipschitzly embed into it. Similar characterizations of superreflexivity using different families of graphs were shown in [Bau], [JS] and [OR]. Baudier et al. in [BCDKRSZ] study similar questions using countably branching diamond graphs and are able to prove some embeddability results among other things. This line of work has been further studied by Swift in [S].

We modified our construction from [BSS] to obtain an analog of countably branching diamond graphs and used it to prove results in a spirit of Ribe's program. Namely, we show that for $1 \leq p < q$ if (X, d) admits a bi-Lipschitz embedding with distortion D into a p-asymptotically midpoint uniformly convex Banach space Y. There exists $\varepsilon := \varepsilon(p, q, D, Y) > 0$ such that if X admits a (ε, q)-thin $\aleph_0$-branching k-diamond substructure then $D = \Omega(k^{1/p-1/q})$. This gives a purely geometric proof that $L_q[0,1]$ does not bi-Lipschitzly embed into any p-asymptotically midpoint uniformly convex Banach space if $q > p \geq 1$. In particular, $L_q[0,1]$ does not bi-Lipschitzly embed into ℓ_p if $q > p \geq 1$.

A classical paper by Kadec and Pełczyński (see [KP]) inspired an open problem that asks if a metric space X that admits a bi-Lipschitz embedding into L_p and into L_q, has to also embed into L_r for every $1 \leq p < r < q < \infty$? Naor and Young gave recently the first partial counter-example to this by producing a Heisenberg type space that does embed into ℓ_1 and into ℓ_q but does not embed into ℓ_r for any $1 < r < 4 \leq q$. However, what happens for ℓ_r in the range $4 \leq r < q$ is not understood. If we could show that the thin Laakso substructures that we study in [BSS] do embed into ℓ_1 then we would have a second counter-example to the metric Kadec-Pełczyński problem which doesn't have any artificial restriction on r.

4 Equivariant coarse embeddings

The classical fact that there is no linear embedding of L_p into ℓ_p follows from a theorem by Pitt, who proved in [P] that every bounded operator $T\colon \ell_p \to \ell_q$ is compact for $1 \leq q < p < \infty$. On the other hand, one can show (see Proposition 6.4.13 in [AK]) using a sequence of independent normalized Gaussians on the interval $[0, 1]$ that ℓ_2 isometrically embeds into L_p for $1 \leq p < \infty$.

A big step in the study of Banach spaces was when people started to look at them from a purely metric point of view and forgot the underlying linear structure. A fundamental result in this setting is a theorem by Mazur and Ulam (see [MU]) that studies isometries $f\colon X \to Y$ between real normed spaces X, Y, that maps 0 to 0. By looking at midpoints of line segments they prove that f is in fact a bounded linear operator. Thus the problem of metric embedding of L_p into ℓ_p can be reduced to the linear setting, which gives us a negative answer.

The standard proof of the nonexistence of bi-Lipschitz embedding from L_p into ℓ_p follows Heinrich and Mankiewicz who in [HM] proved more generally that if $X\colon \to Y$ is a bi-Lipschitz embedding of separable Banach space X into a space Y with the Radon–Nikodym property, then there exists a point of Gâteaux differentiability of f and the derivative at that point is a linear embedding with distortion bounded by distortion of f. Their proof is relying on Rademacher's differentiability theorem, which says that every Lipschitz map $f\colon X \to Y$ from a separable

Banach space X into a Banach space Y with the Radon–Nikodym property is differentiable Haar-almost-everywhere.

More general type of maps that one may consider is quasi-isometric (or coarse Lipschitz) embeddings where linear bounds of the distance between any two points in replaced by an affine bound. In this setting it was Kalton and Randrianarivony who proved in [KR] that L_p does not admit a quasi-isometric embedding into ℓ_p by studying graphs $G_k(\mathbb{M}) = \{\overline{n} = (n_1, \ldots, n_k) \colon n_i \in \mathbb{M}, n_1 < n_2 < \cdots < n_k\}$ with a metric $d(\overline{n}, \overline{m}) = |\{i : n_i \neq m_i\}|$ for any subset $\mathbb{M}$ of the natural numbers. More precisely they estimate the minimal distortion of any Lipschitz embedding of $G_k(\mathbb{M})$ into ℓ_p-like Banach spaces using Ramsey's Theorem.

The most general metric setting that we are going to consider is a coarse category, where the distance in the image is bounded by arbitrary nondecreasing functions of the original distance. In this setting question about embeddability of L_p into ℓ_p has only partial answers. Namely for $1 \leq p < 2$ it follows by work of by Mendel and Naor in [MN] that L_p does embed into ℓ_p. This is done by factoring the embedding through Hilbert space, based on the argument by Nowak who proved in [N] that ℓ_2 coarsely embeds into ℓ_p for $1 \leq p < \infty$.

In [S2], we look at the category of equivariant coarse embeddings - which are coarse embeddings between normed vector spaces that satisfy a certain compatibility condition with a predetermined representation into the isometry group of the target space. In a group theoretic language, such an embedding is equivalent to the existence of a proper, affine, isometric action, where the source space is viewed as a group under addition. We find that every normed, vector space $(V, +)$ that admits an equivariant coarse embedding into ℓ_p is isomorphically embeddable into ℓ_p.

Equivariant coarse embeddings between Banach spaces remain a new and unstudied phenomenon, so there is ample opportunity to produce more results. Our approach relied heavily on the fact that we understand very well the isometry group of ℓ_p so natural candidates for further study include Banach spaces for which the same can be said, for example, infinite p-sums of finite-dimensional spaces, $\ell_p \bigoplus_2 \ell_q$, L_p, L_1 and Orlicz spaces.

It is important to mention that Cornulier, Tessera, Valette proved in [CTV] (inspired by a similar result by Gromov in the discrete setting) that if locally compact, compactly generated, amenable group G coarsely embeds into the Hilbert space, then it also embeds in the coarsely equivariant way. One can hope to generalize this result to some Banach spaces viewed as a topological group under addition. This would let us translate the results about equivariant coarse embeddings into results about coarse embeddings. In particular, we could attack the general question of coarse embeddability of L_p into ℓ_p suing the results stated above.

5 Systolic Complexes

Recall the classical Helly's theorem concerning convex subsets of the Euclidean spaces. Suppose that $X_1, X_2, \ldots, X_n$ is a collection of convex subsets of $\mathbb{R}^d$ (where $n > d$) such that the intersection of every $d+1$ of these sets is nonempty. Then the whole family has a nonempty intersection. This result gave rise to the concept of Helly dimension. For a geodesic metric space X we define its *Helly dimension* $h(X)$ to be the smallest natural number such that any finite family of $(h(X)+1)$-wise non-disjoint convex subsets of X has a non-empty intersection. Clearly, Helly's theorem states that Helly dimension of the Euclidean space $\mathbb{R}^d$ is $\leq d$. It is very easy to find examples showing that it is exactly equal to d.

Systolic complexes were introduced by Januszkiewicz and Świątkowski in [JaS]. They are connected, simply connected simplicial complexes satisfying some additional local combinatorial condition, which is a simplicial analog of nonpositive curvature. Systolic complexes inherit lots of $CAT(0)$-like properties, however being systolic neither implies, nor is implied by nonpositive curvature of the complex equipped with the standard piecewise Euclidean metric.

There is a well-known result for $CAT(0)$ cube complexes which states that regardless of their topological dimension, they all have Helly dimension equal to one. This motivates a question about Helly-like properties of systolic complexes. In [S1] we obtained the following results:

Let X be a 7-systolic complex and let X_1, X_2, X_3 be pairwise intersecting convex subcomplexes. Then there exists a simplex $\sigma \subseteq X$ such that $\sigma \cap X_i \neq \emptyset$ for $i = 1, 2, 3$. Moreover, the dimension of σ is at most two.

In other words, 7-systolic complexes have Helly dimension less or equal to 1. It is easy to see that this is not necessarily true for 6-systolic complexes, but we prove that any systolic complex has Helly dimension less or equal to 2. More precisely:

Let X be a systolic complex and let X_1, X_2, X_3, X_4 be its convex subcomplexes such that every three of them have a nontrivial intersection. Then there exists a simplex $\sigma \subseteq X$ such that $\sigma \cap X_i \neq \emptyset$ for $i = 1, 2, 3, 4$. Moreover, the dimension of σ is at most three.

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