

# HARMONIC EXTENSION TECHNIQUE: PROBABILISTIC AND ANALYTIC PERSPECTIVES

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ABSTRACT. Consider a path of the reflected Brownian motion in the half-plane  $\{y \geq 0\}$ , and erase its part contained in the interior  $\{y > 0\}$ . What is left is, in an appropriate sense, a path of a jump-type stochastic process on the line  $\{y = 0\}$  — the boundary trace of the reflected Brownian motion. It is well known that this process is in fact the 1-stable Lévy process, also known as the Cauchy process.

The PDE interpretation of the above fact is the following. Consider a bounded harmonic function  $u$  in the half-plane  $\{y > 0\}$ , with sufficiently smooth boundary values  $f$ . Let  $g$  denote the normal derivative of  $u$  at the boundary. The mapping  $f \mapsto g$  is known as the Dirichlet-to-Neumann operator, and it is again well known that this operator coincides with the square root of the 1-D Laplace operator  $-\Delta$ . Thus, the Dirichlet-to-Neumann operator coincides with the generator of the boundary trace process.

Molchanov and Ostrovskii proved that isotropic stable Lévy processes are boundary traces of appropriate diffusions in half-spaces. Caffarelli and Silvestre gave a PDE counterpart of this result: the fractional Laplace operator is the Dirichlet-to-Neumann operator for an appropriate second-order elliptic equation in the half-space. Again, the Dirichlet-to-Neumann operator turns out to be the generator of the boundary trace process.

During my talk I will discuss boundary trace processes and Dirichlet-to-Neumann operators in a more general context. My main goal will be to explain the connections between probabilistic and analytical results. Along the way, I will introduce the necessary machinery: Brownian local times and additive functionals, Krein's spectral theory of strings, and Fourier transform methods.

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If you find a mistake, have a comment, or simply find these notes useful, please do send me a message at [mateusz.kwasnicki@pwr.edu.pl](mailto:mateusz.kwasnicki@pwr.edu.pl).

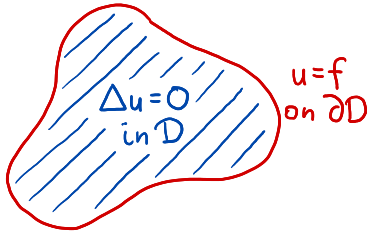


FIGURE 1. The Dirichlet problem.

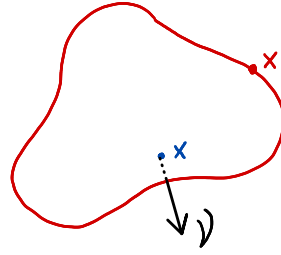


FIGURE 2. Normal derivative at the boundary.

## 1. CAUCHY PROCESS AND THE DIRICHLET-TO-NEUMANN OPERATOR

1.1. **Dirichlet-to-Neumann map.** A function  $u$  on a Euclidean domain  $D \subseteq \mathbb{R}^d$  is harmonic if it is a solution of the Laplace equation  $\Delta u = 0$ , where, as usual,

$$\Delta u = \sum_{j=1}^d \frac{\partial^2 u}{\partial x_j^2}$$

is the Laplace operator. If  $D$  is nice enough (for example, a smooth domain), then for every continuous function  $f$  on  $\partial D$  there is a unique solution of the Dirichlet problem (see Figure 1)

$$\begin{cases} \Delta u = 0 & \text{in } D, \\ u = f & \text{on } \partial D. \end{cases}$$

Suppose that  $f$  is smooth enough (for example,  $C^\infty$ ), and let  $g$  denote the (outward) normal derivative of  $u$  — the corresponding Neumann boundary condition. Then  $g$  is given in terms of  $f$  by a certain linear operator  $K$ :

$$g = Kf,$$

known under a self-explanatory name Dirichlet-to-Neumann operator.

This operator, in a slightly more general form, arises in Calderón's inverse conductivity problem: given the resistance between all pairs of boundary points of a domain  $D$ , can one reconstruct the conductivity inside  $D$ ? In this case the Laplace operator  $\Delta$  is replaced by  $\nabla \cdot (\gamma \nabla u)$ , and the question effectively asks whether the knowledge of  $K$  allows one to determine  $\gamma$ .

If  $P_D(x, x')$  denotes the Poisson kernel of  $D$  and  $\sigma$  is the surface measure on  $\partial D$ , then

$$u(x) = \int_{\partial D} f(x') P_D(x, x') \sigma(dx').$$

It is thus reasonable to expect that the kernel of  $K$  is given by the outward normal derivative at the boundary of the Poisson kernel, say,

$$Kf(x) = - \int_{\partial D} f(x') J(x, x') \sigma(dx')$$

with

$$J(x, x') = \frac{\partial P_D}{\partial \nu_x}(x, x') = \frac{\partial^2 G_D}{\partial \nu_x \partial \nu_{x'}}(x, x')$$

(where  $\nu$  denotes the outward normal vector and  $G_D(x, x')$  is the Green function; see Figure 2; forget about it if you are not already familiar with the Green function and the Poisson kernel, this paragraph is only meant to put things in a broader context).

This is plainly wrong:  $K$  applied to a constant function is zero, so we would rather have something like

$$Kf(x) = \int_{\partial D} (f(x) - f(x'))J(x, x')\sigma(dx'),$$

with the extra  $f(x)$  term originating from the singularity of  $P_D(x, x')$  as  $x$  approaches  $x'$ . The above expression turns out to be correct, as long as the integral is understood in the Cauchy's principal value sense, but we will not pursue this direction.

Before we proceed, let us mention another description of the operator  $K$ . With the notation introduced above, by Green's first identity,

$$\frac{1}{2} \int_D |\nabla u(x)|^2 dx = \int_{\partial D} u(x) \frac{\partial u}{\partial \nu}(x) \sigma(dx) = \int_{\partial D} f(x)g(x)\sigma(dx).$$

Dirichlet's principle states that  $u$  is the unique minimiser of the integral on the left-hand side of the above equation among sufficiently regular functions  $u$  on  $\mathbb{R}^n \times [0, \infty)$  with boundary values  $f$ . Thus,

$$\int_{\partial D} f(x)Kf(x)dx = \min \left\{ \frac{1}{2} \int_D |\nabla v(x)|^2 dx \right\},$$

where the minimum is taken over the class of admissible functions  $v$  just described, and it is attained when  $v = u$  is the harmonic function with boundary values  $f$ .

**1.2. Square root of the Laplace operator.** Throughout these lecture notes we will study the Dirichlet-to-Neumann operator for the  $(d + 1)$ -dimensional half-space  $D = \mathbb{R}^d \times (0, \infty)$ , the boundary of which is naturally identified with  $\mathbb{R}^d$ . This domain is unbounded, and so in order to have a unique solution of the Dirichlet problem, we need to assume, for example, that  $f$  is compactly supported and  $u$  is bounded. From now on we use the symbol  $x$  for the 'horizontal' coordinate  $\mathbb{R}^n$  and  $y \in (0, \infty)$  for the vertical one (see Figure 3).

The Poisson kernel is given by

$$P_D((x, y), x') = P_y(x - x') = c_d \frac{y}{(|x - x'|^2 + y^2)^{(d+1)/2}}$$

(to avoid nested brackets, from now on we switch the notation to  $P_y(x - x')$  when speaking about the Poisson kernel for the half-space), the harmonic extension becomes a convolution with  $P_y$ :

$$u(x, y) = \int_{\mathbb{R}^d} f(x')P_y(x - x')dx' = f * P_y(x),$$

and it is more or less straightforward to see that

$$Kf(x) = -\frac{\partial u}{\partial y}(x, 0) = c_d \text{PV} \int_{\mathbb{R}^d} \frac{f(x) - f(x')}{|x - x'|^{(d+1)/2}} dx'.$$

This operator is sometimes called half-Laplacian and denoted by  $\sqrt{-\Delta}$ . In the sense of spectral theory, it is indeed the square root of  $-\Delta$ , the  $d$ -dimensional Laplace operator (with a minus sign). A somewhat informal way to prove this is as follows: since the  $(d + 1)$ -dimensional Laplace operator of  $u$  is zero, we have

$$K^2 f(x) = \frac{\partial^2 u}{\partial y^2}(x, 0) = -\sum_{j=1}^d \frac{\partial^2 u}{\partial x_j^2}(x, 0) = -\Delta f(x);$$

and by Dirichlet's principle,  $K$  is a positive definite operator. A rigorous argument involves the  $d$ -dimensional Fourier transform:

$$\mathcal{F} f(\xi) = \int_{\mathbb{R}^d} e^{-i\xi x} f(x) dx$$

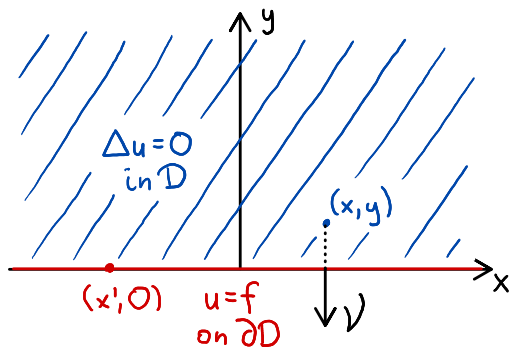


FIGURE 3. The half-plane.

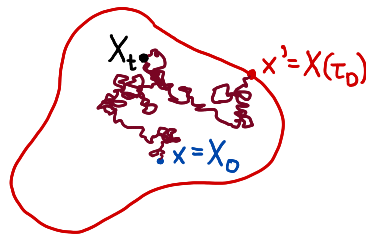


FIGURE 4. Brownian motion and the first exit time.

(here  $\xi x$  is the usual dot product). If we denote by  $\mathcal{F}u(\xi, y)$  the Fourier transform of  $u(\cdot, y)$ , then the Laplace equation is turned into an ODE

$$-|\xi|^2 \mathcal{F}u(\xi, y) + \frac{\partial^2 \mathcal{F}u}{\partial y^2}(\xi, y) = 0,$$

which, given that  $u$  is assumed to be bounded, has a unique solution

$$\mathcal{F}u(\xi, y) = e^{-|\xi|y} \mathcal{F}u(\xi, 0) = e^{-|\xi|y} \mathcal{F}f(\xi).$$

Note that this actually shows that  $e^{-|\xi|y}$  is the Fourier transform of the Poisson kernel  $P_y(\cdot)$ . It follows that

$$\mathcal{F}Kf(\xi) = -\frac{\partial \mathcal{F}u}{\partial y}(\xi, 0) = |\xi| \mathcal{F}u(\xi, 0) = |\xi| \mathcal{F}f(\xi).$$

In particular,  $\mathcal{F}K^2f(\xi) = |\xi|^2 \mathcal{F}f(\xi) = -\mathcal{F}\Delta f(\xi)$ .

**1.3. Boundary trace process.** Up to a factor  $\frac{1}{2}$ , the Laplace operator is the generator of the Brownian motion. The  $d$ -dimensional Brownian motion  $X_t$  is just a vector of independent Wiener processes, and the previous sentence means that

$$\frac{1}{2}\Delta u(x) = \lim_{t \rightarrow 0^+} \frac{\mathbb{E}[u(X_t)|X_0 = x] - u(x)}{t}$$

for sufficiently regular functions  $u$ . The connection is, however, much deeper. For example, Kakutani observed that the Poisson kernel  $P_D(x, \cdot)$  is the density function (with respect to the surface measure) of the distribution of  $X_t$  at the first exit time from  $D$ . Thus, if we denote this exit time by  $\tau_D$ , the solution of the Dirichlet problem is given by (see Figure 4)

$$u(x) = \mathbb{E}[u(X(\tau_D))|X_0 = x].$$

Note that we occasionally write  $X(t)$  instead of  $X_t$  to avoid nested subscripts.

Let us consider a  $(d+1)$ -dimensional Brownian motion  $(X_t, Y_t)$  in the half-space  $D = \mathbb{R}^d \times [0, \infty)$ , reflected at the boundary. Formally, this can be constructed from an unrestricted  $(d+1)$ -dimensional Brownian motion  $(X_t, W_t)$  by setting  $Y_t = |W_t|$ .

What is the trace left by  $(X_t, Y_t)$  on the boundary? Well, it is just a collection of dots — an uncountable one, forming a perfect set with probability one (see Figure 5). But is there a better way to view this trace? It turns out that the answer is yes: we can identify the boundary trace with a certain stochastic process on  $\mathbb{R}^d$ . This, however, requires some care, as  $(X_t, Y_t)$  spends zero time at the boundary.

What we are going to do is the following: we take out all fragments of the paths of  $(X_t, Y_t)$  which lie outside  $\mathbb{R}^d \times [0, \delta]$  (without gluing the pieces together: the resulting process has jumps), we scale the time appropriately, and we hope that in the limit as

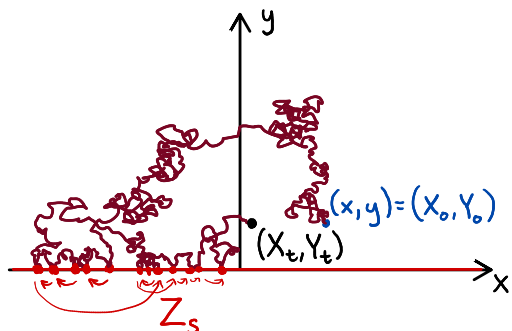


FIGURE 5. Boundary trace.

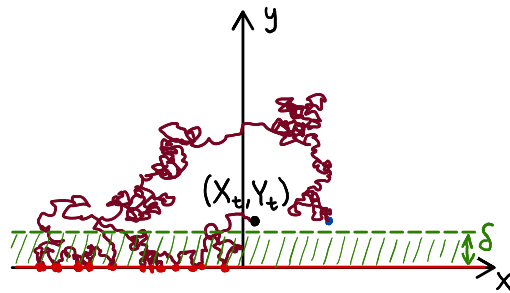


FIGURE 6. Limiting procedure.

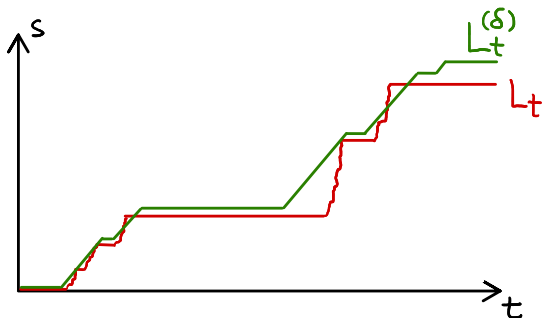


FIGURE 7. Local time.

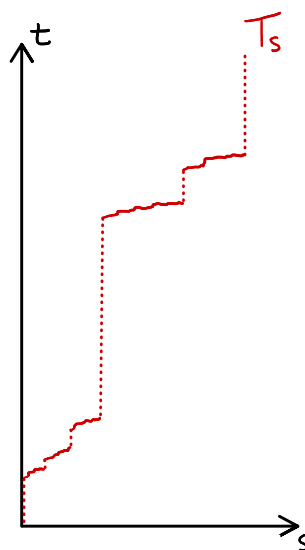


FIGURE 8. Inverse local time.

$\delta \rightarrow 0^+$  we will get a nontrivial process  $Z_t$  on the boundary (see Figure 6). A rigorous construction involves what is called the local time of  $Y_t$  at 0.

For a given  $\delta > 0$  and  $t \geq 0$  let  $L_t^{(\delta)}$  denote the amount of time spend by the process  $(X_s, Y_s)$  up to time  $t$  in the  $\delta$ -neighbourhood of the boundary:

$$L_t^{(\delta)} = \int_0^t \mathbb{1}_{[0, \delta]}(Y_s) ds.$$

Of course,  $L_t^{(\delta)}$  converges to zero as  $\delta \rightarrow 0^+$ . However, it can be proved that  $\delta^{-1}L_t^{(\delta)}$  converges with probability one to a nontrivial continuous process  $L_t$ , called the local time of  $Y_t$  at 0. Clearly,  $L_t$  is non-decreasing, and it is constant on every interval of time where  $Y_t > 0$ . The nontrivial part is that  $L_t$  is finite and nonzero; more precisely,  $Y_t$  indeed does increase at every  $t$  such that  $Y_t = 0$  with probability one. So, in other words,  $dL_t$  is a continuous measure whose support is the set of times  $t$  such that  $Y_t = 0$  (see Figure 7).

The local time is not invertible, but it admits a generalised right-continuous inverse

$$T_s = \max\{t \geq 0 : L_t \leq s\},$$

called simply the inverse local time of  $Y_t$  at 0, and commonly denoted by  $L_s^{-1}$  (see Figure 8). Then  $L(T_s) = s$  for every  $s \geq 0$ , but we only have  $T(L_t) \leq t$  for  $t \geq 0$ . Furthermore, with probability one the range of the process  $T_s$  (augmented by the left limits  $T_{s-}$ , to be precise) coincides with the set of  $t$  such that  $Y_t = 0$ . In particular,  $Y(T_s) = 0$ .

This tells us that the process

$$Z_s = X(T_s)$$

(again: augmented by left limits  $Z_{s-}$ ) is precisely what we are after: it is the trace left by  $(X_t, Y_t)$  on the boundary.

**1.4. Cauchy process.** We already know how we can construct the boundary trace process  $Z_s$ , but what kind of a stochastic process it is? An answer to this question was found by Spitzer in 1957 [20].

It turns out that  $T_s$  is a Lévy process, that is, a càdlàg (right-continuous with left limits) process with stationary and independent increments. Intuitively, this is a consequence of the strong Markov property of  $Y_t$ : the magnitudes of jumps of  $T_s$  are the lengths of excursions of  $Y_t$  away from 0. However, a rigorous proof of this property is more complicated. Since  $T_s$  is non-decreasing, by definition it is a subordinator.

Since  $X_t$  and  $T_s$  are clearly independent processes,  $X_t$  is the Brownian motion, and  $T_s$  is a subordinator, we find that  $Z_s$  is a subordinate Brownian motion. Furthermore, it is known that  $T_s$  is the  $\frac{1}{2}$ -stable subordinator, and so it follows that  $Z_s$  is the isotropic 1-stable Lévy process in  $\mathbb{R}^d$ , known under the name Cauchy process.

The Cauchy process is perhaps the the best understood multidimensional jump-type process. The random variables  $Z_s$  are known to have Cauchy distributions (which explains the name of the process), and hence the density function of  $Z_s$  coincides with the Poisson kernel for the half-space. The generator of  $Z_s$  is the square root of the Laplace operator, that is, it is equal to the Dirichlet-to-Neumann operator  $K$ . The Lévy measure of  $Z_s$ , which describes the intensity of jumps of  $Z_s$  of a given magnitude, has a density function  $c_d|z|^{-d-1}$ , identical to the non-local kernel of the operator  $K$ . Are these properties merely a coincidence, or is there a deeper theory behind them?

In the next section we study a particular modification of what has been discussed above, which already clarifies the situation. A general answer to the above questions is the main result discussed in these notes. But let us make it clear straight away: the first property (equality of the density function of  $Z_s$  and the Poisson kernel) is a serendipity, but the other two manifest certain general principles.

## Exercises.

**1.1.** Let  $K$  be the Dirichlet-to-Neumann operator for the half-space  $\mathbb{R}^d \times (0, \infty)$ . Prove that  $K$  is homogeneous with degree 1: if  $f_\lambda(x) = f(\lambda x)$ , then  $Kf_\lambda(x) = \lambda Kf(\lambda x)$ .

**1.2.** (a) Verify directly that the Poisson kernel for the half-space  $D = \mathbb{R}^d \times (0, \infty)$ :

$$P_y(x - x') = c_d \frac{y}{(|x - x'|^2 + y^2)^{(d+1)/2}},$$

is indeed a harmonic function of  $(x, y)$ .

(b) Prove that there exists a constant  $c_d$  such that

$$\int_{\mathbb{R}^d} P_y(x - x') dx' = 1.$$

(c) With this choice of  $c_d$ , consider a compactly supported continuous function  $f$ , and denote

$$u(x, y) = \int_{\mathbb{R}^d} P_y(x - x') f(x') dx'.$$

Prove that  $u$  is indeed the harmonic extension of  $f$ : a bounded solution to the Dirichlet problem in  $D$  with boundary values  $f$ .

(d) Bonus problem: Find a closed-form expression for  $c_d$ .

**1.3.** Let  $f$  be a compactly supported smooth function on  $\mathbb{R}^d$ . Prove rigorously that the Dirichlet-to-Neumann operator for the half-space is indeed given by

$$\begin{aligned} Kf(x) &= \frac{c_d}{2} \int_{\mathbb{R}^d} \frac{2f(x) - f(x+z) - f(x-z)}{|z|^{(d+1)/2}} dz \\ &= c_d \text{PV} \int_{\mathbb{R}^d} \frac{f(x) - f(x')}{|x-x'|^{(d+1)/2}} dx'. \end{aligned}$$

**1.4.** Let  $f$  be a compactly supported continuous function on  $\mathbb{R}^d$ , and let  $u$  be a bounded harmonic function on  $\mathbb{R}^d \times (0, \infty)$  with boundary values  $f$ . Prove rigorously that  $\mathcal{F}u(\xi, y) = e^{-|\xi|y} \mathcal{F}f(\xi)$  for every  $\xi \in \mathbb{R}^d$  and  $y > 0$ .

**1.5.** Let  $Y_t$  be the reflected Brownian motion in  $[0, \infty)$ , and let  $L_t$  be the local time of  $Y_t$  at zero, defined as the limit of approximate local times  $L_t^{(\delta)}$ .

- (a) Using self-similarity of  $Y_t$ , prove that  $L_t$  is self-similar with Hurst exponent  $\frac{1}{2}$ : the process  $L_{\lambda t}$  has the same law as the process  $\lambda^{1/2}L_t$ .
- (b) Observe that the inverse local time  $T_s$  is thus self-similar with Hurst exponent 2, and therefore it is the  $\frac{1}{2}$ -stable subordinator.
- (c) Conclude that the boundary trace process  $Z_s = X(T_s)$  is self-similar with Hurst exponent 1, and therefore it is the symmetric 1-stable Lévy process (the Cauchy process).

## 2. FRACTIONAL LAPLACIAN AND ISOTROPIC STABLE LÉVY PROCESSES

**2.1. Fractional Laplace operator.** There is little special about the half-Laplacian among other fractional powers of the Laplace operator. These can be defined in terms of the Fourier transform:

$$\mathcal{F}(-\Delta)^{\alpha/2}u(\xi) = |\xi|^\alpha \mathcal{F}u(\xi),$$

or as a principal value integral:

$$(-\Delta)^{\alpha/2}u(x) = c_{d,\alpha} \text{pv} \int_{\mathbb{R}^d} \frac{f(x) - f(x')}{|x - x'|^{(d+1)/2}} dx'.$$

The former definition works for an arbitrary  $\alpha > 0$  (or, in fact, arbitrary complex  $\alpha$ ). The latter one is more common, and it requires  $\alpha \in (0, 2)$  (although a similar expression involving a hyper-singular integral can be given for more general  $\alpha$ ). From now on,  $\alpha \in (0, 2)$  is our standing assumption.

Can one obtain  $(-\Delta)^{\alpha/2}$  as a Dirichlet-to-Neumann operator? Recall that the kernel of  $K$  was the normal derivative at the boundary of the Poisson kernel. Thus, a naive way to go would be to replace the  $(d+1)$ -dimensional Laplace operator by something else, so that the analogue of the Poisson kernel for the half-space  $D = \mathbb{R}^d \times (0, \infty)$  would be, say,

$$P_y(x - x') = c_{d,\alpha} \frac{y}{(|x - x'|^2 + y^2)^{(d+\alpha)/2}}.$$

This fails for obvious reasons: as  $y \rightarrow 0^+$ , the integral of  $P_y(\cdot)$  converges to 0 (if  $\alpha < 1$ ) or to  $\infty$  (if  $\alpha > 1$ ), while we require it to converge to 1. For the rescue, we can try playing with the exponents, and a simple scaling argument brings us to

$$P_y(x - x') = c_{d,\alpha} \frac{y^\alpha}{(|x - x'|^2 + y^2)^{(d+\alpha)/2}}.$$

If  $c_{d,\alpha}$  is chosen appropriately, then  $P_y(\cdot)$  turns out to have integral 1 for every  $y > 0$ , and so indeed  $(-\Delta)^{\alpha/2}$  can be identified with an operator reminiscent of the Dirichlet-to-Neumann map: it is more or less clear that the fractional Laplace operator  $(-\Delta)^{\alpha/2}$  can be expressed as

$$\begin{aligned} Kf(x) &= - \lim_{y \rightarrow 0^+} \frac{u(x, y) - u(x, 0)}{y^\alpha} \\ &= - \lim_{y \rightarrow 0^+} \frac{1}{\alpha y^{\alpha-1}} \frac{\partial u}{\partial y}(x, y), \end{aligned}$$

where

$$u(x, y) = \int_{\mathbb{R}^d} f(x') P_y(x - x') dx'$$

is an extension of  $f$  to the half-space  $\mathbb{R}^d \times (0, \infty)$ . (This last statement is slightly wrong:  $K$  is not exactly equal to  $(-\Delta)^{\alpha/2}$ . Can you spot the error?)

Perhaps the most surprising aspect of the above construction is that...it actually works, in the sense that it is linked to a certain partial differential equation in the half-space. One can directly verify that the Poisson kernel  $P_y(x - x')$ , and therefore also the extension  $u$ , satisfies the following modified Laplace equation:

$$\Delta_{x,y}u(x, y) + \frac{1 - \alpha}{y} \frac{\partial u}{\partial y}(x, y) = 0,$$

which can be written as an equation in divergence form:

$$\nabla_{x,y}(y^{1-\alpha} \nabla_{x,y}u)(x, y) = 0.$$



That is,  $u(x, y)$  is ‘harmonic’ with respect to the operator

$$L = \Delta_{x,y} + \frac{1 - \alpha}{y} \frac{\partial}{\partial y},$$

or, equivalently, with respect to

$$y^{1-\alpha} L = \nabla_{x,y}(y^{1-\alpha} \nabla_{x,y}).$$

While the above construction may seem very natural, it has not been observed until the seminal work by Caffarelli and Silvestre from 2007 [2], and it is known as the Caffarelli–Silvestre extension technique.

We leave the details of the above calculations aside, and instead we discuss an alternative solution that involves the Fourier transform. It allows one to turn the PDE  $Lu = 0$  into an ODE

$$-|\xi|^2 \mathcal{F}u(\xi, y) + \frac{\partial^2 \mathcal{F}u}{\partial y^2}(\xi, y) + \frac{1 - \alpha}{y} \frac{\partial \mathcal{F}u}{\partial y}(\xi, y) = 0.$$

This is a version of the Bessel differential equation, and it has two linearly independent solutions, but only one of them is bounded. It follows that

$$\mathcal{F}u(\xi, y) = c_\alpha (y|\xi|)^{\alpha/2} K_{\alpha/2}(y|\xi|) \mathcal{F}f(\xi),$$

where  $K_{\alpha/2}$  is the modified Bessel function of the second kind, and one can verify directly that

$$\mathcal{F}Kf(\xi) = - \lim_{y \rightarrow 0^+} \frac{\mathcal{F}u(\xi, y) - \mathcal{F}u(\xi, 0)}{y^\alpha} = c_\alpha |\xi|^\alpha \mathcal{F}f(\xi)$$

for appropriate constants  $c_\alpha$  (different from the one in the previous equation; we do not pay much attention to explicitly known constants in these notes).

The Caffarelli–Silvestre extension technique allows one to convert certain non-local problems in  $\mathbb{R}^d$  to local problems in  $\mathbb{R}^d \times (0, \infty)$ . Many results of the theory of PDEs are available only for local problems, either because they generally fail in the non-local setting, or because the methods to tackle their non-local variants have not been developed yet. In these cases the Caffarelli–Silvestre extension technique is the right way to go.

The PDE discussed above appeared already in an article by Muckenaupt and Stein from 1965 [18], but in a much different context. Quite surprisingly, however, a probabilistic counterpart of the above construction goes back to the paper by Molchanov and Ostrovskii from 1969 [17].

**2.2. Isotropic stable Lévy process.** We have seen that the boundary trace of the  $(d + 1)$ -dimensional Brownian motion  $(X_t, Y_t)$  in the half-space is the isotropic 1-stable Lévy process  $Z_s$ , and the generator of  $Z_s$  coincides (up to a sign) with the Dirichlet-to-Neumann operator  $K = \sqrt{-\Delta}$ . What happens if we replace the Laplace operator by the operator  $L$  introduced above? The Dirichlet-to-Neumann operator becomes the fractional Laplace operator  $K = (-\Delta)^{\alpha/2}$ . But what happens to the boundary trace process?

Recall that

$$L = \Delta_x + \left( \frac{\partial^2}{\partial y^2} + \frac{1 - \alpha}{y} \frac{\partial}{\partial y} \right).$$

The operator  $\frac{1}{2}L$  is the generator of a  $(d + 1)$ -dimensional diffusion process  $(X_t, Y_t)$  with independent coordinates  $X_t$  and  $Y_t$ . Furthermore,  $X_t$  is the  $d$ -dimensional Brownian motion (with generator  $\frac{1}{2}\Delta$ ), while as long as it remains positive,  $Y_t$  is an Itô diffusion of the form

$$dY_t = dW_t + \frac{1 - \alpha}{2} \frac{1}{Y_t} dt,$$

where  $W_t$  is some one-dimensional Brownian motion. This is a familiar stochastic differential equation: it defines the Bessel process of (fractional) dimension  $2 - \alpha$ . The point 0

is a regular (that is, accessible and non-absorbing) boundary point for  $Y_t$  precisely when  $\alpha \in (0, 2)$ , and so we may require  $Y_t$  to be reflected at 0. Then there is a way to define the local time  $L_t$  of  $Y_t$  at 0 (with a different scaling of approximate local times, though), and we may again define the inverse local time  $T_s$  and the boundary trace process  $Z_s = X(T_s)$ .

Once again  $Z_s$  is a Lévy process, and a straightforward scaling argument shows that  $Z_s$  is self-similar with Hurst exponent  $\frac{1}{\alpha}$ . Thus,  $Z_s$  is the isotropic  $\alpha$ -stable Lévy process, and so the corresponding generator is, up to a constant factor, the fractional Laplace operator  $-(-\Delta)^{\alpha/2}$ .

The result sketched above was observed and rigorously proved by Molchanov and Ostrovskii already in 1969 [17]. The generator of the boundary trace process  $Z_s$  again coincides with the Dirichlet-to-Neumann operator  $K$ : both are equal to the fractional Laplace operator  $(-\Delta)^{\alpha/2}$ , up to a constant factor. Furthermore, the density of the Lévy measure of  $Z_s$  is again equal to the non-local kernel of  $K$ . However, the density of the distribution of  $Z_s$  is no longer equal to the Poisson kernel: we have evaluated the Fourier transform of  $P_y$  and it is not of the form  $\exp(-s|\xi|^\alpha)$ .

The Molchanov–Ostrovskii boundary trace technique described above is a perfect probabilistic analogue of the Caffarelli–Silvestre extension technique in the theory of PDEs. It allows one to study the isotropic stable Lévy processes using the methods available for diffusions. In the remaining sections we explore to what extent we can generalise the results of Caffarelli and Silvestre on the PDE side, and of Molchanov and Ostrovskii on the probability side, and why these two approaches are in fact essentially equivalent.

### Exercises.

- 2.1.** (a) Verify that the Poisson kernel for the Caffarelli–Silvestre extension indeed satisfies the modified Laplace equation.  
 (b) Check that with an appropriate choice of the constant  $c_{d,\alpha}$ , the Poisson kernel has integral 1.  
 (c) Bonus problem: Find a closed-form expression for this constant  $c_{d,\alpha}$ .
- 2.2.** (a) Prove rigorously that the Dirichlet-to-Neumann operator  $K$  for the Caffarelli–Silvestre extension is, up to a constant factor, indeed the fractional Laplace operator (given as the principal value integral).  
 (b) Bonus problem: Find a closed-form expression for that constant factor.
- 2.3.** Substitute  $y = z^{1/\alpha}$  and prove that in variables  $(x, z)$  the extension  $u$  satisfies

$$\alpha^{-2}y^{2/\alpha-2}\Delta_x u(x, y) + \frac{\partial^2 u}{\partial y^2}(x, y) = 0.$$

Prove that in these coordinates the fractional Laplace operator is, up to a constant factor, again equal to the usual Dirichlet-to-Neumann operator:

$$Kf(x) = -\frac{\partial u}{\partial z}(x, 0),$$

- 2.4.** Bonus problem: Evaluate the constants in the solution of the Caffarelli–Silvestre extension problem involving Fourier transforms.
- 2.5.** (a) Prove that the Bessel process  $Y_t$  is self-similar with Hurst exponent  $\frac{1}{2}$ .  
 (b) The local time of  $Y_t$  at 0 needs to be defined with a different scaling factor:

$$L_t = \lim_{\delta \rightarrow 0^+} \frac{L_t^{(\delta)}}{\delta^{2-\alpha}}.$$

Prove that  $L_t$  is self-similar with Hurst exponent  $\frac{\alpha}{2}$ .

- (c) Observe that the inverse local time  $T_s$  is self-similar with Hurst exponent  $\frac{2}{\alpha}$ , and therefore it is the  $\frac{\alpha}{2}$ -stable subordinator.
- (d) Conclude that the boundary trace process  $Z_s$  is self-similar with Hurst exponent  $\frac{1}{\alpha}$ , and hence it is the isotropic  $\alpha$ -stable Lévy process.

### 3. DIRICHLET-TO-NEUMANN OPERATORS AND KREIN'S STRINGS

**3.1. General nonsense.** By far, our programme is hopefully clear: we study differential operators  $L$  in the half-space  $D = \mathbb{R}^d \times (0, \infty)$ , and reflected diffusions  $(X_t, Y_t)$  with generators  $\frac{1}{2}L$ . We would like to establish a link between the Dirichlet-to-Neumann operator  $K$  corresponding to  $L$ , and the generator of the boundary trace of  $(X_t, Y_t)$ .

What class of operators can we handle? In principle, we should be able to cover essentially arbitrary elliptic second-order differential operators  $L$ , as long as they generate a reasonably nice diffusion process. After all, all that we need is the Poisson kernel on the PDE side, and the local time on the probability side. Although apparently no such general results are known, the task seems feasible. It would likely require a slightly different definition of the approximate local time (with  $\mathbb{1}_{[0,\delta]}(X_s)$  replaced by the indicator of the following event: the last hitting time of 0 up to time  $s$  is greater than the last hitting time of  $\delta$  up to time  $s$ ), and an appropriately modified normalisation of the local time and the Dirichlet-to-Neumann operator. Otherwise, however, the argument should follow a more or less straightforward path. Furthermore, in this case there is in fact no reason to require that  $D$  be the half-space.

Nevertheless, we will not attempt to work with that general operators  $L$ . (If you are interested in doing so, however, do let me know!) The reasons for that are two-fold. First, we have in mind applications to non-local PDEs in  $\mathbb{R}^d$ , and so it is important for us to understand the class of operators  $K$  that we can handle. Second, on the probability side we do want to work with the usual definition of the local time.

**3.2. Elliptic operators.** In what follows we consider elliptic operators of the form

$$L = a(y)\Delta_x + b(y)\frac{\partial}{\partial y} + c(y)\frac{\partial^2}{\partial y^2},$$

where  $a$ ,  $b$  and  $c$  are appropriately regular coefficients. In other words, we require  $L$  to be invariant under isometric transformations (rotations, translations, reflections) of variable  $x$ .

Since we are only concerned with functions harmonic with respect to  $L$ , we may freely multiply  $L$  by a function of  $y$ . On the probability side, this corresponds to a random time change, which obviously does not affect the trace process.

Similarly, we can change the coordinate  $y$  in an arbitrary (possibly nonlinear) way. This change of scale preserves both the Dirichlet-to-Neumann operator and the boundary trace process, as long as we take the change of scale into account in the denominators in the definitions of the Dirichlet-to-Neumann operator  $K$  and the local time  $L_t$ .

By combining the above two operations, and ignoring completely regularity issues, we can cast an arbitrary operator  $L$  described above into a form that is convenient for our needs, essentially reducing the number of coefficients depending on  $y$  from three ( $a$ ,  $b$  and  $c$ ) to one. A natural choice from the point of view of applications in probability is the operator

$$L = \Delta_x + b(y)\frac{\partial}{\partial y} + \frac{\partial^2}{\partial y^2},$$

which is (up to a constant factor  $\frac{1}{2}$ ) the generator of a particularly simple Itô diffusion, the Brownian motion with a state-dependent drift. In PDEs, one would likely prefer the same operator written in divergence form

$$L = \nabla_{x,y}(\beta(y)\nabla_{x,y}).$$

The former one can be obtained from the latter if we divide both sides by  $\beta(y)$  and set  $b(y) = \beta'(y)/\beta(y)$ , and going in the opposite direction is equally simple. These two

forms are the ones that we used in the Caffarelli–Silvestre extension technique and in the Molchanov–Ostrovskii boundary trace technique.

When it comes to actually proving things, it is, however, much more convenient to work with the operators

$$L = a(y)\Delta_x + \frac{\partial^2}{\partial y^2},$$

where  $a$  is a non-negative locally integrable function. Additionally, with this form of  $L$  the Dirichlet-to-Neumann operator is defined in a standard way. Finally, our most general statements require a generalisation that becomes easy to describe only when  $L$  is as above.

The last sentence is to be understood as follows: if we agree to include operators on  $\mathbb{R}^d \times [0, R)$  with a possibly finite  $R \in (0, \infty]$  (and with an additional zero boundary condition on  $\mathbb{R}^d \times \{R\}$  whenever necessary), and if we allow the coefficient  $a$  to be a locally finite non-negative measure on  $[0, R)$ , then we obtain a perhaps surprising ‘if and only if’ result. To keep things simple, though, with few exceptions we will always pretend that  $a$  is a function and  $R = \infty$ . That said, you may verify that everything that follows actually works in the general case, with minor modifications hinted in remarks.

**3.3. Krein’s spectral theory of strings.** After this very long introduction, the fun part begins: we come to the heart of the PDE section of these notes.

Applying the Fourier transform to both sides of the equation  $Lu = 0$ , with  $L$  defined above, leads to the ODE

$$\frac{\partial^2 \mathcal{F}u}{\partial y^2}(\xi, y) = |\xi|^2 a(y) \mathcal{F}u(\xi, y),$$

with initial condition  $\mathcal{F}u(\xi, 0) = \mathcal{F}f(\xi)$ , and with  $\mathcal{F}u(\xi, y)$  bounded with respect to  $y$ .

If we substitute  $\lambda = |\xi|^2$  and  $\varphi(y) = \mathcal{F}u(\xi, y)$ , we arrive at the eigenvalue problem for a second-order ordinary differential operator, studied by Krein:

$$\frac{\partial^2 \varphi}{\partial y^2}(y) = \lambda a(y) \varphi(y).$$

If  $\varphi_\lambda$  is a bounded solution satisfying  $\varphi(0) = 1$ , then the extension  $u$  is given by

$$\mathcal{F}u(\xi, y) = \varphi_\lambda(y) \mathcal{F}f(\xi),$$

where  $\lambda = |\xi|^2$ .

It is a routine application of Picard’s iteration to prove that for every complex  $\lambda$ , there exists a solution  $\varphi$  of Krein’s ODE with any prescribed  $\varphi(0)$  and  $\varphi'(0)$ . Let us denote by  $\varphi_0$  the ‘Dirichlet’ solution with  $\varphi_0(0) = 0$ ,  $\varphi_0'(0) = 1$ , and let  $\varphi_1$  be the ‘Neumann’ solution with  $\varphi_1(0) = 1$ ,  $\varphi_1'(0) = 0$ . Our goal is to construct a solution  $\varphi = \varphi_1 - \mu\varphi_0$  satisfying the initial condition  $\varphi(0) = 1$  and which is bounded on  $[0, \infty)$ . The last condition is a kind of a ‘terminal’ condition at  $\infty$ , so essentially we are solving a variant of the Sturm–Liouville eigenvalue problem.

When  $\lambda \geq 0$ , both fundamental solutions  $\varphi_0$  and  $\varphi_1$  are non-decreasing, and we take

$$\mu = \lim_{y \rightarrow \infty} \frac{\varphi_1(y)}{\varphi_0(y)}.$$

It is fairly easy to see that  $\varphi_1(y)/\varphi_0(y)$  is decreasing:

$$\left( \frac{\varphi_1}{\varphi_0} \right)' = \frac{\varphi_0 \varphi_1' - \varphi_1 \varphi_0'}{(\varphi_0)^2}$$

and since the Wronskian  $\varphi_0 \varphi_1' - \varphi_1 \varphi_0'$  has constant sign, the right-hand side is negative. This property implies that  $\mu$  is well-defined, and additionally  $\varphi_1/\varphi_0 \geq \mu$ , that is,  $\varphi_1 -$

$\mu\varphi_0 \geq 0$ . A slightly more careful analysis reveals that  $\varphi_1 - \mu\varphi_0$  is in fact non-increasing. The argument is a bit awkward, we pass to the limit as  $y^* \rightarrow \infty$  after making the following point: the function  $\varphi_1 - (\varphi_1(y^*)/\varphi_0(y^*))\varphi_0$  is positive up to  $y^*$  and then negative (by monotonicity of  $\varphi_1/\varphi_0$ ), and so also convex up to  $y_0$  and then concave (because it is a solution of Krein's ODE), and this is only possible if it is decreasing.

Since non-increasing and non-negative functions on  $[0, \infty)$  are bounded, we see that

$$\varphi = \varphi_1 - \mu\varphi_0$$

is the solution we have been looking for. Clearly, we have  $\varphi'(0) = -\mu$ . What we have proved so far is summarised in the following statement.

**Proposition 1.** *Given  $\lambda \geq 0$ , there is a unique bounded solution  $\varphi_\lambda$  of Krein's ODE*

$$\frac{\partial^2 \varphi}{\partial y^2}(y) = \lambda a(y)\varphi(y)$$

satisfying  $\varphi_\lambda(0) = 1$ .

Let us comment briefly on the general case. When  $a$  is a measure, Krein's ODE should be understood in the integral form

$$\varphi(y) = \varphi(0) + \varphi'(0)y + \int_0^y \left( \int_{[0, y')} \lambda a(y'')\varphi(y'')dy'' \right) dy'.$$

In particular, if  $a(\{0\}) > 0$ , then  $\varphi'(0)$  has a rather unexpected, and very unfortunate, meaning: the right-hand side derivative of  $\varphi$  at 0 is equal to  $\varphi'(0) + a(\{0\})\varphi(0)$  (for this reason some authors tend to use  $\varphi'(0^-)$  instead). If  $R$  is finite and  $\int_{[0, R)} (R - y)a(dy)$  is finite, then every solution  $\varphi$  is bounded, and the one we are interested in is characterised by the additional Dirichlet boundary condition  $\varphi(R) = 0$ . Otherwise, the argument is exactly the same. These remarks implicitly apply to the proposition given above, as well as to all results that follow.

The measure  $a$  can be understood as the distribution of mass of a (perfectly elastic) vibrating string. For this reason, Krein used the word string for the distribution function of  $a$ , that is,  $y \mapsto a([0, y))$ , and the study of the spectral problem discussed above is widely known as Krein's theory of strings.

We will shortly see that the Dirichlet-to-Neumann operator  $K$  corresponding to the elliptic operator  $L$  is a multiplication operator in the Fourier space, with symbol  $\mu(|\xi|^2)$ , where  $\mu(\lambda) = -\varphi'_\lambda(0)$  (this is the same number  $\mu$  as the one constructed above). So what do we know about the function  $\mu$ ? The answer is provided by the following beautiful and deep result proved by Krein in 1952 [12].

**Theorem 2.** (a) *If  $\mu(\lambda) = -\varphi'_\lambda(0)$ , then  $\mu$  is a complete Bernstein function; that is,  $\mu(\lambda) \geq 0$  for  $\lambda \geq 0$ , and  $\mu$  extends to a holomorphic function on  $\mathbb{C} \setminus (-\infty, 0)$  which preserves the upper and lower complex half-planes. Equivalently:*

$$\mu(\lambda) = p + q\lambda + \int_{(0, \infty)} \frac{\lambda}{\lambda + s} m(ds)$$

for some  $p, q \geq 0$  and a non-negative measure  $m$  for which the above integral is finite.

(b) *Every complete Bernstein function can be represented in the way described above, for a unique coefficient  $a$ .*

We stress that when we say 'for a unique coefficient  $a$ ', what we really mean is: for a unique  $R \in (0, \infty]$  and a unique non-negative locally finite measure  $a(dy)$  on  $[0, R)$ .

We stress even more that Krein's theorem is not constructive. Given the coefficient  $a$ , if one is lucky, one can solve the corresponding ODE and get an expression for the complete Bernstein function  $\mu$ . However, reversing this procedure is practically impossible, unless  $\mu$  has some special form. (For example, if  $\mu$  is a rational function, then the corresponding coefficient  $a$  can be reconstructed using an appropriate continued fraction expansion. In this case, however,  $a$  turns out to be a purely atomic measure.)

Finally, we remark that  $1/\mu$  is known as the spectral function for the Krein's spectral problem.

**3.4. Dirichlet-to-Neumann operators.** Our construction of the solution of Krein's spectral problem given in the first proposition above immediately leads to the following statement.

**Corollary 3.** *Let us write  $\varphi(\lambda, y)$  instead of  $\varphi_\lambda(y)$  for the bounded solution of Krein's ODE satisfying  $\varphi_\lambda(0) = 1$ . Given a square integrable function  $f$  on  $\mathbb{R}^d$ , define  $u$  in terms of the Fourier transform:*

$$\mathcal{F}u(\xi, y) = \varphi(|\xi|^2, y)\mathcal{F}f(\xi).$$

*Then, in an appropriate  $L^2$  sense,  $u$  is a solution of the PDE*

$$a(y)\Delta_x u(x, y) + \frac{\partial^2 u}{\partial y^2}(x, y) = 0,$$

*and the  $L^2$  norm of  $u(\cdot, y)$  does not exceed the  $L^2$  norm of  $f$  for every  $y > 0$ .*

Of course, the 'appropriate  $L^2$  sense' above means in particular that the Fourier transform of  $u$  is a 'classical' solution of Krein's ODE for each parameter  $\xi$ , and at the same time  $y \mapsto u(\cdot, y)$  is a bounded and continuous  $L^2$ -valued function. More can be said about  $u$ ; for example, it satisfies the PDE in the sense of distributions, but we will not go in that direction.

The above proposition in particular means that the Poisson kernel  $P_y$  for the operator  $L$  in the upper half-space has Fourier transform

$$\mathcal{F}P_y(\xi) = \varphi(|\xi|^2, y).$$

In the next section, using probabilistic tools, we will easily verify that  $P_y$  is nonnegative and its integral does not exceed one. Interestingly, a purely analytical proof of this fact seems to require much more effort.

Since  $\varphi_\lambda$  is a convex function, the difference quotient of  $-\varphi_\lambda$  on the interval  $[0, y]$  is positive and it increases to  $\mu(\lambda) = -\varphi'_\lambda(0)$  as  $y$  decreases to 0. Thus, Lebesgue's dominated convergence theorem implies that the Dirichlet-to-Neumann operator  $K$  corresponding to the elliptic operator  $L$  is given by

$$\mathcal{F}Kf(\xi) = -\frac{\partial \mathcal{F}u}{\partial y}(\xi, 0) = -\frac{\partial \varphi}{\partial y}(|\xi|^2, 0)\mathcal{F}f(\xi) = \mu(|\xi|^2)\mathcal{F}f(\xi)$$

as long as the function  $\mu(|\xi|^2)\mathcal{F}f(\xi)$  is square integrable. We have thus essentially proved the first two parts of the following result, given in my joint paper with Jacek Mucha from 2018 [16].

**Proposition 4.** (a) *Dirichlet-to-Neumann operators  $K$  corresponding to elliptic operators of the form*

$$L = a(y)\Delta_x + b(y)\frac{\partial}{\partial y} + c(y)\frac{\partial^2}{\partial y^2}$$

are complete Bernstein functions of  $-\Delta$ . That is, there is a complete Bernstein function  $\mu$  such that

$$\mathcal{F}Kf(\xi) = \mu(|\xi|^2)\mathcal{F}f(\xi)$$

whenever  $f$  and the right-hand side are square integrable.

- (b) Every complete Bernstein function of  $-\Delta$  can be represented in the way described above, for a unique operator of the form

$$L = a(y)\Delta_x + \frac{\partial^2}{\partial y^2}.$$

- (c) For  $L$  as in the previous item, the following Dirichlet's principle holds:

$$\begin{aligned} & \int_{\mathbb{R}^d} f(x)Kf(x)dx \\ &= \min \left\{ \frac{1}{2} \int_{\mathbb{R}^d} \int_0^\infty \left( a(y)|\nabla_x v(x, y)|^2 + \left( \frac{\partial v}{\partial y}(x, y) \right)^2 \right) dy dx \right\}, \end{aligned}$$

where the minimum is taken over all functions  $v$  which are suitable differentiable, converge suitably to 0 as  $y \rightarrow \infty$ , and have boundary values  $f$ .

Complete Bernstein functions have been studied a lot, see Chapters 6 and 7 in the excellent book [19] by Schilling, Song and Vondraček. Complete Bernstein functions of  $-\Delta$  play an important role in the theory of non-local PDEs and in the theory of Lévy processes.

## Exercises.

- 3.1.** Prove that the Dirichlet-to-Neumann operator corresponding to the operator

$$L = \mathbb{1}_{(0,1)}(y)\Delta_x + \frac{\partial^2}{\partial y^2}$$

in  $\mathbb{R}^d \times (0, \infty)$ , is  $K = \sqrt{-\Delta} \tanh \sqrt{-\Delta}$ , in the sense that the complete Bernstein function arising in the corresponding Krein's spectral problem is  $\mu(\lambda) = \sqrt{\lambda} \tanh \sqrt{\lambda}$ . This operator  $K$  arises naturally in the theory of linear water waves.

- 3.2.** Prove that the Dirichlet-to-Neumann operator corresponding to the operator

$$L = \Delta_{x,y} \quad \text{in } \mathbb{R}^d \times (0, 1),$$

with an implicit Dirichlet condition  $u(x, 1) = 0$ , is  $K = \sqrt{-\Delta} / \tanh \sqrt{-\Delta}$ .

- 3.3.** Prove that the Dirichlet-to-Neumann operator corresponding to the operator

$$L = \frac{\partial^2}{\partial y^2}$$

in  $\mathbb{R}^d \times (0, \infty)$  is the zero operator  $K = 0$ .

- 3.4.** Prove that the Dirichlet-to-Neumann operator corresponding to the operator

$$L = \frac{\partial^2}{\partial y^2} \quad \text{in } \mathbb{R}^d \times (0, 1),$$

with an implicit Dirichlet condition  $u(x, 1) = 0$ , is the identity operator  $K = 1$ .

- 3.5.** Prove that the Dirichlet-to-Neumann operator corresponding to the operator

$$L = \delta_1(dy)\Delta_x + \frac{\partial^2}{\partial y^2}$$

in  $\mathbb{R}^d \times (0, \infty)$  is  $K = -\Delta(1 - \Delta)^{-1}$ .



**3.6.** Prove that the Dirichlet-to-Neumann operator corresponding to the operator

$$L = \delta_0(dy)\Delta_x + \frac{\partial^2}{\partial y^2}$$

in  $\mathbb{R}^d \times (0, \infty)$  is, up to a sign, the Laplace operator  $K = -\Delta$ .

**3.7.** Prove that the Dirichlet-to-Neumann operator corresponding to the operator

$$L = \frac{1}{(1+2y)^2} \Delta_x + \frac{\partial^2}{\partial y^2}$$

in  $\mathbb{R}^d \times (0, \infty)$  is the quasi-relativistic operator  $K = \sqrt{-\Delta + 1} - 1$ .

**3.8.** Prove that the Dirichlet-to-Neumann operator corresponding to the operator

$$L = \frac{1}{(1-2y)^2} \Delta_x + \frac{\partial^2}{\partial y^2} \quad \text{in } \mathbb{R}^d \times (0, \frac{1}{2})$$

is  $K = \sqrt{-\Delta + 1} + 1$ . (Note that the Dirichlet condition  $u(x, \frac{1}{2}) = 0$  is automatically satisfied by bounded solutions.)

**3.9.** Bonus problem: Find an elliptic operator  $L$  such that the corresponding Dirichlet-to-Neumann operator is  $K = \sqrt{-\Delta + 1}$ .

## 4. BOUNDARY TRACES OF DIRICHLET-TO-NEUMANN OPERATORS

**4.1. Intuitive argument.** We now study essentially the same problem from the probabilistic perspective: our goal is to find a link between boundary traces of diffusions and Dirichlet-to-Neumann operators  $K$  corresponding to their generators  $L$ . This may appear nearly obvious at an intuitive level, but a direct approach does not seem feasible unless some regularity of the coefficients of  $L$  near the boundary is assumed. For this reason, we tackle the problem in a more analytical fashion. Before we proceed, however, let us make one remark, and then try a different, seemingly more probabilistic approach.

We consider a diffusion process  $(X_t, Y_t)$  in  $\mathbb{R}^d \times [0, \infty)$ , reflected at the boundary. The boundary trace process is given by  $Z_s = X(T_s)$ , where  $T_s$  is the inverse local time of  $Y_t$  at 0. It is clear that the jumps of  $Z_s$  correspond to excursions of  $(X_t, Y_t)$  away from the boundary. One can prove that if  $(X_0, Y_0) = (x, y)$  and if  $\tau_0$  is the hitting time of 0 for  $Y_t$ , then the random variable  $X(\tau_0)$  has a density function  $P_y(x - x')$ . Thus, as  $y \rightarrow 0^+$ , appropriately normalised measures  $P_y(x - x')dx'$  are expected to converge to the distribution of the endpoint of an excursion started at  $(x', 0)$ , which we already identified with the Lévy measure (that is, the intensity of jumps) of  $Z_s$ . On the other hand, appropriately normalised Poisson kernels  $P_y(x - x')$  converge to the non-local kernel of the Dirichlet-to-Neumann operator. This may convince us that indeed these two kernels are the same thing. A rigorous argument of that kind, however, seems to be out of reach.

**4.2. Generalised diffusions.** In the previous section we studied elliptic operators of the form

$$L = a(y)\Delta_x + \frac{\partial^2}{\partial y^2},$$

and now we would like to study diffusion processes generated by these operators (or, strictly speaking, by  $\frac{1}{2}L$ ). Recall, however, that we have much flexibility here: we can change the above operator  $L$  to a different, equivalent form using change of scale and multiplication by a given function. Therefore, it seems reasonable to consider instead an equivalent operator

$$L = \Delta_x + \frac{1}{a(y)} \frac{\partial^2}{\partial y^2},$$

or perhaps the operator which corresponds to the Brownian motion with a state-dependent drift:

$$L = \Delta_x + b(y) \frac{\partial}{\partial y} + \frac{\partial^2}{\partial y^2}.$$

These operators have two separate parts, one of which is simply  $\Delta_x$ , and the other one is given purely in terms of variable  $y$ . Thus, the diffusion  $(X_t, Y_t)$  with generator  $\frac{1}{2}L$  has independent components, and the first one,  $X_t$ , is simply the Brownian motion in  $\mathbb{R}^d$ . This independence immediately tells us what we get: the trace process  $Z_s = X(T_s)$  is the subordinate Brownian motion, determined by the subordinator  $T_s$  obtained as the inverse local time of  $Y_t$  at 0. Therefore, it remains to determine what kind of subordinators  $T_s$  can be obtained in this way.

This question was raised by Itô and McKean in 1965 in their book [10]. While a complete characterisation for Itô diffusions is not known, an answer for a broader class of generalised diffusions, or gap diffusions, was given independently by Knight in 1981 [13] and by Kotani and Watanabe in 1982 [11]: the subordinator  $T_s$  can be obtained as the inverse local time of a reflected generalised diffusion if and only if its characteristic (Laplace) exponent is a complete Bernstein function.

As you have certainly guessed, the generators of generalised diffusions are precisely operators of the form

$$\frac{1}{a(y)} \frac{\partial^2}{\partial y^2},$$

on an infinite or finite interval  $[0, R)$ , where in fact we allow  $a$  to be a locally finite measure. And it will be no surprise to you that the main tool employed in this result is Krein's spectral theory of strings.

While the above argument is in some sense satisfactory, arguably it provides little insight into the relation between the Dirichlet-to-Neumann operator and the generator of the boundary trace process. These two clearly turn out to coincide, but except for the fact that the same tool is used as a black box in the proofs, the two arguments seem very loosely related. It is therefore difficult to apply the same approach if we allow for an extra term in  $L$  of the form  $\sum_{j=1}^d b_j(y) \frac{\partial^2}{\partial x_j \partial y}$ .

Let us make one more remark here. The name gap diffusion refers to the fact that if  $a(y) = 0$  on some maximal interval, then the process  $Y_t$  jumps between the endpoints of that interval and it never enters its interior. If this phenomenon occurs in the vicinity of the boundary point 0, then our intuition with excursions and limits of the Poisson kernel described in the beginning of this section breaks down.

**4.3. MAP.** Quite surprisingly, one gets far better results by studying the same operator as in the previous section:

$$L = a(y)\Delta_x + \frac{\partial^2}{\partial y^2}.$$

The operator  $\frac{1}{2}L$  is the generator of a diffusion  $(X_t, Y_t)$  such that  $X_t$  is no longer the Brownian motion in  $\mathbb{R}^d$ , and  $X_t$  and  $Y_t$  are no longer independent. On the other hand, however,  $Y_t$  is simply the reflected Brownian motion in  $[0, \infty)$ , which regulates the otherwise Brownian evolution of  $X_t$ . This means that the pair  $(Y_t, X_t)$  is a Markov additive process, abbreviated as MAP, with  $Y_t$  being the regulator term, and  $X_t$  the additive term. Fortunately, in this section we will not need any tools from the theory of MAPs. What follows is essentially identical to the calculation carried out in a slightly different context in my article from 2023 [15].

We can construct  $(X_t, Y_t)$  using a pair of independent Brownian motions  $W_t$  (unrestricted in  $\mathbb{R}^d$ ) and  $Y_t$  (reflected in  $[0, \infty)$ ) by setting

$$dX_t = \sqrt{a(Y_t)} dW_t.$$

Clearly, the quadratic variation of  $X_t$  is

$$\langle X \rangle_t = \int_0^t a(Y_s) ds.$$

It turns out, however, that there is a better way to go.

Let us again start with the same pair  $(W_t, Y_t)$ , and denote by

$$A_t = \int_0^t a(Y_s) ds$$

the non-decreasing additive functional of  $Y_t$  (with Revuz measure  $a(y)dy$  if one is familiar with this concept). Now we set

$$X_t = W(A_t).$$

Then  $X_t$  is a continuous martingale with quadratic variation  $\langle X \rangle_t = A_t$ , and so the pair  $(X_t, Y_t)$  is exactly what we need.

It is the right moment to mention that when  $a$  is a measure rather than a function, then the above definitions get slightly more complicated: in order to properly define  $A_t$ ,

we integrate with respect to  $a(dy)$  the family of local times of  $Y_t$  at all points  $y \geq 0$ . But this is a standard procedure in this business (in sharp contrast to the problems that occur with the other possible definition  $dX_t = \sqrt{a(Y_t)} dW_t$ ).

**4.4. Itô calculus.** The main advantage of the above construction is that it allows us to calculate various quantities related to  $(X_t, Y_t)$  in an effective way, and link them with the objects that we used in the previous section. In order to do so, we get rid of  $X_t$  and the underlying Brownian motion  $W_t$  by considering a probabilistic counterpart of the Fourier transform with respect to variable  $x$ . Namely, for a given value of the Fourier parameter  $\xi$  we evaluate the ‘partial’ expectation of

$$\exp(-i\xi X_t)$$

with respect to the process  $W_t$ , while leaving the process  $Y_t$  random. This is formally written as the conditional expectation with respect to the  $\sigma$ -algebra generated by  $Y_t$ , but we prefer to use a simpler (and more common sense) notation  $\mathbb{E}_W$ . Also, here and below, with no loss of generality, we assume that  $(X_0, Y_0) = (0, 0)$ . Thus, we define

$$\Xi_t = \mathbb{E}_W[\exp(-i\xi X_t)].$$

Since  $X_t = W(A_t)$  and  $A_t$  is a functional of  $Y_t$ , which is independent from  $W_t$ , we simply have

$$\Xi_t = \mathbb{E}_W[\exp(-i\xi W(A_t))] = \exp(-\frac{1}{2}|\xi|^2 A_t).$$

In other words,  $\Xi_t$  is a non-increasing process started at  $\Xi_0 = 1$ , satisfying

$$\begin{aligned} d\Xi_t &= -\frac{1}{2}|\xi|^2 \Xi_t dA_t \\ &= -\frac{1}{2}|\xi|^2 \Xi_t a(Y_t) dt. \end{aligned}$$

Now we only have one source of randomness left: the reflected Brownian motion  $Y_t$  in  $[0, \infty)$ . The term  $|\xi|^2 a(y)$  on the right-hand side of the expression for  $d\Xi_t$  has already appeared above, in the right-hand side of Krein’s ODE

$$\varphi''(y) = |\xi|^2 a(y) \varphi(y)$$

studied in the previous section. So let us denote by  $\varphi$  the bounded solution which satisfies  $\varphi(1) = 0$ , denoted by  $\varphi(|\xi|^2, \cdot)$  in the previous section, and let us inspect the process

$$\Phi_t = \varphi(Y_t).$$

By Itô’s lemma,

$$\begin{aligned} d\Phi_t &= \varphi'(Y_t) dY_t + \frac{1}{2} \varphi''(Y_t) dt \\ &= \varphi'(Y_t) dY_t + \frac{1}{2} |\xi|^2 a(Y_t) dt. \end{aligned}$$

Therefore, the bounded variation terms in  $\Xi_t$  and  $\Phi_t$  will cancel out if we multiply these two processes. Indeed: using the integration by parts formula for Itô integrals:

$$d(\Xi\Phi)_t = \Xi_t d\Phi_t + \Phi_t d\Xi_t + \langle \Xi, \Phi \rangle_t,$$

and the fact that  $\Xi$  is of bounded variation, we arrive at

$$\begin{aligned} d(\Xi\Phi)_t &= \Xi_t \cdot \left( \varphi'(Y_t) dY_t + \frac{1}{2} |\xi|^2 a(Y_t) dt \right) + \Phi_t \cdot \left( -\frac{1}{2} |\xi|^2 \Xi_t a(Y_t) dt \right) + 0 \\ &= \Xi_t \varphi'(Y_t) dY_t. \end{aligned}$$

So it turns out that  $\Xi_t \Phi_t$  is almost a martingale:  $Y_t$  is the reflected Brownian motion in  $[0, \infty)$ , so  $dY_t$  is essentially the differential of the Brownian motion, except when  $Y_t = 0$ . More precisely, if  $L_t$  is the local time of  $L_t$  at zero, then, by Tanaka’s formula,  $\tilde{Y}_t = Y_t - L_t$  is the (unrestricted) Brownian motion. Therefore, the Doob–Meyer decomposition of  $\Xi_t \Phi_t$  is given by

$$d(\Xi\Phi)_t = \Xi_t \varphi'(Y_t) d\tilde{Y}_t + \Xi_t \varphi'(Y_t) dL_t,$$

the former term being the differential of a martingale, and the latter one the differential of a bounded variation process. If we integrate over  $t$  from 0 to  $T_s$  and apply optional stopping theorem, the expectation of the martingale part vanishes, and we arrive at

$$\mathbb{E}_Y[\Xi(T_s)\Phi(T_s)] = \mathbb{E}_Y[\Xi_0\Phi_0] + \mathbb{E}_Y\left[\int_0^{T_s} \Xi_t\varphi'(Y_t)dL_t\right]$$

(we write  $\mathbb{E}_Y$  to emphasise that we have already integrated out over the law of  $W_t$ ). To take a breath before we move on, let us appreciate the fact that we have just used four significant results from the stochastic analysis toolkit: Itô's lemma, integration by parts, Tanaka's formula and optional stopping theorem, all in a single paragraph.

We are almost done. Since we assumed that  $X_0 = 0$  and  $Y_0 = 0$ , we have  $\Xi_0 = \mathbb{E}_W \exp(-i\xi X_0) = 1$ , and  $\Phi_0 = \varphi(Y_0) = 1$ . Substituting  $L_t = r$  in the integral on the right-hand side (which a Lebesgue–Stieltjes integral, not an Itô one), and recalling that  $\Phi_t = \varphi(Y_t)$ , we find that

$$\mathbb{E}_Y[\Xi(T_s)\varphi(Y(T_s))] = 1 + \mathbb{E}_Y\left[\int_0^s \Xi(T_r)\varphi'(Y(T_r))dr\right].$$

However,  $\Xi_t = \mathbb{E}_W[\exp(-i\xi X_t)]$ ,  $X(T_s) = Z_s$  is the boundary trace process and  $Y(T_s) = 0$  by the definition of the inverse local time  $T_s$ . Finally, we have  $\varphi(0) = 1$  and  $\varphi'(0) = -\mu(|\xi|^2)$ , where  $\mu$  is a complete Bernstein function (the reciprocal of the spectral function for Krein's spectral problem). Therefore, our equation reduces to

$$\mathbb{E}_Y[\mathbb{E}_W[\exp(-i\xi Z_s)] \cdot 1] = 1 + \mathbb{E}_Y\left[\int_0^s \mathbb{E}_W[\exp(-i\xi Z_r)] \cdot (-\mu(|\xi|^2))dr\right].$$

Using Fubini's theorem and the fact that  $\mathbb{E}_Y\mathbb{E}_W = \mathbb{E}$ , we conclude that

$$\mathbb{E}[\exp(-i\xi Z_s)] = 1 - \mu(|\xi|^2) \int_0^s \mathbb{E}[\exp(-i\xi Z_r)]dr.$$

The above equation is the integral form of a simple first-order ODE for  $\psi(s) = \mathbb{E}[\exp(-i\xi Z_s)]$ , namely,

$$\psi'(s) = -\mu(|\xi|^2)\psi(s)$$

with initial condition  $\psi(0) = 1$ . A solution is an appropriate exponential function, and we conclude that

$$\mathbb{E}[\exp(-i\xi Z_s)] = \exp(-s\mu(|\xi|^2)).$$

But hey, this is the Lévy–Khintchine formula for the Lévy process  $Z_s$ ! We have just proved that  $\mu(|\xi|^2)$  is the characteristic exponent of the boundary trace process  $Z_s$ , and this is exactly what we wanted.

I recommend you to follow the above argument once more, bearing in mind the final result. Our conclusion may have appeared in an unexpected way from seemingly unrelated calculations. On the second reading, however, the derivation of the Lévy–Khintchine formula for  $Z_s$  should appear much clearer and way more natural.

## Exercises.

**4.1.** Follow the recommendation from the last paragraph: read again Section 4.4 and try to figure out why it is in a sense natural to consider the process  $\Phi_t = \varphi(Y_t)$  and multiply it by  $\Xi_t$ .

**4.2.** (a) Redo the calculation from Section 4.4 with an arbitrary starting point  $(X_0, Y_0) = (x, y)$ , and stop the process  $\Xi_t\Phi_t$  at the first hitting time of 0 for  $Y_t$  (which just happens to agree with  $T_0$ ) rather than at  $T_s$ .

- (b) Prove that the characteristic function of  $X(T_0)$  is equal to  $\exp(-\varphi(|\xi|^2, y))$ , and so the distribution of  $X(T_0)$  is equal to  $P_y(x - x')dx'$ , where  $P_y$  is the Poisson kernel discussed in the previous section.
- (c) Show that if  $f$  is non-negative, then also its extension  $u$ , introduced in the previous section, is non-negative.
- (d) Prove that if  $f$  is a compactly supported continuous function, then its extension  $u$  is continuous.

**4.3.** Let  $(X_n, Y_n)$  be a  $(d + 1)$ -dimensional simple random walk on the lattice  $\mathbb{Z}^d \times \{0, 1, \dots\}$ , reflected on the boundary. (There are many ways to actually implement the reflection mechanism and you are welcome to choose your favourite one.) Define the local time  $L_n$  of  $Y_n$  at 0 in the obvious way:

$$L_n = \sum_{j=0}^n \mathbb{1}_{\{0\}}(Y_j),$$

and the inverse local time  $T_k$  in an equally natural manner:

$$T_k = \max\{n \geq 0 : L_n \leq k\}.$$

Find as much information as possible about the boundary trace process  $Z_k = X(T_k)$ .

## 5. EXTENSIONS

**5.1. More general symmetric operators and processes.** It is fairly clear that the Laplace operator  $\Delta_x$  in the Caffarelli–Silvestre extension technique can be replaced by an essentially arbitrary positive definite self-adjoint operator  $L_x$  on a Hilbert space  $H$ . One just replaces the Fourier transform  $\mathcal{F}$  by the spectral resolution of  $L_x$  to turn the extension problem into the same Krein’s ODE. While the idea is clear, the details may be overwhelming if  $L_x$  is, say, a differential operator, unless one agrees to work in the context of  $H$ -valued functions  $y \mapsto u(\cdot, y)$ . We refer to a recent preprint by Hauer and Lee [4] for further discussion.

On the probability side, the above extension corresponds to replacing the Brownian motion  $X_t$  by a more general (diffusion) process. This was carried out, with a slightly different approach, in a paper by Assing and Herman from 2021 [1].

**5.2. Non-symmetric operators and diffusions.** As it was suggested in the previous section, the evaluation of the Lévy–Khintchine exponent of the boundary trace process actually works for a broader class of operators  $L$ . We still need to assume that the coefficients only depend on  $y$ , so that  $L$  is invariant under translations, but invariance under rotations is not really important. There is no reason to expect any reasonable class of boundary trace processes that can be obtained in this way in higher dimensions, but for  $d = 1$  one gets the following, perhaps unexpected, result. We state it simultaneously in two flavours: a PDE variant and its probabilistic reformulation.

**Theorem 5.** (a) *Dirichlet-to-Neumann operators  $K$  corresponding to elliptic second-order differential operators on  $\mathbb{R} \times (0, \infty)$  of the form*

$$L = a(y) \frac{\partial^2}{\partial x^2} + b(y) \frac{\partial^2}{\partial x \partial y} + c(y) \frac{\partial^2}{\partial y^2} + d(y) \frac{\partial}{\partial x} + e(y) \frac{\partial}{\partial y}$$

*have completely monotone kernels: they are given by*

$$Kf(x) = \alpha f''(x) + \beta f'(x) + \gamma f(x) + \int_{-\infty}^{\infty} (f(x+z) - f(x) - zf'(x)\mathbf{1}_{(-1,1)}(z))J(z)dz,$$

*where  $\alpha, \gamma \geq 0$ ,  $\beta \in \mathbb{R}$ ,  $\min\{1, x^2\}J(x)$  is integrable, and both  $J(x)$  and  $J(-x)$  are completely monotone functions of  $x > 0$ .*

- (b) *The correspondence between  $L$  and  $K$  is onto, and if we restrict our attention to the case  $d(y) = e(y) = 0$ , then it is additionally one-to-one.*
- (c) *The trace of a diffusion process on  $\mathbb{R} \times [0, \infty)$ , which is reflected at the boundary and which is invariant under horizontal translations, can be identified with a Lévy process with completely monotone jumps (that is, with Lévy measure  $J(x)dx$  for a function  $J$  as in the first item).*
- (d) *Conversely, every Lévy process with completely monotone jumps is the boundary trace process for a uniquely determined diffusion in  $\mathbb{R} \times [0, \infty)$  which is additionally a martingale and the vertical component of which is the reflected Brownian motion.*

The first two statements are given in my paper from 2022 [14], the other two in a paper of mine from 2023 [15]. The proof requires some more advanced tools. Most notably, it hinges on an extension of Krein’s theory developed by Eckhardt and Kostenko in 2016 [3] (and built upon the results proved by Louis de Branges in 1960–62 [5, 6, 7, 8, 9]).

It is difficult to predict whether the above theorems will find applications, but working out their proofs brought me a lot of joy and satisfaction. And it finally made me to learn some stochastic calculus. . .

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