Bell-shaped functions

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Karlovasi, July 19, 2019

Definition

A function f is bell-shaped if

- $f \to 0$ at $\pm\infty$,
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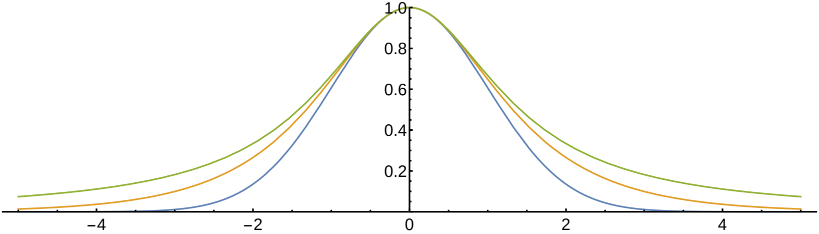

Examples: $\exp(-x^2)$, $\dfrac{1}{\cosh x}$, $\dfrac{1}{(1+x^2)^p}$ $(p > 0)$.

Problem

Describe the class of bell-shaped functions.

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A new class of bell-shaped functions
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Simpler Problem

Describe functions f such that $f^{(n)}$ have no zeroes.

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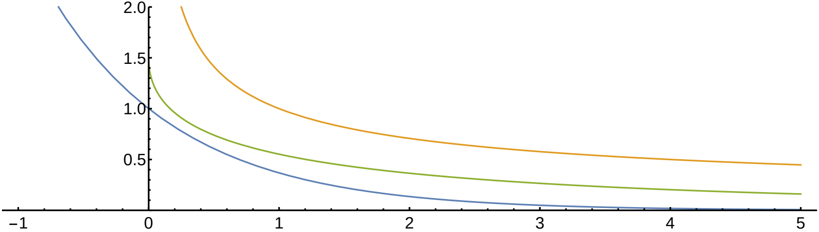
if $(-1)f^{(n)} \geqslant 0$ for $n = 0, 1, 2, \ldots$



Examples in $\mathcal{CM}$: $\exp(-x)$, $\dfrac{1}{x^p}$ $(p > 0)$, $\exp(-\sqrt{x})$.

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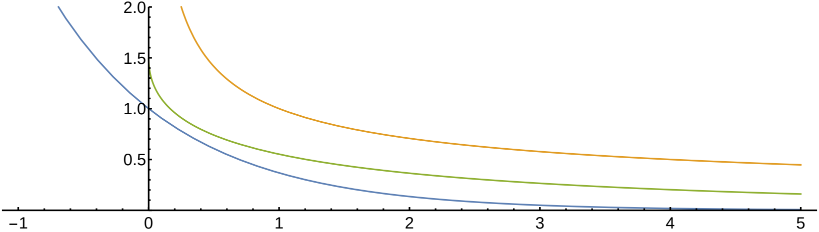
A function f is completely monotone on D ($\mathcal{CM}_D$) if $(-1)f^{(n)} \geqslant 0$ in D for $n = 0, 1, 2, \ldots$



Examples in $\mathcal{CM}_{(0,\infty)}$: $\exp(-x)$, $\dfrac{1}{x^p}$ $(p > 0)$, $\exp(-\sqrt{x})$.

Definition

A function f is absolutely monotone on D ($\mathcal{AM}_D$) if $f^{(n)} \geqslant 0$ in D for $n = 0, 1, 2, \ldots$

Fact

$$f(x) \text{ is } \mathcal{CM} \iff f(-x) \text{ is } \mathcal{AM}.$$

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$$f^{(n)} \text{ have no zeroes} \iff \pm f \in \mathcal{AM} \text{ or } \pm f \in \mathcal{CM}.$$

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Theorem (Bernstein, 1928)

$$f \in \mathcal{CM}_{(0,\infty)} \iff f(x) = \int_{[0,\infty)} e^{-sx} \mu(dx).$$

Theorem (Hirschman, 1950; *Schoenberg's conjecture*)

There are no bell-shaped functions in a finite interval.

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Stable densities are bell-shaped if $\alpha = 2, 1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \frac{1}{5}, \ldots$

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Theorem (Jedidi–Simon, 2015)

1-D diffusion hitting time densities are bell-shaped.

Definition

A function f has at least n sign changes ($\mathcal{V}(f) \geqslant n$) if

$$f(x_0), f(x_1), f(x_2), \ldots, f(x_n) \text{ have alternating signs}$$

for some $x_0 < x_1 < x_2 < \ldots < x_k$.

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Definition

A probability density function f is a variation diminishing kernel if

$$\mathcal{V}(f * g) \leqslant \mathcal{V}(g)$$

for every bounded g.

Definition

A function f is a Pólya frequency function ($\mathcal{PFF}$) if it is a convolution of a Gaussian and countably many densities of centred $\pm$exponential distributions.

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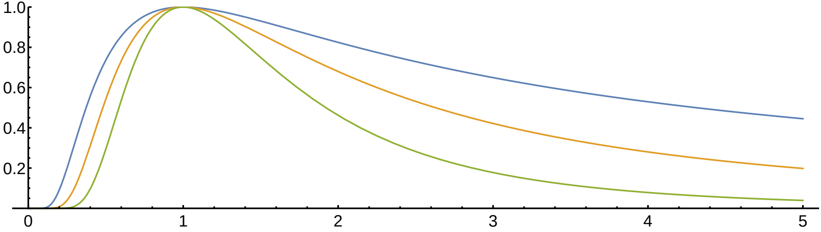
Theorem (Schoemberg, 1930–48)

$f \in \mathcal{PFF} \iff f$ is a variation diminishing kernel.

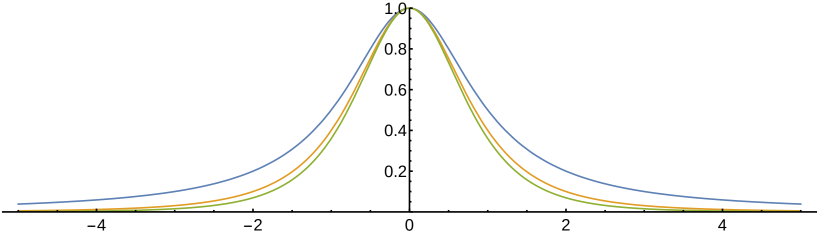
Definition

A smooth function $f \geqslant 0$ is bell-shaped ($\mathcal{BS}$) if

- $f \to 0$ at $\pm\infty$,
- $\mathcal{V}(f^{(n)}) = n$ for $n = 0, 1, 2, \ldots$



Examples: $\dfrac{e^{-1/x}}{x^p} \mathbb{1}_{(0,\infty)}(x)$ ($p > 0$).



Examples:

- $f(x) = \dfrac{1}{1+x^2}$
- $g(x) = \dfrac{1}{1+x^2}\,\dfrac{1}{4+x^2}$
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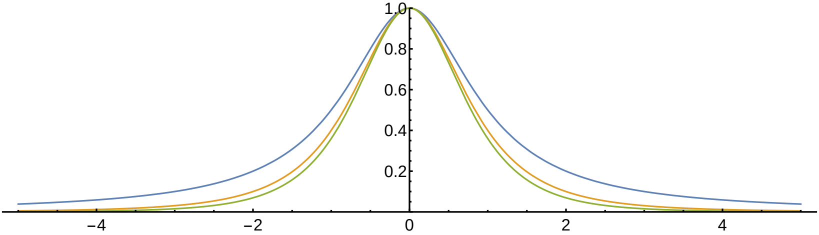

Examples:

- $f(x) = \dfrac{1}{1+x^2}$ is bell-shaped;
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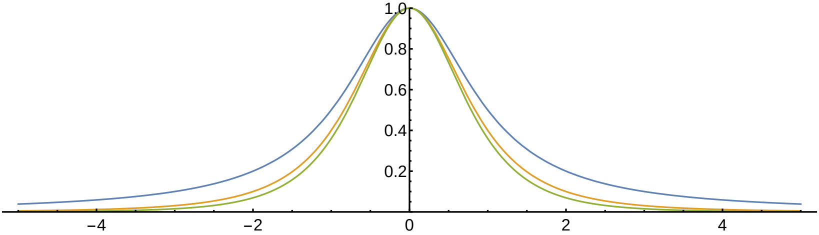

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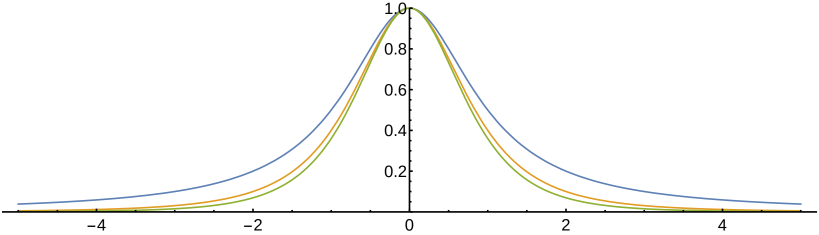

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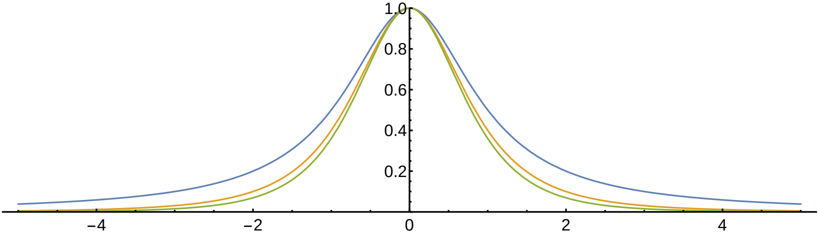

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- $h(x) = \dfrac{1}{1+x^2}\,\dfrac{1}{4+x^2}\,\dfrac{1}{9+x^2}$ is not bell-shaped:

  $\mathcal{V}(h^{(n)}) = n$ for $n = 0, 1, 2, \ldots, 56$, but $\mathcal{V}(h^{(57)}) = 61$.

Definition

A function or a measure $f \geqslant 0$ is weakly bell-shaped ($\overline{\mathcal{BS}}$) if $f * g$ is bell-shaped for any Gaussian g.

Fact

$$\mathcal{BS} = \overline{\mathcal{BS}} \cap \mathcal{C}^\infty.$$

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$g \in \mathcal{PFF}$

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A function f is $\mathcal{AM}$-then-$\mathcal{CM}$ ($\mathcal{AM}$-$\mathcal{CM}$) if

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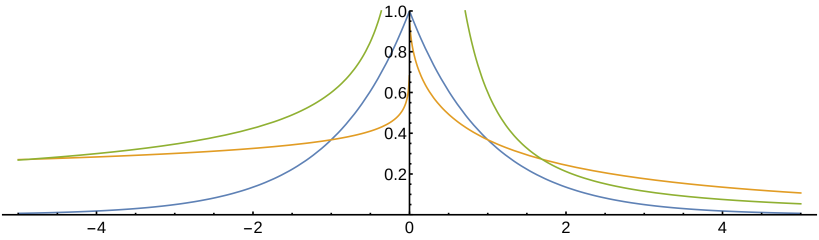

Sample $\mathcal{AM}$-$\mathcal{CM}$ functions.

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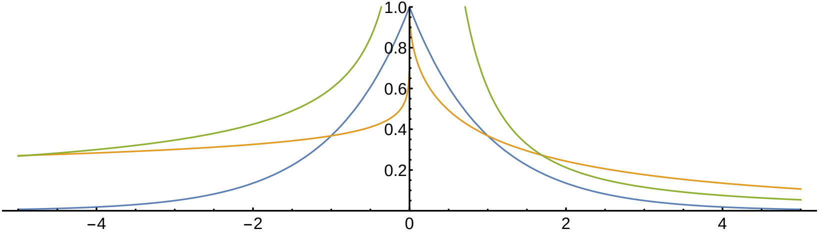

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Corollary (K, 2019)

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The class $\mathcal{AM}\text{-}\mathcal{CM} * \mathcal{PFF}$ includes:

- all previously known bell-shaped functions;

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The class $\mathcal{AM}\text{-}\mathcal{CM} * \mathcal{PFF}$ includes:

- all previously known bell-shaped functions;
- stable distributions,
- extended generalised gamma convolutions ($\mathcal{EGGC}$).

Definition (de Finetti, 1929)

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Definition (Rogers, 1983)

A Lévy process (X_t) has completlely monotone jumps if ν has an $\mathcal{AM}$-$\mathcal{CM}$ density function.

Examples: stable Lévy processes,
'complete' subordinate Brownian motions.

Notation

$\mathcal{ID}_{\mathcal{AM}\text{-}\mathcal{CM}} = \{$p.d.f.'s of X_1 for Lévy processes (X_t) with completely monotone jumps$\}$.

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Notation

$f_{a,b,\mu} \in \mathcal{ID}$ with Gaussian coefficients a, b, and the Lévy measure:

$$\nu(dz) = \begin{cases} \left(\int_{(0,\infty)} e^{-sz}\mu(ds)\right) dz & \text{on } (0,\infty), \\ \left(-\int_{(-\infty,0)} e^{-sz}\mu(ds)\right) dz & \text{on } (0,\infty), \end{cases}$$

Here $\mu \geqslant 0$ on $(0,\infty)$ and $\mu \leqslant 0$ on $(-\infty,0)$.

Fact

$$\mathcal{ID}_{\mathcal{AM}\text{-}\mathcal{CM}} = \{f_{a,b,\mu}\}.$$

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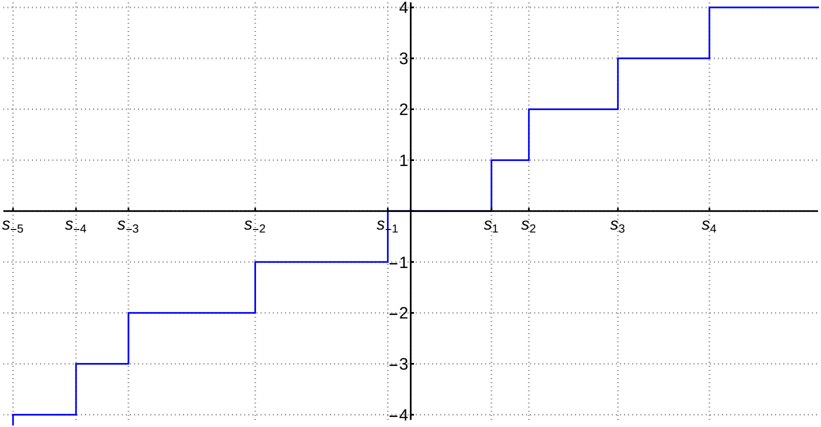
Fact

The measure μ is the boundary value of
the imaginary part of
the analytic extension to $\{\operatorname{Re}\xi > 0\}$ of
the characteristic exponent: $-\log \hat{f}_{a,b,\mu}$.

(This is why we prefer to have $\mu \leqslant 0$ on $(-\infty, 0)$.)

Proposition

$$\mathcal{PFF} = \{f_{a,b,\mu} : \mu(ds) = \varphi(s)ds,\ \varphi : \mathbb{R} \to \mathbb{Z} \text{ increasing}\}.$$



Sample φ.

Proposition

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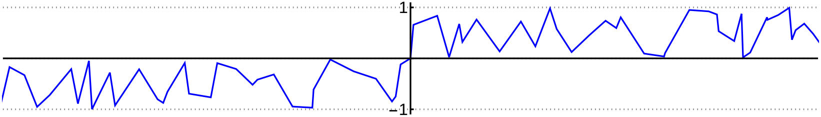

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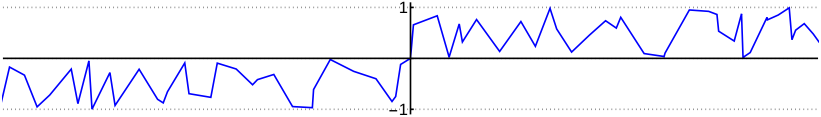


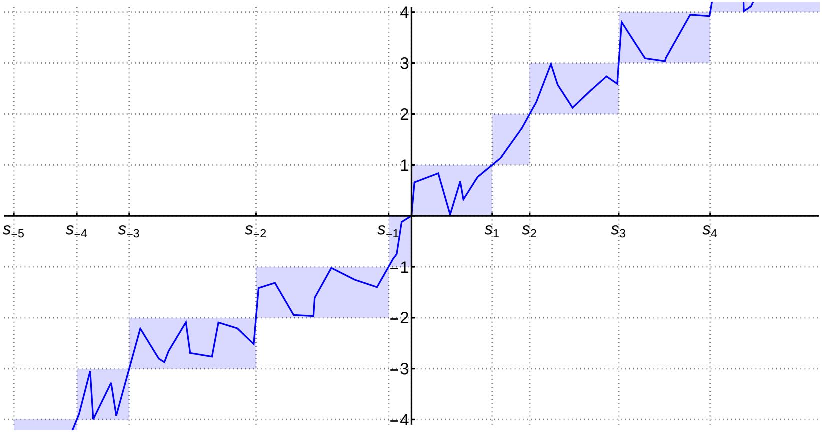
Sample φ.

Fact

$$f_{a_1,b_1,\mu_1} * f_{a_2,b_2,\mu_2} = f_{a_1+a_2,b_1+b_2,\mu_1+\mu_2}.$$

Proposition

$\mathcal{AM}\text{-}\mathcal{CM} * \mathcal{PFF} = \{f_{a,b,\mu} : \mu(ds) = \varphi(s)ds,\ \varphi$ as below$\}$.



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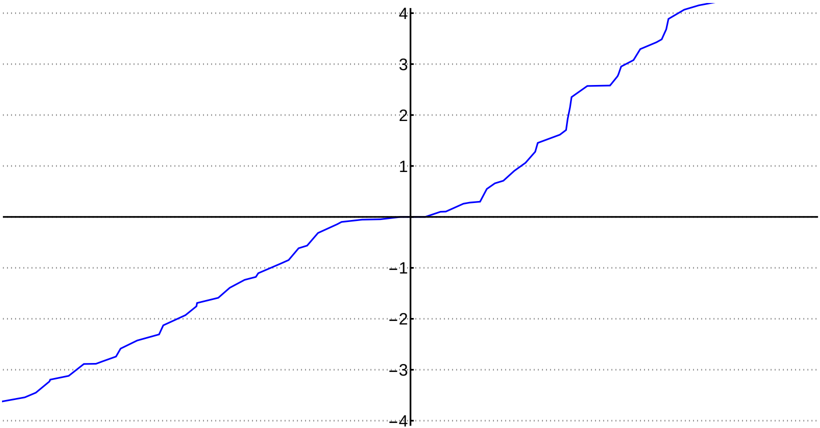
$$\mathcal{EGGC} = \{f_{a,b,\mu} : \mu(ds) = \varphi(s)ds,\ \varphi \text{ increasing}\}.$$



Sample φ.

Summary

$$\{\text{stable}\} \subseteq \mathcal{EGGC} \subseteq (\mathcal{AM}\text{-}\mathcal{CM} * \mathcal{PFF}) \subseteq \overline{\mathcal{BS}}.$$

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Theorem (K, Simon, 2019+)

$$(\mathcal{AM}\text{-}\mathcal{CM} * \mathcal{PFF}) = \overline{\mathcal{BS}}.$$

A complete surprise to me!

Corollary

If $f \in \overline{\mathcal{BS}}$ is a p.d.f., then $f \in \mathcal{ID}_{\mathcal{AM}\text{-}\mathcal{CM}}$.

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Corollary

$\{(X_t) \text{ Lévy} : X_t \in \overline{\mathcal{BS}} \text{ for } t > 0\} = \{(X_t) \text{ Lévy} : X_1 \in \mathcal{EGGC}\}$.

Proposition (K, Simon, 2019+)

If $f \in \mathcal{BS}$ is a p.d.f., then

$$\lim_{n\to\infty} \frac{1}{n}\{\text{zeroes of } f^{(n)}\} = \{1/s_k : k \in \mathbb{Z} \setminus \{0\}\}.$$

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If $f \in \mathcal{BS}$, then

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Corollary

$$\frac{1 - 2px + x^2}{(1+x^2)^2} \in \mathcal{BS} \iff p = 0 \text{ or } p = \frac{1}{\pi k}, k \in \mathbb{Z} \setminus \{0\}.$$