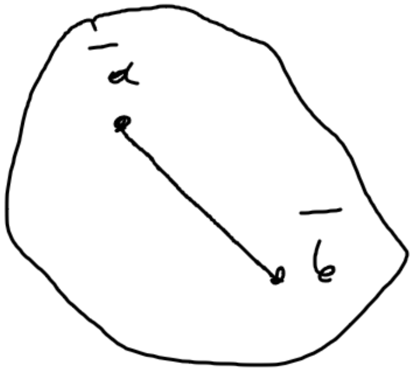
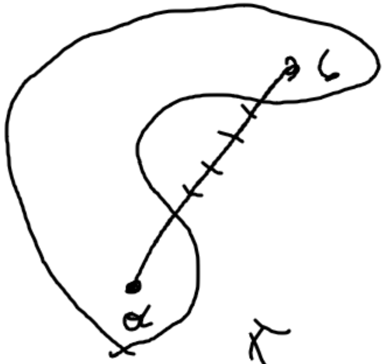
CONVEX SUBSETS

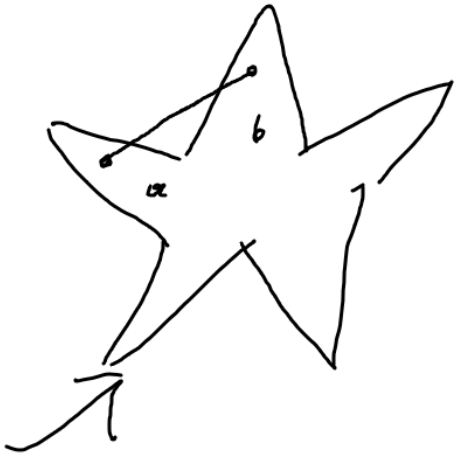
$X \subset \mathbb{R}^n$ $\quad (x_1 \dots x_n) = \bar{x} \in X$

X IS CONVEX IF FOR ANY PAIR POINTS $\bar{a}, \bar{b} \in X$ THE LINE SEGMENT $[\bar{a}, \bar{b}] \subset X$



CONVEX

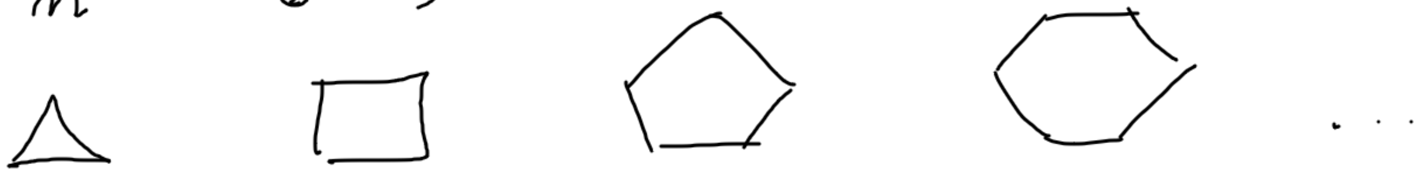




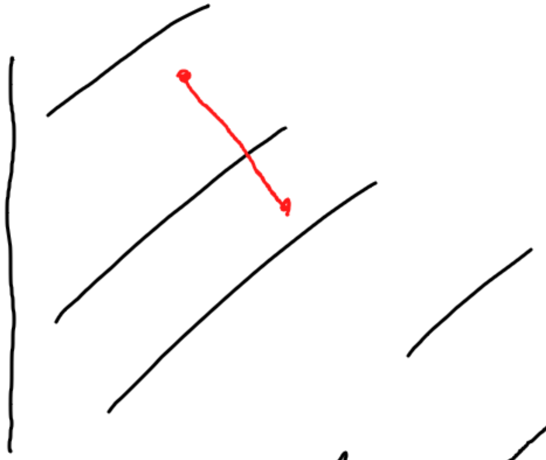
NOT CONVEX

EX. • BALLS OF RADIUS r ARE CONVEX SUBSETS

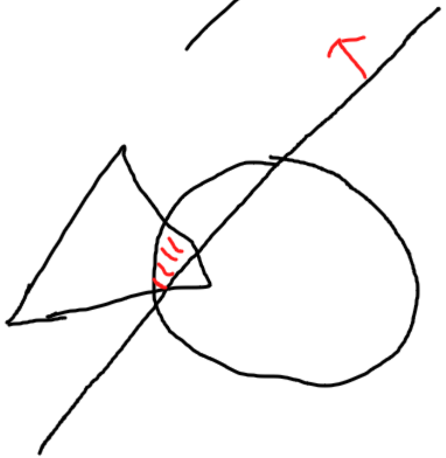
• n - GONS ARE CONVEX



• HALF-SPACES



• TH THE INTERSECTION OF ANY FAMILY OF CONVEX SETS IS CONVEX



PROOF $\{X_i\}_{i\in I} \subset \mathbb{R}^n$

WE WANT TO SHOW THAT $\bigcap_{i\in I} X_i$ IS CONVEX

$a, b \in \bigcap_{i\in I} X_i$ WTS $[a,b] \in \bigcap_{i\in I} X_i$ i.e.

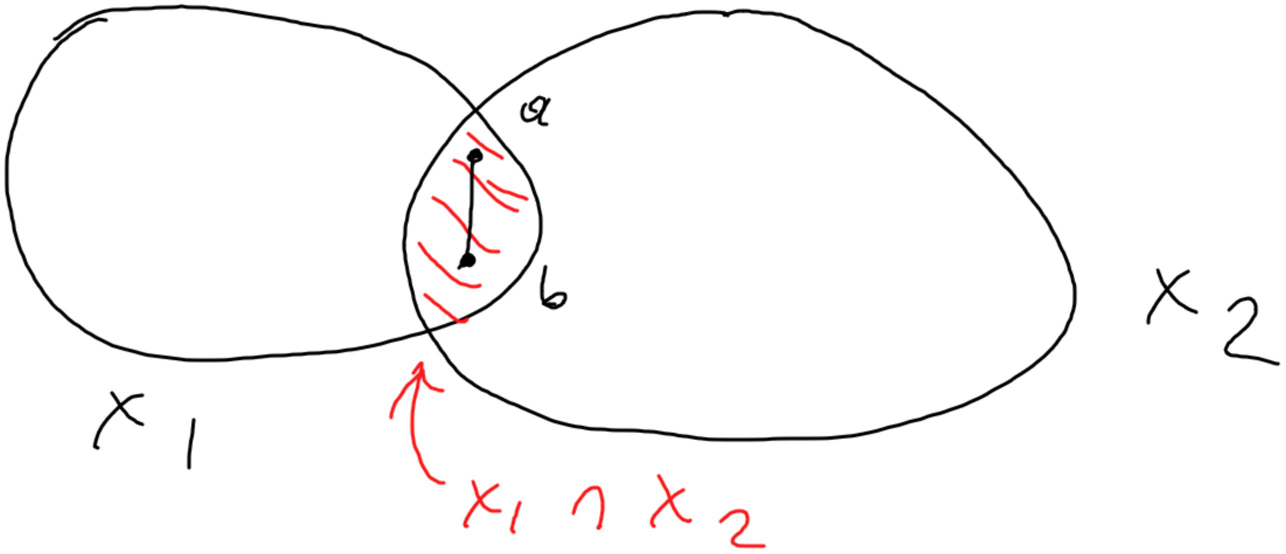
FOR ANY $i \in I$ $[a,b] \in X_i$

BUT SINCE $a, b \in \bigcap_{i\in I} X_i \Rightarrow a, b \in X_i$

BUT X_i IS CONVEX, SO $[a,b] \in X_i$

SO $[a,b] \subset \bigcap X_i$, HENCE $\bigcap X_i$ IS CONVEX

$\blacksquare$



$x \in X_1 \cap X_2$

$\Updownarrow$

$x \in X_1$

AND

$x \in X_2$

- $a, b \in X_1$

  SINCE X_1 IS CONVEX

  $[a, b] \subset X_1$

- $a, b \in X_2$

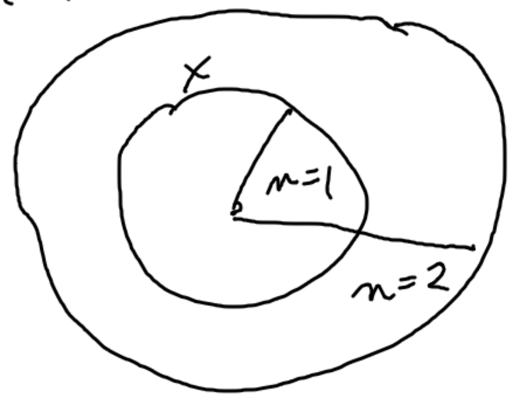
  SINCE X_2 IS CONVEX

  $[a, b] \subset X_2$

$[a,b] \subset X_1$ AND $[a,b] \subset X_2 \Longrightarrow [a,b] \subset X_1 \cap X_2$

• $\lambda \in \mathbb{R}$ $\qquad \lambda \cdot \underline{X}$ is also convex

$\underline{X}$ - convex



• A - convex subset $\qquad \bar{b} \in \mathbb{R}^n$

$A + \bar{b} = \{ \bar{a} + \bar{b} \mid \bar{a} \in A \}$

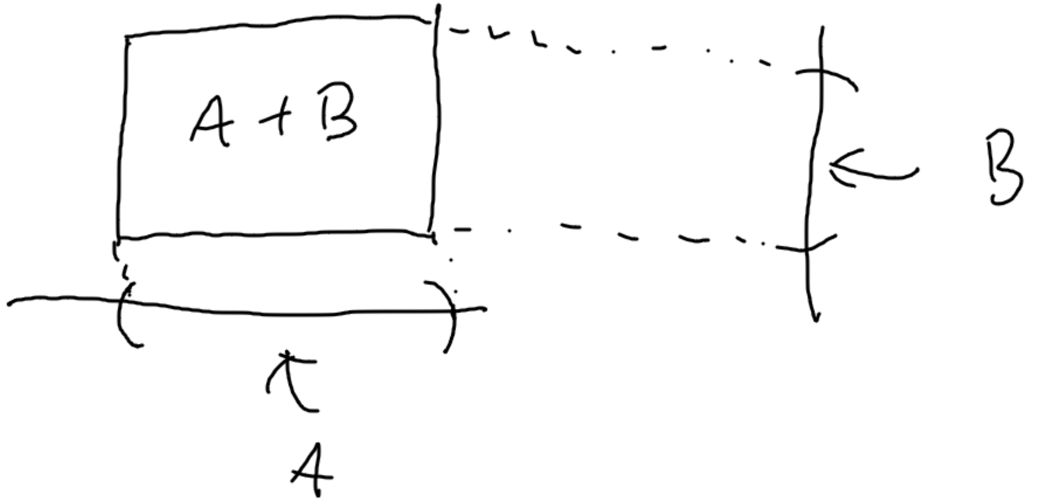
is also convex

- MORE GENERALLY:

A, B — CONVEX SUBSETS

$$A + B = \{ \bar{a} + \bar{b} \mid a \in A \quad b \in B \}$$

IS ALSO CONVEX



CONVEX HULL

$A \subset \mathbb{R}^n$ ANY SUBSET

$\operatorname{CONV}(A) \subset \mathbb{R}^n$ THE SMALLEST CONVEX SET THA CONTAINS A.

WHY THIS IS WELL DEFINED ?

FOR ANY $A \subset \mathbb{R}^n$, $\mathbb{R}^n$ ~~IS~~ CONTAIN A AND IS CONVEX. SO THE ~~RE~~ FAMILLY OF CONVEX SETS $\mathcal{F}_A$ WHICH CONTAINS SET A IS NON-EMPTY. CONSIDER $\bigcap_{F \in \mathcal{F}_A} F$ BY OUR TH. THIS INTERSECTION IS CONVEX AND $A \subset F$ FOR ALL $F \subset \mathcal{F}_A$

$A \subset \bigcap_{F \in \mathcal{F}_A} F \neq \emptyset$. $\operatorname{CONV}(A) = \bigcap_{F \in \mathcal{F}_A} F$

• RECALL $\bar{a}, \bar{b} \in \mathbb{R}^n$

THEN $\{\lambda_1 \bar{a} + \lambda_2 \bar{b}\}$ WHERE $\lambda_1 + \lambda_2 = 1$, $\lambda_1, \lambda_2 \in \mathbb{R}$

IS AN INTERVAL BTW $\bar{a}, \bar{b}$

<u>TH</u> LET $A \subset \mathbb{R}^n$ SUBSET

THEN THE CONVEX HULL

$$\mathrm{CONV}(A) = \{\lambda_1 \bar{a} + \lambda_2 \bar{b} \mid a, b \in A,\ \lambda_1 + \lambda_2 = 1\}$$

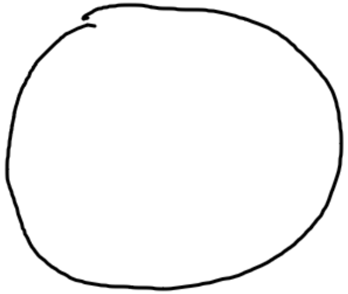
CONVEX COMBINATION

"PROOF": • $A \subset \{\lambda_1 \bar{a} + \lambda_2 b \mid \lambda_1 = 1,\ \lambda_2 = 0,\ a = a,\ b \in A\}$

a

• IT'S SMALLEST BC IF CONV(a, b) DIDN'T CONTAIN $[a, b]$ THEN IT WOULDN'T BE CONVEX

EX

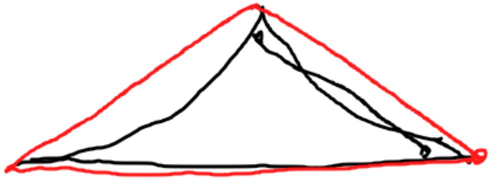


$A \subset \mathbb{R}^2$ is convex

$\text{conv}(A) = A$

since A is convex and $A \subset A$

$A \subset \mathbb{R}^2$ not convex



$\text{conv}(A)$

$\text{conv}(A)$

A

HELLY'S THEOREM

LET $\mathcal{F}$ BE A FAMILY OF CONVEX SETS IN $\mathbb{R}^n$ $|\mathcal{F}| \geq n+1$. ASSUME THAT ANY $n+1$ SUBSETS OF $\mathcal{F}$ HAVE A NON-TRIVIAL INTERSECTION. THEN $\mathcal{F}$ HAS A NON-TRIVIAL INTERSECTION.

IT THIS PROVED USING INDUCTION ON THE NUMBER OF ELEMENTS IN $\mathcal{F}$. AND THEN THE ARGUMENT RELIES ON CONVEX HULLS.

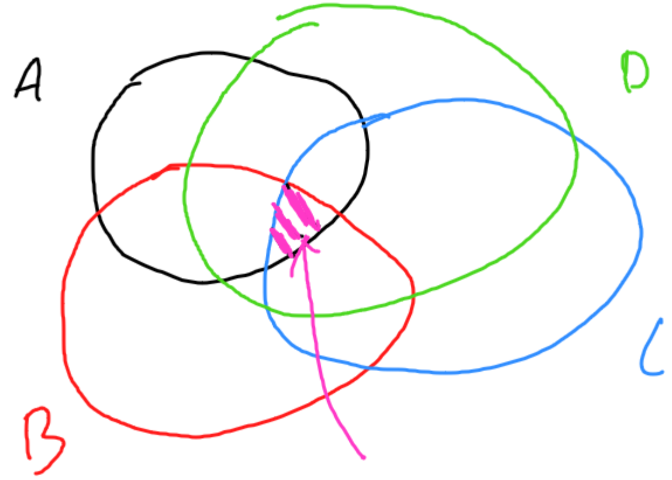
- $n = 1$ $\quad \mathcal{F} = \{A, B, C\}$

$A \cap B \neq \emptyset \qquad A \cap C \neq \emptyset \qquad B \cap C \neq \emptyset$

$\Rightarrow \quad A \cap B \cap C \neq \emptyset$

$A \cap B \cap C \neq \emptyset$

$n = 2 \qquad \{A, B, C, D\} \subset \mathbb{R}^2$



$A \cap B \cap C \cap D \neq \emptyset$

$A \cap B \cap C \neq \emptyset$

$A \cap C \cap D \neq \emptyset$

$A \cap B \cap D \neq \emptyset$

$B \cap C \cap D \neq \emptyset$

$\Longrightarrow$

$A \cap B \cap C \cap D \neq \emptyset$

STAR CENTER

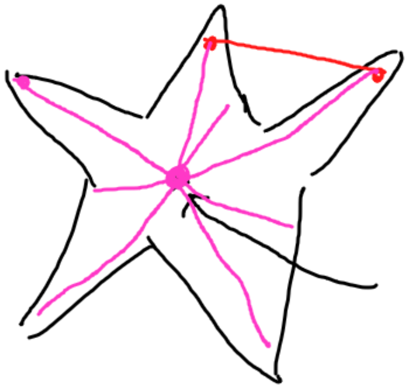
$a \in A \subset \mathbb{R}^n$ a IS A STAR CENTER OF A IF FOR ANY $b \in A$ $[a, b] \subset A$ i.e. $\lambda a + (1-\lambda) b \subset A$ FOR ANY λ

EX FOR CONVEX SUBSETS EVERY POINT IS A STAR-CENTER

- A SET A IS CALLED STAR SHAPED IF ONE OF ITS POINTS IS A STAR CENTER

EX CONVEX SETS ARE STAR SHAPED

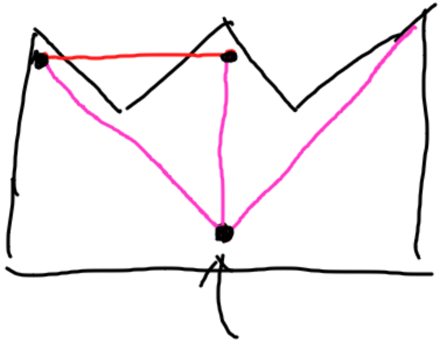
EX



IS NOT CONVEX

STAR CENTER, SO THIS SUBSET IS STAR SHAPED

EX



STAR CENTER

STAR SHAPED BUT NOT CONVEX

TH (KRASNOSSELSKY'S)

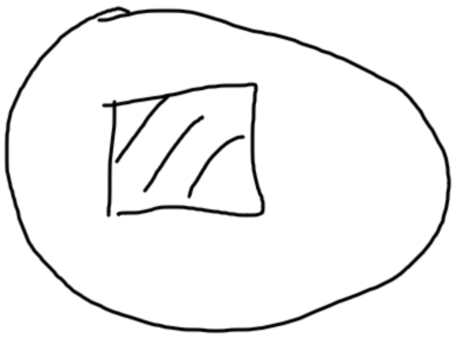
LET $A \subset \mathbb{R}^n$ BE A CLOSED, BOUNDED SUBSET. SUPPOSE THAT FOR EACH $n+1$ POINTS OF A THERE IS A POINT OF A FROM WHICH ALL THESE POINTS ARE VISIBLE. THEN A IS STRA SHAPED.

CLOSED = CONTAINS THE BOUNDARY

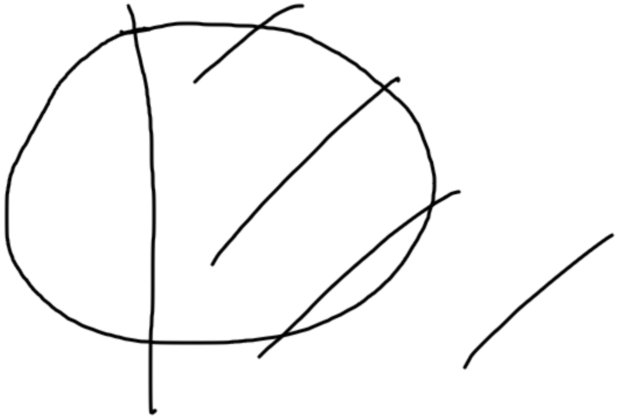
NOT CLOSED

CLOSED

BOUNDED MEANS CONTAINED IN SOME FINITE RADIUS BALL



BOUNDED



NOT BOUNDED