Game Theory and Applications
Problem set 4

1. Find all pure-strategy Nash equilibria in a two-player game with strategy sets $X = Y = [0, 1]$ and utility functions $u_1(x, y) = x^3 - 9x^2y + 15xy^2$, $u_2(x, y) = 2\ln(y) - 9y^2$.

2. Find all pure-strategy Nash equilibria in a two-player game with strategy sets $X = Y = [0, 1]$ and utility functions $u_1(x, y) = \frac{4}{3}x^3 + xy^2 - 2x^2y - x^2 + xy$, $u_2(x, y) = 2y - 2xy - y^2$.

3. Consider zero-sum game with strategy sets $X = Y = [0, 1]$ and utility function of player 1 $u(x, y) = -(x - y)^2$. This is a simplified version of the location game from the lecture. Show that this game does not have a pure Nash equilibrium. Further, find its unique mixed-strategy equilibrium. Its form is rather intuitive. What properties of the utility functions will always imply the existence of Nash equilibrium with similar structure?

4. Consider two-player game with strategy sets $X = Y = [0, 1]$ and utility functions $u_1(x, y) = -(x - y)^2$, $u_2(x, y) = \begin{cases} y & \text{for } x < \frac{1}{2} \\ 1 - y & \text{for } x \geq \frac{1}{2} \end{cases}$. Show that this game has no (either pure or mixed) Nash equilibria.
Hint: First show that the best response of player 1 against any mixed strategy of player 2 is pure.

5. Consider an asymmetric variant of the Cournot duopoly game from the lecture where one of the players (player 1) is a producer of the good he sells, while the other is only a reseller. Thus, the cost player 1 has to pay for x_1 units of the good is $\max\{c_0, c_1x_1\}$, where c_0 is some fixed cost he has to pay to get the production underway. The cost for player 2 is c_2x_2 with $c_2 > c_1$. Write down the utility functions and find the Nash equilibrium in this game.

6. Stackelberg duopoly game is a variant of Cournot's quantity competition where players have different priority. Player 1 chooses his strategy first, then Player 2, knowing the strategy chosen by his opponent, chooses his. This results in different procedure used to compute equilibria: we first compute the best response of Player 2 to any strategy of Player 1, we insert it into the utility of Player 1 and find strategies maximizing this new utility. Compute the equilibrium in Stackelberg duopoly game corresponding to Cournot game from the lecture. How do utilities of the players at the equilibrium change with respect to those at Cournot-Nash equilibrium? What about the price of the good? Is it more profitable for the consumer to have priority among the players?

7. Consider the following game: A bank has a two-level security system to protect the data of its clients. Breaking level 2 of the system is only possible if level 1 is already broken. Stealing any important data from the bank is only possible if both levels of defence are broken. Some of the bank's systems work only behind the first security level, so breaking it also causes some damage. Assume that the losses of the bank when the first level is broken are around 1 million Euros, while when the second level is broken, they are around 5 million. Both the hackers (player 1) and the bank (player 2) decide how much resources (identified with two-dimensional vectors summing up to 1) they assign to attack (protect) each of the levels. The probability of breaking through level i when the resources assigned to it by the hackers and the bank are respectively x_i and y_i equals $(\alpha x_i - y_i)^+$, where $\alpha \in \mathbb{R}^+$ is a coefficient describing difference between resources available to both sides. The utility of the hackers (and minus the utility of the bank) is the expected loss of the bank. Write down the strategy sets for the players and the utility function for player 1. Find all Nash equilibria in this game for each value of coefficient α.

8. Consider the following game between the Police and a criminal. Police tries to find a place in criminal's home to plant a bug. The criminal suspects that his house is wired, so he tries to find a place where he can talk safely. The goal of the Police is to maximize the probability of obtaining some useful information, while the criminal wants to minimize this probability. As in other of our simplified models, we identify the house of the criminal with the interval $[0, 1]$. The probability of obtaining some useful information from the bug is $1 - e^{d-1}$, where d is the distance between the bug and criminals talking.

   (a) What are the optimal strategies for the Police and the criminal?

   (b) How will they change if the Police tries to find places for n bugs instead of one? (The probability of obtaining the information can then be computed as $1 - e^{\sum_{i=1}^{n}(d_i - 1)}$, where d_i is the distance between the criminal and the ith bug).

   (c) Assume that the Police tries to optimize the number of bugs. By that they mean maximizing the probability of obtaining the information minus the unit cost of a bug c multiplied by the number of bugs. What will be the optimal number of bugs then?