Spectral decomposition of integro-differential operators related to one-dimensional Lévy processes in domains

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Goal

Study the spectral theory of nonlocal operators on $L^2(D)$ for $D=(0,\infty)$, $D=\mathbf{R}\setminus\{0\}$, $D=(-1,1)$.

Joint project with **Kamil Kaleta**, **Tadeusz Kulczycki**, **Jacek Małecki**, **Michał Ryznar**, **Andrzej Stós**

Outline:

- Motivation: classical results
- Eigenfunction expansions (ordinary and generalized)
- Lévy operators
- Half-line $(0,\infty)$
- Complement of a point $\mathbf{R}\setminus\{0\}$
- Interval $(-1,1)$

Note: this is a **1-D** talk.

Part 1

Motivation: classical results

One-dimensional Dirichlet Laplace operator

- **Laplace operator**: $\Delta f = f''$
- Δ generates the **heat semigroup**

$$P_t = \exp(t\Delta)$$

- **Dirichlet Laplace operator** in D:
  Friedrichs extension of Δ restricted to $C_c^\infty(D)$
- Δ_D generates the **Dirichlet heat semigroup**

$$P_t^D = \exp(t\Delta_D)$$

Probabilistic viewpoint

- Δ is the generator of the **Brownian motion** X_t
- P_t are the **transition operators**:

$$P_t f(x) = \mathbf{E}_x f(X_t)$$

- Here: $\mathbf{E}_x X_t = x$, $\operatorname{Var}_x X_t = 2t$

- Δ_D is the generator of X_t **killed** upon leaving D
- P_t^D are the **transition operators**:

$$P_t^D f(x) = \mathbf{E}_x (f(X_t) \mathbf{1}_{t < \tau_D})$$

$$\tau_D = \inf\{t \ge 0 : X_t \notin D\}$$

Fourier transform and Δ

- Fourier transform:

$$\mathcal{F}f(s) = \int_{-\infty}^{\infty} e^{-isx} f(x) dx$$

- Inverse transform:

$$f(x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{isx} \mathcal{F}f(s) ds$$

- Spectral representation of Δ:

$$\mathcal{F}(\Delta f)(s) = -s^2 \mathcal{F}f(s)$$

$$\mathcal{F}(P_t f)(s) = e^{-ts^2} \mathcal{F}f(s)$$

Fourier sine transform and $\Delta_{(0,\infty)}$

- Fourier sine transform:

$$\mathcal{F}_{\sin} f(s) = \int_0^\infty \sin(sx) f(x) dx$$

- Inverse transform:

$$f(x) = \frac{2}{\pi} \int_0^\infty \sin(sx) \mathcal{F}_{\sin} f(s) ds$$

- Spectral representation of $\Delta_{(0,\infty)}$:

$$\mathcal{F}_{\sin}(\Delta_{(0,\infty)} f)(s) = -s^2 \mathcal{F}_{\sin} f(s)$$

$$\mathcal{F}_{\sin}(P_t^{(0,\infty)} f)(s) = e^{-ts^2} \mathcal{F}_{\sin} f(s)$$

Fourier series and $\Delta_{(-1,1)}$

- Fourier series coefficients:

$$f_n = \int_{-1}^{1} \sin(\tfrac{n\pi}{2}(x+1)) f(x)dx$$

- Fourier series:

$$f(x) = \sum_{n=1}^{\infty} f_n \sin(\tfrac{n\pi}{2}(x+1))$$

- Spectral representation of $\Delta_{(-1,1)}$:

$$(\Delta_{(-1,1)}f)_n = -(\tfrac{n\pi}{2})^2 f_n$$

$$(P_t^{(-1,1)}f)_n = e^{-t(\frac{n\pi}{2})^2} f_n$$

Part 2

Eigenfunction expansions (ordinary and generalized)

(Ordinary) eigenfunction expansion

- Self-adjoint operator $\mathbf{A}$ on, say, $L^2((-1, 1))$
- Eigenfunctions φ_n and eigenvalues λ_n
- φ_n form a complete orthonormal set
- **Eigenfunction expansion** (EE):

$$f(x) = \sum_{n=1}^{\infty} a_n \varphi_n(x)$$

$$\mathbf{A} f(x) = \sum_{n=1}^{\infty} \lambda_n a_n \varphi_n(x)$$

$$a_n = \int_{-1}^{1} f(x) \varphi_n(x) dx$$

Fourier series as EE

- Recall that

$$f(x) = \sum_{n=1}^{\infty} a_n \sin(\tfrac{n\pi}{2}(x+1))$$

$$\Delta_{(-1,1)} f(x) = \sum_{n=1}^{\infty} -(\tfrac{n\pi}{2})^2 a_n \sin(\tfrac{n\pi}{2}(x+1))$$

$$a_n = \int_{-1}^{1} \sin(\tfrac{n\pi}{2}(x+1)) f(x) dx$$

- Here:

$$\varphi_n(x) = \sin(\tfrac{n\pi}{2}(x+1)) \qquad \text{and} \qquad \lambda_n = -(\tfrac{n\pi}{2})^2$$

Generalized eigenfunction expansion

- Self-adjoint operator **A** on, say, $L^2((0,\infty))$
- **Generalized** eigenfunction expansion (GEE):

$$f(x) = \int a_s \varphi_s(x) m(ds)$$

$$\mathbf{A} f(x) = \int \lambda_s a_s \varphi_s(x) m(ds)$$

$$a_s = \int_0^\infty f(x) \varphi_s(x) dx$$

- **Generalized** eigenfunctions φ_s and eigenvalues λ_s
- φ_s typically fail to be in $L^2((0,\infty))$

Fourier sine transform as GEE

- Recall that

$$f(x) = \frac{2}{\pi}\int_0^\infty a_s \sin(sx)ds$$

$$\Delta_{(0,\infty)} f(x) = \frac{2}{\pi}\int_0^\infty -s^2 a_s \sin(sx)ds$$

$$a_s = \int_0^\infty \sin(sx)f(x)dx$$

- Here:

$$\varphi_s(x) = \sin(sx)$$

$$\lambda_s = -s^2$$

$$m(ds) = \tfrac{2}{\pi}ds$$

Some general results

Theorem (Gårding, 1950s)

If $\mathbf{A}$ is self-adjoint and, for some nonzero h, $h(\mathbf{A})$ is a **Carleman's operator**, then $\mathbf{A}$ admits GEE.

Theorem (Getoor, 1959)

If $\mathbf{A}$ is the generator of a Markov process with bounded transition density function, then $\mathbf{A}$ admits GEE.

- Little information about the eigenfunctions and eigenvalues
- Limited applicability

Part 3

Lévy operators

Setting

Assumption

$$\mathbf{A}f(x) = bf''(x) + \mathrm{pv}\int_{-\infty}^{\infty}(f(y)-f(x))\nu(y-x)dy$$

- $b \geq 0$
- $\nu(z) \geq 0$, $\nu(z) = \nu(-z)$, $\int_{-\infty}^{\infty}\min(1,z^2)\nu(z)dz < \infty$
- ν is completely monotone on $(0,\infty)$
  (i.e. $(-1)^n\nu^{(n)}(z) \geq 0 \quad \text{for} \quad z > 0,\ n = 0,1,2,\ldots$)

- $\mathbf{A}_D$ is the Friedrichs extension of $\mathbf{A}$ restricted to $C_c^\infty(D)$
- This corresponds to the Dirichlet **exterior** condition

Examples

$$\mathbf{A}f(x) = bf''(x) + \mathrm{pv}\int_{-\infty}^{\infty}(f(y) - f(x))\nu(y-x)dy$$

- $\mathbf{A} = -(-\Delta)^{\alpha/2}$

  $b = 0,\ \nu(z) = c|z|^{-1-\alpha}$

- $\mathbf{A} = -(-\Delta + 1)^{1/2} + 1$

  $b = 0,\ \nu(z) = c|z|^{-1}K_1(|z|)$

- $\mathbf{A} = c_1\Delta - c_2(-\Delta)^{\alpha/2}$

  $b = c_1,\ \nu(z) = c|z|^{-1-\alpha}$

- $\mathbf{A} = -\log(-\Delta + 1)$

  $b = 0,\ \nu(z) = |z|^{-1}e^{-|z|}$

- $\mathbf{A} = -((-\Delta)^{-1} + 1)^{-1}$

  $b = 0,\ \nu(z) = \frac{1}{2}e^{-|z|}$

Complete Bernstein functions

$$\mathbf{A}f(x) = bf''(x) + \mathrm{pv}\int_{-\infty}^{\infty} (f(y) - f(x))\nu(y-x)dy$$

Lemma

Our assumption:

$$b \geq 0,\ \nu(z) = \nu(-z),$$
$$\nu \text{ completely monotone on } (0,\infty)$$

is equivalent to $\mathbf{A} = -\psi(-\Delta)$ for a **complete Bernstein function** ψ:

$$\psi(s) = bs + \int_0^{\infty} \frac{s}{u+s}\mu(du)$$

- Hence $\mathbf{A}$ is a Fourier multiplier with symbol $-\psi(s^2)$

Lévy processes

$$\mathbf{A}f(x) = bf''(x) + \mathrm{pv}\int_{-\infty}^{\infty} (f(y) - f(x))\nu(y - x)dy$$

- $\mathbf{A}$ is the generator of a symmetric Lévy process X_t
- b is the diffusion coefficient
- There is no drift
- ν is the density of the Lévy measure (measures the intensity of jumps)
- $\mathbf{A}_D$ is the generator of X_t killed upon leaving D

$$\tau_D = \inf\{t \geq 0 : X_t \notin D\}$$

$$P_t^D f(x) = \mathbf{E}_x(f(X_t)\mathbf{1}_{t<\tau_D})$$

$$\mathbf{A}_D f = \lim_{t\to 0} \frac{P_t^D f - f}{t}$$

Part 4

Half-line

$$D = (0, \infty)$$

Summary

- We prove GEE for $\mathbf{A}_{(0,\infty)}$
- Eigenfunctions are given fairly explicitly
- Results suitable for numerical computations
- Applications to fluctuation theory

Tadeusz Kulczycki, K., Jacek Małecki, Andrzej Stós
Spectral properties of the Cauchy process...
Proc. London Math. Soc. 101(2) (2010)

K.
Spectral analysis of subordinate Brownian motions...
Studia Math. 206(3) (2011)

K., Jacek Małecki, Michał Ryznar
First passage times for subordinate Brownian motions
arXiv:1110.0401

GEE

Theorem (part 1)

$\mathbf{A}_{(0,\infty)}$ admits GEE:

$$f(x) = \frac{2}{\pi} \int_0^\infty a_s F_s(x) ds$$

$$\mathbf{A}_{(0,\infty)} f(x) = \frac{2}{\pi} \int_0^\infty -\psi(s^2) a_s F_s(x) ds$$

$$a_s = \int_0^\infty F_s(x) f(x) dx$$

- This corresponds to:

$$\varphi_s(x) = F_s(x)$$

$$\lambda_s = -\psi(s^2)$$

$$m(ds) = \tfrac{2}{\pi} ds$$

Eigenfunctions

Theorem (part 2)

$$F_s(x) = \sin(sx + \vartheta_s) - \int_{(0,\infty)} e^{-xu} g_s(du)$$

for $\vartheta_s \in [0, \frac{\pi}{2})$:

$$\vartheta_s = \frac{1}{\pi} \int_0^\infty \frac{s}{s^2 - v^2} \log \frac{\psi(s^2) - \psi(v^2)}{\psi'(s^2)(s^2 - v^2)} dv$$

and g_s positive, finite:

$$g_s(du) = \frac{1}{\pi} \left(\operatorname{Im} \frac{s\psi'(s^2)}{\psi(s^2) - \psi^+(-u^2)} \right) \times \exp\left(\frac{1}{\pi} \int_0^\infty \frac{u}{u^2 + v^2} \log \frac{\psi(s^2) - \psi(v^2)}{\psi'(s^2)(s^2 - v^2)} dv \right) du$$

Laplace transform of eigenfunctions

Theorem (part 3)

$$\mathcal{L}F_s(u) = \int_0^\infty F_s(x) e^{-ux} dx$$
$$= \frac{s}{s^2 + u^2} \exp\left(\frac{1}{\pi} \int_0^\infty \frac{u}{u^2 + v^2} \log \frac{\psi'(s^2)(s^2 - v^2)}{\psi(s^2) - \psi(v^2)} dv \right)$$

- Remember that
$$F_s(x) = \sin(sx + \vartheta_s) - G_s(x)$$
with G_s positive, bounded, integrable and completely monotone
- Tauberian theorems give estimates of F_s

Better notation (3 slides in 1)

$$\psi_s(u^2) = \frac{\psi'(s^2)(s^2-u^2)}{\psi(s^2)-\psi(u^2)}$$

$$\psi_s^*(u) = \exp\left(\frac{1}{\pi}\int_0^\infty \frac{u}{u^2+v^2}\log\psi_s(v^2)dv\right)$$

$$F_s(x) = \sin(sx+\vartheta_s) - \int_{(0,\infty)} e^{-xu} g_s(du)$$

$$\vartheta_s = \operatorname{Arg}(\psi_s^*(is))$$

$$g_s(du) = \frac{1}{\pi}\,\frac{s}{s^2+u^2}\,\frac{\operatorname{Im}(\psi_s)^+(-u^2)}{\psi_s^*(u)}\,du$$

$$\mathcal{L}F_s(u) = \frac{s}{s^2+u^2}\,\psi_s^*(u)$$

Example: $\mathbf{A} = \Delta$, $\psi(s) = s$

$$\psi_s(u^2) = \frac{\psi'(s^2)(s^2 - u^2)}{\psi(s^2) - \psi(u^2)} = 1$$

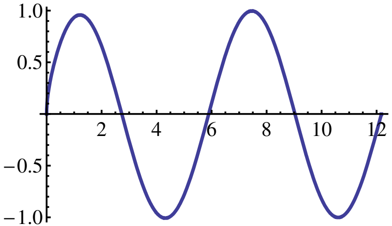
$$\psi_s^*(u) = \exp\left(\frac{1}{\pi}\int_0^\infty \frac{u}{u^2 + v^2} \log \psi_s(v^2) dv\right) = 1$$

$$\vartheta_s = \operatorname{Arg}(\psi_s^*(is)) = 0$$

$$g_s(du) = \frac{1}{\pi} \frac{s}{s^2 + u^2} \frac{\operatorname{Im}(\psi_s)^+(-u^2)}{\psi_s^*(u)} du = 0$$

$$F_s(x) = \sin(sx + \vartheta_s) - \int_{(0,\infty)} e^{-xu} g_s(du) = \sin(sx)$$

Example: $\mathbf{A} = -(-\Delta)^{\alpha/2}$, $\psi(s) = s^{\alpha/2}$



$F_1(x)$ for $\alpha = 1$, $\mathbf{A} = -(-\Delta)^{1/2}$

$$F_s(x) = \sin\left(sx + \tfrac{(2-\alpha)\pi}{8}\right) - \int_0^\infty e^{-sxu} g(u) du \qquad \left(\vartheta_s = \tfrac{(2-\alpha)\pi}{8}\right)$$

$$g(u) = \frac{\sqrt{2\alpha}\sin\frac{\alpha\pi}{2}}{2\pi} \frac{s^\alpha}{1 + s^{2\alpha} - 2s^\alpha \cos\frac{\alpha\pi}{2}} \times \exp\left(\frac{1}{\pi}\int_0^\infty \frac{1}{1+v^2} \log\frac{1-u^2v^2}{1-u^\alpha v^\alpha} dv\right)$$

Example: $\mathbf{A} = -((-\Delta)^{-1} + 1)^{-1}$, $\psi(s) = s/(1+s)$

$$\vartheta_s = \arctan s$$

$$g_s(u) = 0$$

$$F_s(x) = \sin(sx + \arctan s)$$

- F_s does not vanish at 0
- $\mathbf{A} = -\psi(-\Delta)$ is bounded
- General rule: $F_s(x) \sim c_s \sqrt{\psi(\frac{1}{x^2})}$ as $x \to 0$

Application: first passage times (part 1)

- **First passage time**: $\tau_x = \inf\{t \geq 0 : X_t \geq x\}$
- $\mathbf{P}_0(\tau_x > t) = \mathbf{P}_x(\tau_{(0,\infty)} > t) = P_t^{(0,\infty)} \mathbf{1}(x)$
- For $f = \mathbf{1}$:

$$a_s = \int_0^\infty F_s(x) f(x) dx = \mathcal{L} F_s(0) = \frac{\psi_s^*(0)}{s} = \sqrt{\frac{\psi'(s^2)}{\psi(s^2)}}$$

- By GEE:

$$P_t^{(0,\infty)} \mathbf{1}(x) = \frac{2}{\pi} \int_0^\infty e^{-t\psi(s^2)} a_s F_s(x) ds$$

- Integrability issues: $f \notin L^2((0,\infty))$

Application: first passage times (part 2)

Theorem

Assume in addition that

$$\sup_{s>0} \frac{-s\psi''(s)}{\psi'(s)} < 2$$

$$\int_1^\infty e^{-t\psi(s^2)} \sqrt{\frac{\psi'(s^2)}{\psi(s^2)}} \, ds < \infty$$

Then:

$$\mathbf{P}_0(\tau_x > t) = \frac{2}{\pi} \int_0^\infty e^{-t\psi(s^2)} \sqrt{\frac{\psi'(s^2)}{\psi(s^2)}} F_s(x) ds$$

- Extra assumptions assert minimal regularity and at least logarithmic growth of ψ at ∞

Part 5

Complement of a point

$D = \mathbf{R} \setminus \{0\}$

Summary

- We develop a similar theory for $\mathbf{A}_{\mathbf{R}\setminus\{0\}}$
- $\mathbf{R} \setminus \{0\}$ is simpler than $(0, \infty)$, with one exception
- These result are in fact more general
  (they hold e.g. for truncated stable processes,
  $\nu(z) = c|z|^{-1-\alpha} \mathbf{1}_{(-1,1)}(z)$)
- Applications are partially still work in progress
- Note: we assume that $\mathbf{A}_{\mathbf{R}\setminus\{0\}} \neq \mathbf{A}$
  (i.e. X_t hits single points)

K.
Spectral theory for one-dimensional symmetric...
arXiv:1110.5894

GEE

Theorem (part 1)

$\mathbf{A}_{\mathbf{R}\setminus\{0\}}$ admits GEE:

$$f(x) = \frac{1}{\pi} \int_0^\infty a_s^{(1)} F_s(x) ds + \frac{1}{\pi} \int_0^\infty a_s^{(2)} \sin(sx) ds$$

$$\mathbf{A}_{\mathbf{R}\setminus\{0\}} f(x) = \frac{1}{\pi} \int_0^\infty -\psi(s^2) a_s^{(1)} F_s(x) ds + \frac{1}{\pi} \int_0^\infty -\psi(s^2) a_s^{(2)} \sin(sx) ds$$

$$a_s^{(1)} = \int_{-\infty}^\infty F_s(x) f(x) dx \qquad a_s^{(2)} = \int_{-\infty}^\infty \sin(sx) f(x) dx$$

- $F_s(x)$ are even functions, $\sin(sx)$ are odd functions

Eigenfunctions

Theorem (part 2)

$$F_s(x) = \sin(s|x| + \vartheta_s) - \int_{(0,\infty)} e^{-|x|u} g_s(du)$$

for $\vartheta_s \in [0, \frac{\pi}{2})$:

$$\vartheta_s = \arctan\left(\frac{1}{\pi}\int_0^\infty \left(\frac{2s}{s^2 - v^2} - \frac{2s\psi'(s^2)}{\psi(s^2) - \psi(v^2)}\right) dv\right)$$

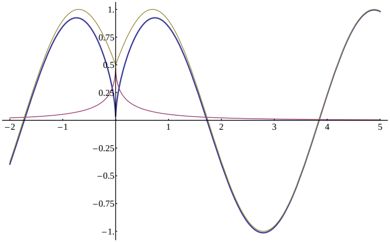
and g_s positive, finite:

$$g_s(du) = \frac{1}{\pi}\operatorname{Im}\frac{2s\psi'(s^2)}{\psi(s^2) - \psi^+(-u^2)} du$$

Theorem (part 3)

$$\mathcal{F}F_s(u) = \cos\vartheta_s \operatorname{pv}\frac{2s\psi'(s^2)}{\psi(s^2) - \psi(v^2)} + \pi\sin\vartheta_s(\delta_s(u) + \delta_{-s}(u))$$

Example: $\mathbf{A} = -(-\Delta)^{\alpha/2}$, $\psi(s) = s^{\alpha/2}$, $\alpha > 1$



$F_1(x)$, $\sin(|x| + \frac{\pi}{6})$ and the difference between the two for $\alpha = \frac{3}{2}$, $\mathbf{A} = -(-\Delta)^{3/4}$

$$F_s(x) = \sin(s|x| + \tfrac{\pi}{\alpha} - \tfrac{\pi}{2}) - \int_{(0,\infty)} e^{-sxu} g(u) du \quad (\vartheta_s = \tfrac{\pi}{\alpha} - \tfrac{\pi}{2})$$

$$g(u) = \frac{\alpha \sin \frac{\alpha\pi}{2} \sin \frac{\pi}{\alpha}}{\pi} \frac{s^\alpha}{1 + s^{2\alpha} - 2s^\alpha \cos \frac{\alpha\pi}{2}}$$

Application: hitting time of a point

- **Hitting time of a point**: $\tau_x = \inf\{t \geq 0 : X_t = x\}$
- $\mathbf{P}_0(\tau_x > t) = \mathbf{P}_x(\tau_{\mathbf{R}\setminus\{0\}} > t)$

Theorem

$$\mathbf{P}_0(\tau_x \in (t, \infty)) = \frac{1}{\pi} \int_0^\infty e^{-t\psi(s^2)} \frac{2s\psi'(s^2) \cos \vartheta_s}{\psi(s^2)} F_s(x) ds$$

Part 6

Interval

$D = (-1, 1)$

Summary

- EE is standard
- No closed-form expressions for eigenvalues or eigenfunctions
- Even estimates of eigenvalues are problematic
- General result due to Chen, Song:

$$c\psi(-\tilde{\lambda}_n) \le -\lambda_n \le \psi(-\tilde{\lambda}_n)$$

  where
  - λ_n are the eigenvalues of $\mathbf{A}_D$
  - $\tilde{\lambda}_n$ are the eigenvalues of Δ_D
  - D is regular enough
- Extremely few finer results even for the interval (Bañuelos, DeBlassie, Kulczycki, Méndez-Hernández, Siudeja)

Two-term Weyl-type formula

- For many ψ and for $D = (-1, 1)$ we prove that
$$-\lambda_n = \psi(\tfrac{n\pi}{2} - \vartheta) + \mathcal{O}(\tfrac{1}{n})$$
where
$$\vartheta = \lim_{s \to \infty} \vartheta_s$$
with ϑ_s the phase shift for the half-line $(0, \infty)$
- Note: $-\tilde{\lambda}_n = \frac{n\pi}{2}$
- Our method gives rather explicit bounds for λ_n

Example: $\mathbf{A} = -(-\Delta)^{1/2}$, $\psi(s) = \sqrt{s}$

Theorem

The eigenvalues λ_n of $\mathbf{A}_{(-1,1)}$ satisfy:

$$-\lambda_n = \frac{n\pi}{2} - \frac{\pi}{8} + \mathcal{O}(\tfrac{1}{n})$$

$$\left|-\lambda_n - (\tfrac{n\pi}{2} - \tfrac{\pi}{8})\right| < \frac{1}{n}$$

In particular, λ_n are simple.

Tadeusz Kulczycki, K., Jacek Małecki, Andrzej Stós
Spectral properties of the Cauchy process...
Proc. London Math. Soc. 101(2) (2010)

Example: $\mathbf{A} = -(-\Delta)^{\alpha/2}$, $\psi(s) = s^{\alpha/2}$

Theorem

The eigenvalues λ_n of $\mathbf{A}_{(-1,1)}$ satisfy:

$$-\lambda_n = \left(\frac{n\pi}{2} - \frac{(2-\alpha)\pi}{8}\right)^{\alpha} + \mathcal{O}\left(\frac{1}{n}\right)$$

K.
Eigenvalues of the fractional Laplace operator...
J. Funct. Anal. 262(5) (2012)

Example: $\mathbf{A} = -(-\Delta + 1)^{1/2} + 1$, $\psi(s) = \sqrt{s+1} - 1$

Theorem

The eigenvalues λ_n of $\mathbf{A}_{(-a,a)}$ are simple and satisfy:

$$-\lambda_n = \frac{n\pi}{2a} - \frac{\pi}{8a} + \mathcal{O}(\tfrac{1}{n})$$

Uniform bounds for λ_n up to $\mathcal{O}(\frac{1}{n})\mathcal{O}(\frac{1}{a}e^{-a/4})$ are given

Kamil Kaleta, K., Jacek Małecki
One-dimensional quasi-relativistic particle in the box
arXiv:1110.5887