Coupling Property and Gradient Estimates of Lévy Processes Via the Symbol

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Abstract. We derive explicitly the coupling property for the transition semigroup of a Lévy process and gradient estimates for the associated semigroup of transition operators. This is based on the asymptotic behaviour of the symbol or the characteristic exponent near zero and infinity, respectively. Our results can be applied to a large class of Lévy processes, including stable Lévy processes, layered stable processes, tempered stable processes and relativistic stable processes.

Keywords: Coupling; gradient estimates; Lévy process; symbol.

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1. Introduction and Main Results

Let $X_t$ be a pure jump Lévy process on $\mathbb{R}^d$ with the symbol (or characteristic exponent)

$$
\Phi(\xi) = \int_{z \neq 0} \left( 1 - e^{i\xi \cdot z} + i\xi \cdot z \mathbb{1}_{B(0,1)}(z) \right) \nu(dz),
$$

where $\nu$ is the Lévy measure, i.e. a $\sigma$-finite measure on $\mathbb{R}^d \setminus \{0\}$ such that $\int_{z \neq 0} (1 \wedge |z|^2) \nu(dz) < \infty$. There are many papers studying regularity properties of Lévy processes in terms of the symbol $\Phi$. For example, recently [14, Theorem 1] points out the relations between the classic Hartman-Wintner condition (see [9] or (1.1) below) and some smoothness properties of the transition density for Lévy processes. In particular, the condition that the symbol $\Phi(\xi)$ of the Lévy process $X_t$ satisfies

$$
\liminf_{|\xi| \to \infty} \frac{\text{Re} \Phi(\xi)}{\log(1 + |\xi|)} = \infty
$$

is equivalent to the statement that for all $t > 0$ the random variables $X_t$ have a transition density $p_t(y)$ such that $\nabla p_t \in L_1(\mathbb{R}^d) \cap C_\infty(\mathbb{R}^d)$, where $C_\infty(\mathbb{R}^d)$ denotes the set of all continuous functions which vanish at infinity. The main purpose of this paper is to derive an explicit coupling property and gradient estimates of Lévy processes directly from the corresponding symbol $\Phi$.

Let $(X_t)_{t \geq 0}$ be a Markov process on $\mathbb{R}^d$ with transition probability function $\{P_t(x, \cdot)\}_{t \geq 0, x \in \mathbb{R}^d}$. An $\mathbb{R}^{2d}$-valued process $(X'_t, X''_t)_{t \geq 0}$ is called a coupling of the Markov process $(X_t)_{t \geq 0}$, if both $(X'_t)_{t \geq 0}$ and $(X''_t)_{t \geq 0}$ are Markov processes which

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have the same transition functions $P_t(x,\cdot)$ but possibly different initial distributions. In this case, $(X_t^1)_{t\geq 0}$ and $(X_t^2)_{t\geq 0}$ are called the marginal processes of the coupling process; the coupling time is defined by $T := \inf\{t \geq 0 : X_t^1 = X_t^2\}$. The coupling $(X_t^1, X_t^2)_{t\geq 0}$ is said to be successful if $T$ is a.s. finite. If for any two initial distributions $\mu_1$ and $\mu_2$, there exists a successful coupling with marginal processes starting from $\mu_1$ and $\mu_2$, respectively, we say that $X_t$ has the coupling property (or admits successful couplings). According to [15] and the proof of [23, Theorem 4.1], the coupling property is equivalent to the statement that

$$\lim_{t \to \infty} \|P_t(x,\cdot) - P_t(y,\cdot)\|_{\text{var}} = 0 \quad \text{for any } x,y \in \mathbb{R}^d,$$

where $P_t(x,\cdot)$ is the transition function of the Markov process $(X_t)_{t\geq 0}$. By $\|\mu\|_{\text{var}}$ we denote the total variation norm of the signed measure $\mu$. We know from [23, Theorem 4.1] that every Lévy process whose transition functions have densities for large enough $t > 0$ has the coupling property. In this case, the transition probability function satisfies

$$(1.2) \quad \|P_t(x,\cdot) - P_t(y,\cdot)\|_{\text{var}} \leq C(1 + |x-y|) / \sqrt{t} \quad \text{for } t > 0 \text{ and } x,y \in \mathbb{R}^d.$$ 

It is clear that for any $x,y \in \mathbb{R}^d$ and $t \geq 0$, $\|P_t(x,\cdot) - P_t(y,\cdot)\|_{\text{var}} \leq 2$, and $\|P_t(x,\cdot) - P_t(y,\cdot)\|_{\text{var}}$ is decreasing with respect to $t$. This shows that it is enough to estimate $\|P_t(x,\cdot) - P_t(y,\cdot)\|_{\text{var}}$ for large values of $t$. We will call any estimate for $\|P_t(x,\cdot) - P_t(y,\cdot)\|_{\text{var}}$ an estimate of the coupling time. The rate $1/\sqrt{t}$ in (1.2) is not optimal for general Lévy processes which admit successful couplings. For example, for rotationally invariant $\alpha$-stable Lévy processes we can prove, see [1, Example 2.3], that

$$\|P_t(x,\cdot) - P_t(y,\cdot)\|_{\text{var}} \approx \frac{1}{t^{1/\alpha}} \quad \text{as } t \to \infty,$$

where for any two non-negative functions $g$ and $h$, the notation $g \asymp h$ means that there are two positive constants $c_1$ and $c_2$ such that $c_1 g \leq h \leq c_2 g$.

Let $P_t(x,\cdot)$ and $P_t$ be the transition function and the semigroup of the Lévy process $X_t$, respectively. We begin with coupling time estimates of Lévy processes which satisfy the following Hartman-Wintner condition

$$\liminf_{|\xi| \to \infty} \frac{\Re \Phi(\xi)}{|\xi| \log(1 + |\xi|)} > 0;$$

this condition actually ensures that the transition function of the Lévy process $X_t$ is, for sufficiently large $t > 0$, absolutely continuous, see e.g. [9] or [14].

**Theorem 1.1.** Suppose that (1.3) holds and

$$\Re \Phi(\xi) \asymp f(|\xi|) \quad \text{as } |\xi| \to 0,$$

where $f : [0,\infty) \to \mathbb{R}$ is a strictly increasing function which is differentiable near zero and which satisfies

$$\liminf_{r \to 0} f(r)|\log r| < \infty$$

and

$$\limsup_{s \to 0} f^{-1}(2s)/f^{-1}(s) < \infty.$$ 

Then the corresponding Lévy process $X_t$ has the coupling property, and for any $x,y \in \mathbb{R}^d$,

$$\|P_t(x,\cdot) - P_t(y,\cdot)\|_{\text{var}} = O\left(f^{-1}(1/t)\right) \quad \text{as } t \to \infty.$$
It can be seen from the above remark on rotationally invariant $\alpha$-stable Lévy processes that the estimate in Theorem 1.1 is sharp.

Now we turn to explicit gradient estimates for the semigroup of a Lévy process. For a function $u \in B_b(\mathbb{R}^d)$ we define 

$$\|\nabla u(x)\| := \limsup_{y \to x} \frac{|u(y) - u(x)|}{|y - x|}, \quad x \in \mathbb{R}^d.$$ 

If $u$ is differentiable at $x$, then $|\nabla u(x)|$ is just the norm of the gradient of $u$ at $x$. We are interested in sub-Markov semigroups $P_t$ on $B_b(\mathbb{R}^d)$ which satisfy that for some positive function $\phi$ on $(0, \infty)$

$$\|\nabla P_t u\|_{\infty} \leq \|u\|_{\infty} \phi(t), \quad t > 0, u \in B_b(\mathbb{R}^d).$$ 

Similar uniform gradient estimates for Markov semigroups have attracted a lot of attention in analysis and probability, e.g. see [17] and references therein. Because of the Markov property of the semigroup $P_t$, $\phi(t)$ is decreasing with respect to $t$. Thus, it is enough to obtain sharp estimates for $\phi(t)$ both as $t \to 0$ and $t \to \infty$. For Lévy processes, we have

**Theorem 1.2.** Assume that (1.1) holds. If there is a strictly increasing function $f$ which is differentiable near infinity and which satisfies 

$$\limsup_{s \to \infty} f^{-1}(2s)/f^{-1}(s) < \infty,$$

and 

$$\Re \Phi(\xi) \asymp f(|\xi|) \quad \text{as } |\xi| \to \infty,$$

then there exists a constant $c > 0$ such that for $t > 0$ small enough,

$$\|\nabla P_t u\|_{\infty} \leq c \|u\|_{\infty} f^{-1}(1/t), \quad u \in B_b(\mathbb{R}^d). \quad (1.4)$$

Similarly, let $f$ be a strictly increasing function which is differentiable near zero and which satisfies 

$$\liminf_{r \to 0} f(r) |\log r| < \infty, \quad \limsup_{s \to 0} f^{-1}(2s)/f^{-1}(s) < \infty$$

and 

$$\Re \Phi(\xi) \asymp f(|\xi|) \quad \text{as } |\xi| \to 0.$$ 

Then there exists a constant $c > 0$ such that (1.4) holds for $t > 0$ large enough.

We will see in Remark 3.3 below that Theorem 1.2 is also sharp for rotationally invariant $\alpha$-stable Lévy processes. Roughly speaking, Theorems 1.1 and 1.2 show that the gradient estimate (1.4) for a Lévy process for small $t \ll 1$ depends on the asymptotic behaviour of the symbol $\Phi$ near infinity, while (1.4) for large $t \gg 1$ relies on the asymptotic behaviour of the symbol $\Phi$ near zero. This situation is familiar from estimates of the coupling time of Lévy processes. More details can be found in the examples given below.

In order to illustrate the power of Theorems 1.1 and 1.2 we present two examples.

**Example 1.3.** Let $X_t$ be a subordinate Brownian motion with symbol $f(|\xi|^2)$, where $f(\lambda) = \lambda^{\alpha/2} (\log(1 + \lambda))^{\beta/2}$, $\alpha \in (0, 2)$ and $\beta \in (-\alpha, 2 - \alpha)$. To see that $f$ is indeed
a Bernstein function we observe that \( \lambda, \log(1 + \lambda) \) and \( \lambda/\log(1 + \lambda) \) are complete Bernstein functions and that for \( \alpha, \beta > 0 \)

\[ \lambda^{\alpha/2} \cdot \left( \log(1 + \lambda) \right)^{\beta/2} \]

is a complete Bernstein function if \( \frac{\alpha}{2} + \frac{\beta}{2} \leq 1 \),

while for \(-\alpha \leq \beta \leq 0 \leq \alpha\)

\[ \lambda^{(\alpha-\beta)/2} \cdot \left( \frac{\lambda}{\log(1 + \lambda)} \right)^{\beta/2} \]

is a complete Bernstein function if \( \frac{\alpha}{2} + \frac{\beta}{2} \leq 1 \).

This follows easily from [22, (Proof of) Proposition 7.10], see also [24, Examples 5.15, 5.16].

For all \( x, y \in \mathbb{R}^d \),

\[ \| P_t(x, \cdot) - P_t(y, \cdot) \|_{\text{Var}} = O \left( t^{-1/(\alpha+\beta)} \right) \text{ as } t \to \infty, \]

and there exists \( c > 0 \) such that for all \( u \in B_b(\mathbb{R}^d) \),

\[ \| \nabla P_t u \|_{\infty} \leq \begin{cases} 
  c \| u \|_{\infty} t^{-1/2} & \text{for small } t \ll 1; \\
  c t^{-1/(\alpha+\beta)} \| u \|_{\infty} & \text{for large } t \gg 1.
\end{cases} \]

**Example 1.4.** Let \( \mu \) be a finite nonnegative measure on the unit sphere \( S \) and assume that \( \mu \) is nondegenerate in the sense that its support is not contained in any proper linear subspace of \( \mathbb{R}^d \). Let \( \alpha \in (0, 2), \beta \in (0, \infty) \) and assume that the Lévy measure \( \nu \) satisfies that for some constant \( r_0 > 0 \) and any \( A \in \mathcal{B}(\mathbb{R}^d) \),

\[ \nu(A) \geq \int_0^{r_0} \int_S \mathbb{1}_A(s\theta) s^{-1-\alpha} ds \mu(d\theta) + \int_{r_0}^{\infty} \int_S \mathbb{1}_A(s\theta) s^{-1-\beta} ds \mu(d\theta). \]

Then, for all \( x, y \in \mathbb{R}^d \),

\[ \| P_t(x, \cdot) - P_t(y, \cdot) \|_{\text{Var}} = O \left( t^{-1/(\beta^2)} \right) \text{ as } t \to \infty, \]

and there exists a constant \( c > 0 \) such that for all \( u \in B_b(\mathbb{R}^d) \),

\[ \| \nabla P_t u \|_{\infty} \leq \begin{cases} 
  c \| u \|_{\infty} t^{-1/\alpha} & \text{for small } t \ll 1; \\
  c \| u \|_{\infty} t^{-1/(\beta^2)} & \text{for large } t \gg 1.
\end{cases} \]

The remaining part of this paper is organized as follows. In Section 2 we first present estimates for the derivatives of the density for infinitely divisible distributions in terms of the corresponding Lévy measure; this part is of some interest on its own. Then we use these estimates to investigate derivatives of the density for Lévy processes, whose Lévy measures have (modified) bounded support. In Section 3 we give the proofs of all the theorems and examples stated in Section 1, by using the results of Section 2. Some remarks and examples are also included here to illustrate the optimality and the efficiency of Theorems 1.1 and 1.2.
2. Derivatives of Densities for Infinitely Divisible Distributions

Let $\pi$ be an infinitely divisible distribution. It is well known that its characteristic function $\hat{\pi}(\xi) := \int_{\mathbb{R}^d} e^{i\xi y} \pi(dy)$ is of the form $\exp(-\Phi(\xi))$, where

$$\Phi(\xi) = \int_{y \neq 0} \left(1 - e^{i\xi y} + i\xi \cdot y \mathbb{1}_{B(0,1)}(y)\right) \nu(dy),$$

and $\nu$ is a Lévy measure on $\mathbb{R}^d \setminus \{0\}$ such that $\int_{y \neq 0} (1 \land |y|^2) \nu(dy) < \infty$. In this section we first aim to study estimates for derivatives of the density of $\pi$. As usual, we denote for every $n \in \mathbb{N}_0$ by $C^n_b(\mathbb{R}^d)$ the set of all $n$-times continuously differentiable functions on $\mathbb{R}^d$ which are, together with all their derivatives, bounded; for $n = 0$ we use the convention that $C^0_b(\mathbb{R}^d) = C_b(\mathbb{R}^d)$ denotes the set of continuous and bounded functions on $\mathbb{R}^d$.

**Proposition 2.1.** If for some $n$, $m \in \mathbb{N}_0$,

$$\int e^{-\text{Re} \Phi(\xi)} (1 + |\xi|)^{n+m} \, d\xi < \infty,$$  \hspace{1cm} (2.5)

and

$$\int_{|y| > 1} |y|^{2vn} \nu(dy) < \infty,$$  \hspace{1cm} (2.6)

then $\pi$ has a density $p \in C^{n+m}_b(\mathbb{R}^d)$ such that for every $\beta \in \mathbb{N}_0^d$ with $|\beta| \leq m$,

$$|\partial^\beta p(y)| \leq \psi(n, m, \nu)(1 + |y|)^{-n}, \quad y \in \mathbb{R}^d,$$

where for $n \geq 0$

$$\psi(n, m, \nu) = C(n, d) \left(1 + \int_{|y|^2 + |y|^{2vn}} \nu(dy)\right)^n \int e^{-\text{Re} \Phi(\xi)} (1 + |\xi|)^{n+m} \, d\xi.$$

**Proof of Proposition 2.1.** The existence of the density $p \in C^{n+m}_b(\mathbb{R}^d)$ is a consequence of (2.5) and [19, Proposition 28.1] or [16, Proposition 0.2].

To prove the second assertion, we recall some necessary facts and notations. Given a function $f \in L^1(\mathbb{R}^d)$, its Fourier transform is given by

$$\hat{f}(\xi) = \int_{\mathbb{R}^d} f(y)e^{i\xi y} \, dy.$$

For $\xi \in \mathbb{R}^d$ and a multiindex $\beta = (\beta_1, \beta_2, \ldots, \beta_d) \in \mathbb{N}_0^d$, we set $M_{\beta}(\xi) := \xi^\beta = \xi_1^{\beta_1} \xi_2^{\beta_2} \cdots \xi_d^{\beta_d}$. If $\hat{f} \in C^N(\mathbb{R}^d)$ and $\partial^\gamma (M_{\beta} \hat{f}) \in L^1(\mathbb{R}^d)$ for $N \in \mathbb{N}_0$ and every $\gamma \in \mathbb{N}_0^d$ such that $|\gamma| \leq N$, then, using the inverse Fourier transform and the integration by parts formula, we obtain that for every $\delta \in \mathbb{N}_0^d$ with $|\delta| \leq N$

$$y^\delta \partial^\delta f(y) = (2\pi)^{-d} (-1)^{|\beta|} (i)^{|\beta| - |\delta|} \int \partial^\delta \left[M_{\beta} \hat{f}\right](\xi) e^{-iy \cdot \xi} \, d\xi.$$

This yields

$$|y^\delta \partial^\delta f(y)| \leq (2\pi)^{-d} \int \left|\partial^\delta \left[M_{\beta} \hat{f}\right](\xi)\right| \, d\xi.$$  \hspace{1cm} (2.7)

In particular, for every $n \in \mathbb{N}_0$,

$$|y_k^n \partial^\delta f(y)| \leq (2\pi)^{-d} \int \left|\frac{\partial^n}{\partial \xi_k^n} \left[M_{\beta} \hat{f}\right](\xi)\right| \, d\xi.$$  \hspace{1cm} (2.8)
For $n = 0$, the required assertion immediately follows from (2.7) if we use $f = p$ and $\delta = 0$. If $n > 0$, then for every $\beta \in \mathbb{N}_d^*$ such that $|\beta| = 1$ we have

$$\partial^\beta \Phi(\xi) = -i \int y^\beta (e^{i\xi \cdot y} - 1_{B(0,1)}(y)) \nu(dy).$$

By the Hölder inequality,

$$|\partial^\beta \Phi(\xi)| \leq \left[ \int |y|^2 \nu(dy) \right]^{1/2} \left[ 2 \int_{B(0,1)} (1 - \cos(\xi \cdot y)) \nu(dy) + \nu(B(0,1)^c) \right]^{1/2} \tag{2.9}$$

$$\leq \int |y|^2 \nu(dy) \cdot (|\xi|^2 + 1)^{1/2}$$

$$\leq (1 + |\xi|) \int |y|^2 \nu(dy).$$

On the other hand, for $1 < |\beta| \leq n$, we have

$$\partial^\beta \Phi(\xi) = -(i)^{|\beta|} \int y^\beta e^{i\xi \cdot y} \nu(dy),$$

and so

$$|\partial^\beta \Phi(\xi)| \leq \int |y|^{|\beta|} \nu(dy). \tag{2.10}$$

For symmetric Lévy measures $\nu$ similar estimates are due to Hoh [10], see also [12, Theorem 3.7.13].

Let $k \in \{1, \ldots, d\}$ and $M \in \mathbb{N}$ with $M \leq n$. We use Faa di Bruno’s formula, see [6], to obtain

$$\frac{\partial^M}{\partial \xi_k^M} \tilde{p}(\xi) = M! \exp(-\Phi(\xi)) \sum_{j=1}^{M} \sum_{u(M,j)} \prod_{l=1}^{M} \frac{\left( \frac{\partial^l \Phi(\xi)}{\partial \xi_l^l} \right)^{\lambda_l}}{(\lambda_l!)^{(l!)^{\lambda_l}}},$$

where

$$u(M,j) = \left\{ (\lambda_1, \ldots, \lambda_M) : \lambda_l \in \mathbb{N}_0, \sum_{l=1}^{M} \lambda_l = j, \sum_{l=1}^{M} l\lambda_l = M \right\}.$$
This, (2.9) and (2.10) yield
\[
\left| \frac{\partial^M}{\partial \xi^M_k} \hat{\mu}(\xi) \right| \leq |\exp(-\Phi(\xi))| \sum_{j=1}^{M} \sum_{\lambda(M,j)}^{M} \frac{M!}{(\lambda_l)!} \left( 1 + |\xi| \right) \int_{\mathbb{R}^d} |y|^{1+2n} \nu(dy) \right)^{\lambda_l} \\
\leq e^{-Re \Phi(\xi)} \sum_{j=1}^{M} \left( 1 + |\xi| \right) \int_{\mathbb{R}^d} (|y|^2 + |y|^{2\nu(n)}) \nu(dy) \right)^{j} \sum_{\lambda(M,j)}^{M} \frac{M!}{(\lambda_l)!} \left( 1 + |\xi| \right) \int_{\mathbb{R}^d} (|y|^2 + |y|^{2\nu(n)}) \nu(dy) \right)^{j} \\
\leq c_1(n) e^{-Re \Phi(\xi)} (1 + |\xi|)^{n+m} \left[ 1 + \int_{\mathbb{R}^d} (|y|^2 + |y|^{2\nu(n)}) \nu(dy) \right]^n.
\]

We note that this inequality remains valid for \( M = 0. \)

For \( \beta \in \mathbb{N}_0^d \) with \( |\beta| \leq m \), we can use the Leibniz rule to get
\[
\left| \frac{\partial^n}{\partial \xi^n_k} (M_\beta \hat{\mu}(\xi)) \right| \leq \sum_{j=0}^{n} \binom{n}{j} \left| \frac{\partial^j}{\partial \xi^j_k} M_\beta(\xi) \frac{\partial^{n-j}}{\partial \xi^{n-j}_k} \hat{\mu}(\xi) \right| \\
\leq (1 + |\xi|)^{|\beta|} \sum_{j=0}^{n} \binom{n}{j} \left| \frac{\partial^{n-j}}{\partial \xi^{n-j}_k} \hat{\mu}(\xi) \right| \\
\leq c_2(n) e^{-Re \Phi(\xi)} (1 + |\xi|)^{n+m} \left[ 1 + \int_{\mathbb{R}^d} (|y|^2 + |y|^{2\nu(n)}) \nu(dy) \right]^n.
\]

By (2.8), we see
\[
|y|^n |\partial^\beta p(y)| \leq c_2(n) \left[ 1 + \int_{\mathbb{R}^d} (|y|^2 + |y|^{2\nu(n)}) \nu(dy) \right]^n \int e^{-Re \Phi(\xi)} (1 + |\xi|)^{n+m} d\xi.
\]

Finally,
\[
(1 + |y|)^n \leq 2^{n-1}(1 + |y|^n) \leq 2^{n-1}d^{n/2} \left( 1 + \sum_{k=1}^{d} |y_k|^n \right),
\]
and the required assertion follows with \( C(n, d) = 2^{n-1}d^{n/2}(d + 1)c_2(n). \)

We will now study the derivatives of transition densities for Lévy processes with (modified) bounded support. For this we need Proposition 2.1. Let \( \Phi \) be the symbol (i.e. the characteristic exponent) of a Lévy process and consider for every \( r > 0 \) the semigroup of infinitely divisible measures \( \{P^t, t \geq 0\} \) whose Fourier transform is of the form \( \hat{P}^t(\xi) = \exp(-t\Phi_r(\xi)) \), where
\[
\Phi_r(\xi) = \int_{|y| \leq r} \left( 1 - e^{\xi \cdot y} + i \xi \cdot y \right) \nu(dy),
\]
(\( \nu \) is the Lévy measure of the symbol \( \Phi \)). For \( \rho > 0 \) and \( t > 0 \), we define
\[
(2.11) \quad \varphi(\rho) = \sup_{|\eta| \leq \rho} Re \Phi(\eta) \quad \text{and} \quad h(t) := \frac{1}{\varphi^{-1}(1/t)}.
\]

**Proposition 2.2.** Assume that (1.3) holds, and for some \( m \in \mathbb{N}_0 \),
\[
(2.12) \quad \int \exp \left( -t Re \Phi(\xi) \right) |\xi|^m d\xi = O \left( \varphi^{-1} \left( \frac{1}{t} \right)^{m+d} \right) \quad \text{as} \quad t \to \infty.
\]
Then there is a \( t_0 = t_0(m, d) > 0 \) such that for any \( t \geq t_0 \), there exists a density \( p_t^{h(t)} \in C^m_b(\mathbb{R}^d) \) of \( P_t^{h(t)} \), and for every \( n \in \mathbb{N}_0 \) and \( \beta \in \mathbb{N}_0^d \) with \( |\beta| \leq m - n \),

\[
|\partial_\beta p_t^{h(t)}(y)| \leq c(m, n, |\beta|, \Phi) \left( \varphi^{-1}(t^{-1}) \right)^{d+|\beta|} \left( 1 + \varphi^{-1}(t^{-1})|y| \right)^{-n}, \quad y \in \mathbb{R}^d.
\]

**Proof.** Step 1. For \( \xi \in \mathbb{R}^d \),

\[
|\hat{\hat{P}}_t(\xi)| = \exp \left( -t \int_{|y|<r} (1 - \cos(\xi \cdot y)) \nu(dy) \right)
\]

\[
= \exp \left( -t \left( \Re \Phi(\xi) - \left( \int_{|y|\geq r} (1 - \cos(\xi \cdot y)) \nu(dy) \right) \right) \right)
\]

\[
\leq \exp \left( -t \left( \Re \Phi(\xi) \right) \right) \exp \left( 2t\nu(B(0, r)^c) \right).
\]

By (1.3) and [19, Proposition 28.1], it follows that there exists \( t_1 := t_1(d) > 0 \) such that for all \( r > 0 \) and for any \( t \geq t_1 \), the measure \( P_t^r \) has a density \( p_t^r \in C_b(\mathbb{R}^d) \).

Step 2. For \( t \geq t_1 \), we define \( g_t(y) = h(t)^d p_t^{h(t)}(h(t)y) \). We consider the infinitely divisible distribution \( \pi_t(dy) = g_t(y)dy \). Its Fourier transform is given by

\[
\hat{\pi}_t(\xi) = (h(t))^d \int e^{i\xi \cdot y} p_t^{h(t)}(h(t)y) dy
\]

\[
= \int e^{i\xi \cdot y/h(t)} p_t^{h(t)}(y) dy
\]

\[
= \exp \left( -t \int_{|y|\leq h(t)} \left( 1 - e^{i\xi \cdot y/h(t)} + \frac{i\xi \cdot y}{h(t)} \right) \nu(dy) \right)
\]

\[
= \exp \left( -t \int_{|y|\leq 1} \left( 1 - e^{i\xi \cdot y} + i\xi \cdot y \right) \lambda_t(dy) \right)
\]

\[
= \exp \left( -G_t(\xi) \right),
\]

where \( \lambda_t \) is the Lévy measure of \( \pi_t \), i.e. for any Borel set \( B \subset \mathbb{R}^d \setminus \{0\} \),

\[
\lambda_t(B) = t \int_{|y|\leq h(t)} 1_B(y/h(t)) \nu(dy).
\]
For $n \geq 2$, we have

$$
\int |y|^n \lambda_t(dy) = t \int_{|y| \leq h(t)} \left( \frac{|y|}{h(t)} \right)^n \nu(dy)
\leq t \int_{|y| \leq h(t)} \left( \frac{|y|}{h(t)} \right)^2 \nu(dy)
\leq 2t \int_{|y| \leq h(t)} \frac{(|y|/h(t))^2}{1 + (|y|/h(t))^2} \nu(dy)
\leq 2t \int \frac{(|y|/h(t))^2}{1 + (|y|/h(t))^2} \nu(dy)
= 2t \int \left( 1 - \cos \left( \frac{y}{h(t)} \cdot \xi \right) \right) f_d(\xi) d\xi \nu(dy)
= 2t \int \left( 1 - \cos \left( \frac{y \cdot \xi}{h(t)} \right) \right) \nu(dy) f_d(\xi) d\xi
= 2t \int \Re \Phi \left( \frac{\xi}{h(t)} \right) f_d(\xi) d\xi,
$$

where

$$
f_d(\xi) = \frac{1}{2} \int_0^\infty (2\pi \rho)^{-d/2} e^{-|\xi|^2/(2\rho)} e^{-\rho/2} d\rho.
$$

Obviously, $f_d(\xi)$ possesses all moments, see e.g. [20, (2.5) and (2.6)].

By using several times the subadditivity of $\eta \mapsto \sqrt{\Re \Phi(\eta)}$, we can easily find, see e.g. the proof of [21, Lemma 2.3],

$$
\Re \Phi \left( \frac{\xi}{h(t)} \right) \leq 2 \left( 1 + |\xi|^2 \right) \sup_{|\eta| \leq 1/h(t)} \Re \Phi(\eta) = 2 \left( 1 + |\xi|^2 \right)^{1/2}.
$$

So,

$$
\int \Re \Phi \left( \frac{\xi}{h(t)} \right) f_d(\xi) d\xi \leq 2 \sup_{|\eta| \leq 1/h(t)} \Re \Phi(\eta) \int (1 + |\xi|^2) f_d(\xi) d\xi =: \frac{c_0}{t}.
$$

According to the definition of $h(t)$, we get that for any $t > 0$,

$$
(2.15) \quad \int |y|^n \lambda_t(dy) \leq 2c_0.
$$

**Step 3.** It is easily seen from (2.14) that the characteristic exponent of $\pi_t$ is $G_t(\xi)$, and

$$
\Re G_t(\xi) = t \Re \left( \Phi_{h(t)}^{-1}(h(t)^{-1}\xi) \right).
$$

Thus,

$$
\Re G_t(\xi) = t \left[ \Re \left( \Phi(h(t)^{-1}\xi) \right) - \int_{|y| > h(t)} \left( 1 - \cos (h(t)^{-1}\xi \cdot y) \right) \nu(dy) \right]
\geq t \Re \left( \Phi(h(t)^{-1}\xi) \right) - 2t \nu(B(0, h(t)^c)).
$$
For any \( t > 0 \),
\[
\nu(B(0, h(t))^c) \leq 2 \int_{|y| > h(t)} \frac{(|y|/h(t))^2}{1 + |y|^2/h(t)^2} \nu(dy)
\]
\[
\leq 2 \int \frac{(|y|/h(t))^2}{1 + (|y|/h(t))^2} \nu(dy)
\]
\[
= 2 \iint \left(1 - \cos \left( h(t)^{-1} \xi \cdot y \right) \right) \nu(dy) f_d(\xi) \, d\xi
\]
\[
= 2 \int \Re \Phi \left( \frac{\xi}{h(t)} \right) f_d(\xi) \, d\xi
\]
\[
\leq 2c_0 \sup_{|y| \leq 1/h(t)} \Re \Phi(\eta),
\]
where the last two lines follow from the same arguments as those leading to (2.15). Hence, for any \( t > 0 \), we have
\[
t \nu(B(0, h(t))^c) \leq 2c_0 \, t \sup_{|y| \leq 1/h(t)} \Re \Phi(\eta) = 2c_0.
\]
By (2.12), for \( m \in \mathbb{N}_0 \) and \( c_1 > 0 \), there exists \( t_0 := t_0(m, \Phi, c_1) > t_1 \) such that for any \( t \geq t_0 \),
\[
\int \exp \left(-t \Re \Phi(\xi)\right) |\xi|^m \, d\xi \leq c_1 h(t)^{-(m+d)}.
\]
Therefore, we obtain
\[
\int \exp \left[-\Re(G_t(\xi))\right] |\xi|^m \, d\xi \leq e^{4c_0} \int \exp \left[-t \Re \left( \frac{\Phi(\xi/h(t))}{\eta} \right)\right] |\xi|^m \, d\xi
\]
\[
= e^{4c_0} h(t)^{m+d} \int \exp \left[-(t \Re \Phi(\xi))\right] |\xi|^m \, d\xi
\]
\[
= c_1 e^{4c_0} < \infty.
\]

**Step 4.** According to (2.15), (2.16) and Proposition 2.1, \( g_t \in C^m_b(\mathbb{R}^d) \) for any \( t \geq t_0 \), and for every \( n \in \mathbb{N}_0 \) and \( \beta \in \mathbb{N}^d_0 \) with \(|\beta| \leq m - n\) we get
\[
|\partial^\beta_y g_t(y)| \leq c(m, n, |\beta|, \Phi)(1 + |y|)^{-n}, \quad y \in \mathbb{R}^d.
\]
This finishes the proof since \( \partial^\beta_y g_t(y) = h(t)^{d+|\beta|} \partial^\beta_y p_t^{h(t)}(h(t)y) \).

The following result is the counterpart of Proposition 2.2, which presents estimates for the derivatives of the densities \( p_t^{h(t)} \) for small time. Recall the definitions of \( \varphi \) and \( h \) from (2.11).

**Proposition 2.3.** Assume that (1.1) is satisfied, and for some \( m \in \mathbb{N}_0 \),
\[
\int \exp \left[-(t \Re \Phi(\xi))\right] |\xi|^m \, d\xi = O \left( \varphi^{-1}(1/t)^{m+d} \right) \quad \text{as} \quad t \to 0.
\]
Then there is some \( t_0 > 0 \) such that for all \( t \leq t_0 \), there exists a density \( p_t^{h(t)} \in C^m_b(\mathbb{R}^d) \) of \( F_t^{h(t)} \). Moreover, for every \( n \in \mathbb{N}_0 \) and \( \beta \in \mathbb{N}^d_0 \) with \(|\beta| \leq m - n\),
\[
|\partial^\beta_y p_t^{h(t)}(y)| \leq c(m, n, |\beta|, \Phi, t_0) \left( \varphi^{-1}(t^{-1}) \right)^{d+|\beta|} \left( 1 + \varphi^{-1}(t^{-1}) |y| \right)^{-n}, \quad y \in \mathbb{R}^d.
\]
Proof. The proof is similar to that of Proposition 2.2, and we only sketch some key differences. We continue to use the notations of the proof of Proposition 2.2. According to (2.13) and (1.1), for all \( r > 0 \) and \( t > 0 \), the measure \( P^r_t \) is absolutely continuous with respect to Lebesgue measure. Denote by \( P' \) its density. Following the argument of Proposition 2.2, we find that (2.15) is still valid, and according to (2.17), there exists some \( t_0 > 0 \) such that (2.16) holds for all \( 0 < t \leq t_0 \). The required assertion follows now from Proposition 2.1. \( \square \)

3. Proofs of the Main Theorems and Further Examples

We will now give the proofs for Theorem 1.1 and 1.2. For this we need to estimate the coupling time of a general Lévy process. We will use the functions \( \varphi \) and \( h \) defined in (2.11).

**Theorem 3.1.** Assume that (1.3) holds, and

\[
\int \exp \left[ - (t \text{Re} \Phi(\xi)) \right] |\xi|^{d+2} d\xi = O \left( \varphi^{-1}(1/t)^{2d+2} \right) \quad \text{as} \quad t \to \infty.
\]

Then, the Lévy process \( X_t \) has the coupling property, and for any \( x, y \in \mathbb{R}^d \),

\[
\|P_t(x, \cdot) - P_t(y, \cdot)\|_{\text{Var}} = O \left( \varphi^{-1}(1/t) \right) \quad \text{as} \quad t \to \infty.
\]

**Proof.** Set

\[
\Phi_r(\xi) = \int_{|y| \leq r} \left( 1 - e^{i\xi \cdot y} + i\xi \cdot y \right) \nu(dy)
\]

and

\[
\Psi_r(\xi) := \Phi(\xi) - \Phi_r(\xi) = \int_{|y| > r} \left( 1 - e^{i\xi \cdot y} \right) \nu(dy) - i\xi \cdot \int_{1 < |y| \leq r} y \nu(dy).
\]

Let \( Y_t \) and \( Z_t \) be two independent Lévy processes whose symbols are \( \Phi_r(\xi) \) and \( \hat{\Phi}_r(\xi) \), respectively. Denote by \( Q_t \) and \( Q_t(x, \cdot) \) the semigroup and the transition function of \( Y_t \). Similarly, \( R_t \) and \( R_t(x, \cdot) \) stand for the semigroup and the transition function of \( Z_t \). Then,

\[
\|P_t(x, \cdot) - P_t(y, \cdot)\|_{\text{Var}} = \sup_{\|f\|_{\infty} \leq 1} \|P_t f(x) - P_t f(y)\|
\]

\[
= \sup_{\|f\|_{\infty} \leq 1} \left| Q_t R_t f(x) - Q_t R_t f(y) \right|
\]

\[
\leq \sup_{\|g\|_{\infty} \leq 1} \left| Q_t g(x) - Q_t g(y) \right|
\]

\[
= \|Q_t(x, \cdot) - Q_t(y, \cdot)\|_{\text{Var}}.
\]

Now, we take \( r = h(t) \). Then, according to (3.18) and Proposition 2.2, there exists \( t_0 > 0 \) such that for any \( t \geq t_0 \), the kernel \( Q_t \) has a density \( q_t \in C^{d+2}_{b}(\mathbb{R}^d) \), and for all \( y \in \mathbb{R}^d \),

\[
|\nabla q_t(y)| \leq c(d, \Phi) h(t)^{-(d+1)} \left( 1 + h(t)^{-1} |y| \right)^{-(d+1)}.
\]

\[\text{We prove here actually that } \|P_t(x, \cdot) - P_t(y, \cdot)\|_{\text{Var}} \leq c|x - y|/h(t) \text{ and in this form we use it later in the proof of Theorem 3.2. Maybe it would be better to write the estimate explicitly here.}\]
Thus, for any $t \geq t_0$,
\begin{equation}
\|Q_t(x, \cdot) - Q_t(y, \cdot)\|_{\mathcal{V}_{\var}} = \sup_{\|f\|_{\infty} \leq 1} \left| \int f(z) Q_t(x, dz) - \int f(z) Q_t(y, dz) \right|
\end{equation}

(3.22)
\begin{align*}
&= \sup_{\|f\|_{\infty} \leq 1} \left| \int f(z) q_t(z - x) \, dz - \int f(z) q_t(z - y) \, dz \right| \\
&= \int |q_t(z - x) - q_t(z - y)| \, dz.
\end{align*}

Let $t \geq t_0$. Assume that $|x - y| > h(t)$. Then,
\[
\int |q_t(z - x) - q_t(z - y)| \, dz \leq 2 \int |q_t(z - x) - q_t(z - y)| \, dz \leq \frac{2|x - y|}{h(t)}.
\]

If $|x - y| \leq h(t)$, then, by (3.21),
\[
\int |q_t(z - x) - q_t(z - y)| \, dz \\
= \int_{|z-x|>2h(t)} |q_t(z - x) - q_t(z - y)| \, dz + \int_{|z-x|\leq 2h(t)} |q_t(z - x) - q_t(z - y)| \, dz \\
\leq c(d, \Phi) \frac{|x - y|}{h(t)^{d+1}} \int_{|z-x|>2h(t)} \left[ h(t)^{d+1} \sup_{w \in B(z-x,|y-x|)} |\nabla q_t(w)| \right] \, dz + \int_{|z-x|\leq 2h(t)} dz \\
\leq c(d, \Phi) \frac{|x - y|}{h(t)^{d+1}} \int \left[ 1 + \left( \frac{|z-x|}{2h(t)} \right)^{d-1} \right] \, dz + c_d (2h(t))^d \\
= c(d, \Phi) \frac{|x - y|}{h(t)^{d+1}} \int \left( 1 + \frac{|z-x|}{2h(t)} \right)^{d-1} \, dz + 2^d c_d \frac{|x - y|}{h(t)} \\
\leq \frac{C |x - y|}{h(t)}.
\]

Therefore, there exists $C > 0$ such that for all $x, y \in \mathbb{R}^d$ and $t \geq t_0$,
\begin{equation}
(3.23) \quad \int |q_t(z - x) - q_t(z - y)| \, dz \leq \frac{C |x - y|}{h(t)}.
\end{equation}

The assertion follows now from (3.20), (3.22) and (3.23). \hfill \Box

Next, we turn to the proof of Theorem 1.1.

Proof of Theorem 1.1. Under the conditions of Theorem 1.1 we can suppose that for any $\xi \in \mathbb{R}^d$,
\[\Re \Phi(\xi) \geq F(|\xi|),\]
where $F(r)$ is a strictly increasing and differentiable function on $(0, \infty)$ such that
\[F(r) = \begin{cases} \frac{c_1 f(r)}{c_3 \log(c_4 + c_5 r)} & \text{if } r \in (0, c_2); \\
\frac{c_3 \log(c_4 + c_5 r)}{c_3 \log(c_4 + c_5 r)} & \text{if } r \in [c_2, \infty),\end{cases}\]
for some constants $c_i > 0$, $i = 1, 2, 3, 4, 5$. Thus,
\[ \int \exp \left[ -t \Re \Phi(\xi) \right] |\xi|^{d+2} \, d\xi \leq \int \exp \left[ -t F(|\xi|) \right] |\xi|^{d+2} \, d\xi \]
\[ = c_d \int_0^\infty e^{-tr} \left[ F^{-1}(r) \right]^{2d+1} \, dF^{-1}(r) \]
\[ = \frac{c_d}{2(d+1)} \int_0^\infty e^{-tr} \left[ F^{-1}(r) \right]^{2(d+1)}. \]

Since $\lim \inf_{r \to 0} f(r) |\log r| < \infty$ and $\lim \sup_{s \to 0} f^{-1}(2s)/f^{-1}(s) < \infty$, we have
\[ \lim \inf_{r \to 0} F(r) |\log r| < \infty \]
and
\[ \lim \sup_{s \to 0} F^{-1}(2s)/F^{-1}(s) < \infty. \]

Note that the function $F$ also satisfies
\[ \lim_{s \to \infty} F(s)/\log s = c_3. \]

Then, following the proof of [1, Theorem 2.1], we obtain that for $t \to \infty$,
\[ \int_0^\infty e^{-tr} \left[ F^{-1}(r) \right]^{2(d+1)} \propto \left[ F^{-1}(1/t) \right]^{2(d+1)} \]
\[ = \left[ f^{-1}(1/t) \right]^{2(d+1)} \propto \left[ \varphi^{-1}(1/t) \right]^{2(d+1)}. \]

The desired assertion follows from Theorem 3.1.

The following result is the short-time analogue of Theorem 3.1 which gives, additionally, gradient estimates for general Lévy processes.

**Theorem 3.2.** Assume that (1.1) holds and let $\phi$ and $h$ be as in (2.11). If
\[ (3.24) \int \exp \left( -t \Re \Phi(\xi) \right) |\xi|^{d+2} \, d\xi = O \left( \varphi^{-1}(1/t)^{2d+2} \right) \quad \text{as} \quad t \to 0 \quad [t \to \infty] \]
then there exists a constant $c > 0$ such that for all $u \in B_b(\mathbb{R}^d)$
\[ (3.25) \|\nabla P_t u\|_\infty \leq c \|u\|_\infty \left( \varphi^{-1}(1/t) \right)^{d+1} \quad \text{for all} \quad t \ll 1 \quad [t \gg 1]. \]

**Proof.** We will treat the short- and large-time cases separately.

Recall the notations used in the proof of Theorem 3.1: $Q_t$ and $R_t$ are the semigroups corresponding to $\Phi_t(\xi)$ and $\Psi_t(\xi)$, respectively. According to (3.24) and Proposition 2.3, for small enough $t \ll 1$, and $r = h(t)$, the measure $Q_t$ has a density $q_t \in C_b^{d+2}(\mathbb{R}^d)$ such that for any $y \in \mathbb{R}^d$,
\[ (3.26) \|\nabla q_t(y)\| \leq c(d, \Phi) \left( \varphi^{-1}(t^{-1}) \right)^{d+1} (1 + \varphi^{-1}(t^{-1})|y|)^{-(d+1)}. \]

---

2 I added $c_d$ here.

3 I replaced 1 by $c_3$ here.
Then, for all \( u \in B_b(\mathbb{R}^d) \),

\[
\sup_{\|u\|_\infty \leq 1} \| \nabla Q_t u \|_\infty = \sup_{\|u\|_\infty \leq 1} \sup_{x \in \mathbb{R}^d} | \nabla Q_t u(x) | \\
= \sup_{\|u\|_\infty \leq 1} \sup_{x \in \mathbb{R}^d} \nabla \int q_t(z - x) \cdot u(z) \, dz \\
= \sup_{x \in \mathbb{R}^d} \sup_{\|u\|_\infty \leq 1} \left| \int \nabla q_t(z - x) \cdot u(z) \, dz \right| \\
= \int \left| \nabla q_t(z) \right| \, dz \\
\leq c \varphi^{-1}(t^{-1}),
\]

(3.27)

where we used (3.26) and dominated convergence. This calculation shows

\[
\| \nabla Q_t u \|_\infty \leq c \varphi^{-1}(t^{-1}) \| u \|_\infty.
\]

Therefore,

\[
\| \nabla P_t u \|_\infty = \| \nabla Q_t (R_t u) \|_\infty \leq c \varphi^{-1}(t^{-1}) \| R_t u \|_\infty \leq c \varphi^{-1}(t^{-1}) \| u \|_\infty,
\]

(3.28)

which finishes the proof for small \( t \ll 1 \).

If \( t \gg 1 \) is sufficiently large, we can apply (3.19), to find for any \( u \in B_b(\mathbb{R}^d) \) with \( \| u \|_\infty = 1 \),

\[
| \nabla P_t u(x) | \leq \limsup_{y \to x} \frac{| P_t u(x) - P_t u(y) |}{| y - x |} \\
\leq \limsup_{y \to x} \frac{\sup_{\|w\|_\infty \leq 1} | P_t w(x) - P_t w(y) |}{| y - x |} \\
\leq \limsup_{y \to x} \frac{\| P_t (x, \cdot) - P_t (y, \cdot) \|_{\text{Var}}}{| y - x |} \\
\leq C \varphi^{-1}(t^{-1}).
\]

This finishes the proof for large \( t \gg 1 \). \( \square \)

**Remark 3.3.** Let \( X_t \) be a rotationally invariant \( \alpha \)-stable Lévy process on \( \mathbb{R}^d \), and \( p_t \) be its density function. By the scaling property, for any \( t > 0 \) and \( x \in \mathbb{R}^d \), \( p_t(x) = t^{-d/\alpha} p_1(t^{-1/\alpha} x) \). On the other hand, it is well known that, see e.g. [8],

\[
| \nabla p_t(x) | \leq \frac{c}{1 + |x|^{d+\alpha}}.
\]

Denote by \( P_t \) the semigroup of \( X_t \). Then, according to the proof of (3.27), we have

\[
\sup_{\|u\|_\infty \leq 1} \| \nabla P_t u \|_\infty = t^{-1/\alpha} \int | \nabla p_1(z) | \, dz.
\]

This implies that Theorem 1.2 is optimal.

We can now use Theorem 3.2 to prove Theorem 1.2.
Proof of Theorem 1.2. The second assertion easily follows from Theorem 1.1 and the proof of Theorem 3.2. It is therefore enough to consider the first conclusion. Under the conditions assumed in Theorem 1.2, we know that for any $\xi \in \mathbb{R}^d$,

$$\Re \Phi(\xi) \geq F(|\xi|),$$

where $F(r)$ is an increasing function on $(0, \infty)$ such that

$$F(r) = \begin{cases} 
0 & \text{if } r \in (0, c_1]; \\
c_2 f(r) & \text{if } r \in (c_1, \infty),
\end{cases}$$

for some constants $c_i > 0$, $i = 1, 2$, and $f$ is strictly increasing and differentiable on $(c_1, \infty)$. Therefore,

$$\int \exp \left[ -t \Re \Phi(\xi) \right] |\xi|^{d+2} d\xi \leq \int \exp \left[ -t F(|\xi|) \right] |\xi|^{d+2} d\xi.$$

Since $\limsup_{s \to \infty} f^{-1}(2s)/f^{-1}(s) < \infty$, we can choose $c > 2$ and $s_0 > 0$ such that we have $f^{-1}(2s) \leq cf^{-1}(s)$ for all $s \geq s_0$. For any $k \geq 1$ the monotonicity of $f^{-1}$ gives

$$f^{-1}(2^k s) \leq c^k f^{-1}(s) = 2^{k\alpha} f^{-1}(s),$$

where we use $\alpha = \log_2 c$. Then, for sufficiently small $t \ll 1$,

$$\int \exp \left[ -t F(|\xi|) \right] |\xi|^{d+2} d\xi
\leq \begin{array}{l}
\int_{|\xi| < c_1} |\xi|^{d+2} d\xi + c_d \int_{c_1}^{\infty} e^{-c_2 f(r)} r^{2d+1} dr \\
\leq C_1 + c_d \int_{0}^{\infty} e^{-c_2 s} d_s \left[ f^{-1} ((s/t)) \right]^{2d+2} \\
\leq C_1 + c_d \left\{ \int_{0}^{1} + \sum_{n=1}^{\infty} \int_{2^{n-1}}^{2^n} \right\} e^{-c_2 s} d_s \left[ f^{-1} ((s/t)) \right]^{2d+2} \\
\leq C_1 + c_d \left[ f^{-1} (1/t) \right]^{2d+2} + c_d \sum_{n=1}^{\infty} \exp \left[ -c_2 2^{n-1} \right] \left[ f^{-1} (2^n/t) \right]^{2d+2} \\
\leq C_1 + c_d \left[ 1 + \sum_{n=1}^{\infty} \exp \left[ -c_2 2^{n-1} \right] 2^{n\alpha(2d+2)} \right] \left[ f^{-1} (1/t) \right]^{2d+2} \\
\leq C_1 + c_d \left[ f^{-1} (1/t) \right]^{2d+2}.
\end{array}$$

Because of (1.1) and $\Re \Phi(\xi) \sim f(|\xi|)$ as $|\xi| \to \infty$, we find $f^{-1} (1/t) \to \infty$ as $t \to 0$. Thus,

$$\int \exp \left[ -t \Re \Phi(\xi) \right] |\xi|^{d+2} d\xi \leq \int \exp \left[ -t F(|\xi|) \right] |\xi|^{d+2} d\xi
\leq C_5 \left[ f^{-1} (1/t) \right]^{2d+2}.$$

Now the assertion follows from Theorem 3.2. $\square$

Remark 3.4. If we assume in the statement of Theorem 1.2 that $f^{-1}(s) = s^\alpha \ell(s)$ for some $\alpha > 0$ and some positive function $\ell$ which is slowly varying at $\infty$—that is, $\lim_{s \to \infty} \ell(\lambda s)/\ell(s) = 1$ for every $\lambda > 0$, then standard Abelian and Tauberian
arguments (see, e.g. [2, Theorems 1.7.1 and 1.7.1’] or [7, Chapter XIII.5, Theorems 1 and 3]), we can obtain that
\[
\int \exp \left[ -t F(|\xi|) |\xi|^{d+2} \right] d\xi \asymp \left[ f^{-1} (1/t) \right]^{2d+2} \quad \text{as } t \to 0.
\]

Let us finally turn to the examples from Section 1.

**Proof of Example 1.3.** The symbol of the subordinate Brownian motion here satisfies
\[
\Re \Phi(\xi) \asymp |\xi|^\alpha + \beta \quad \text{as } |\xi| \to 0,
\]
and
\[
\Re \Phi(\xi) \asymp |\xi|^{\alpha \left( \log(1 + |\xi|) \right) + \beta / 2} \quad \text{as } |\xi| \to \infty.
\]

For \( r > 0 \), set \( f(r) = r^\alpha (\log(1 + r))^{\beta / 2} \) and \( g(r) = \left( r(\log(1 + r))^{-\beta / 2} \right)^{1/\alpha} \). Then, for \( r \to \infty \), we have
\[
f(g(r)) = r \left( \log(1 + r) \right)^{-\beta / 2} \left[ \log \left( 1 + \left( r(\log(1 + r))^{-\beta / 2} \right)^{1/\alpha} \right) \right]^{\beta / 2}
\]
\[
\asymp r \left( \log r - \beta \log \log r \right)^{\beta / 2} / \alpha 
\]
\[
\asymp r.
\]

This shows that \( f^{-1}(r) \asymp g(r) \) for \( r \to \infty \), and now Theorems 1.1 and 1.2 apply. \( \square \)

**Proof of Example 1.4.** Let \( Y_t \) and \( Z_t \) be Lévy processes whose Lévy measures are given by
\[
\nu^Y(A) := \int_0^\infty \int_S \mathbb{1}_A(s\theta) s^{-1-\alpha} ds \mu(d\theta) + \int_0^\infty \int_S \mathbb{1}_A(s\theta) s^{-1-\beta} ds \mu(d\theta)
\]
and
\[
\nu^Z(dz) := \nu(dz) - \nu^Y(dz) \geq 0,
\]
respectively. After some elementary calculations, we see that the symbol \( \Phi^Y \) of \( Y_t \) satisfies \( \Re \Phi^Y(\xi) \asymp |\xi|^\alpha \) as \( |\xi| \to \infty \) and \( \Re \Phi^Y(\xi) \asymp |\xi|^{\beta / 2} \) as \( |\xi| \to 0 \). Let \( P^Y_t(x, \cdot) \) and \( Q^Y_t(x, \cdot) \) denote the transition function and the semigroup of \( Y_t \). According to Theorems 1.1 and 1.2, we can prove the claim first for \( Y \) (if we replace in these Theorems \( \Phi, P_t \) and \( P_t(x, \cdot) \) by the corresponding objects \( \Phi^Y, P^Y_t \) and \( P^Y_t(x, \cdot) \)). To come back to the original process resp. semigroup we can now use (3.20) and (3.28).

Example 1.4 applies to a large number of interesting and important Lévy processes, whose Lévy measures is of the following polar coordinates form:
\[
\nu(A) = \int_0^\infty \int_S \mathbb{1}_A(s\theta) Q(\theta, s) ds \mu(d\theta);
\]
\( Q(\theta, s) \) is a nonnegative function on \( S \times (0, \infty) \). For instance, Example 1.4 is applicable for the following processes:

1. Stable Lévy processes ([25]):
\[
Q(\theta, s) \asymp s^{-\alpha-1},
\]
where \( \alpha \in (0, 2) \).
Layered stable processes ([11]):

\[ Q(\theta, s) \asymp s^{-\alpha}1_{(0,1]}(s) + s^{-\beta}1_{[1,\infty)}(s), \]

where \( \alpha \in (0, 2) \) and \( \beta \in (0, \infty) \).

Tempered stable processes ([18]):

\[ Q(\theta, s) \asymp s^{-\alpha-1}e^{-cs}, \]

where \( \alpha \in (0, 2) \) and \( c > 0 \).

Relativistic stable processes ([5, 3]):

\[ Q(\theta, s) \asymp s^{-\alpha-1}(1 + s)^{\frac{d-\alpha}{2}}e^{-s}, \]

where \( \alpha \in (0, 2) \).

Lamperti stable processes ([4]):

\[ Q(\theta, s) = s^{-\alpha-1}\exp(sf(\theta))s^{1+\alpha}(e^{s} - 1)^{1+\alpha}, \]

where \( \alpha \in (0, 2) \) and \( f : S \to \mathbb{R} \) such that \( \sup_{\theta \in S}f(\theta) < 1 + \alpha \).

Truncated stable processes ([13]):

\[ Q(\theta, s) \asymp s^{-\alpha-1}1_{(0,1]}(s), \]

where \( \alpha \in (0, 2) \).

Motivated by Example 1.4, we can present a short proof of (one part of) F.-Y. Wang’s result on explicit gradient estimates for the semigroup of a general Lévy processes.

**Theorem 3.5** (F.-Y. Wang [26, Theorem 1.1]). Let \( X_t \) be a Lévy process on \( \mathbb{R}^d \) with Lévy measure \( \nu \). Assume that there exists some \( r \in (0, \infty] \) such that

\[ \nu(dz) \geq |z|^{-d}f(|z|^{-2})1_{\{|z|\leq r\}}dz, \]

where \( f \) is Bernstein function such that

\[ \liminf_{r \to \infty} \frac{f(r)}{\log r} = \infty \quad \text{and} \quad \limsup_{s \to \infty} \frac{f^{-1}(2s)}{f^{-1}(s)} \in (0, \infty). \]

Then there exists a constant \( c > 0 \) such that for any \( t > 0 \),

\[ \|\nabla P_t u\|_{\infty} \leq c\|u\|_{\infty} f^{-1}\left(\frac{1}{t \wedge 1}\right), \quad u \in B_b(\mathbb{R}^d). \]

**Proof.** According to the proofs of Example 1.4 and [26, Theorem 1.1], we see that \( X_t \) can be decomposed into two independent Lévy processes \( Y_t \) and \( Z_t \), such that the symbol \( \Phi^Y(\xi) \) of \( Y_t \) satisfies \( \Phi^Y(\xi) \asymp f(|\xi|^2) \) as \( |\xi| \to \infty \). Now we can apply Theorem 1.2 and the claim follows. \( \square \)

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